MA3484 Methods of Mathematical Economics
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Simplex Method Example

We consider the following linear programming problem.

minimize

$$3x_1 + 4x_2 + 2x_3 + 9x_4 + 5x_5$$

subject to the following constraints:

$5x_1 + 3x_2 + 4x_3 + 7x_4 + 3x_5 = 11;$

$4x_1 + x_2 + 3x_3 + 8x_4 + 4x_5 = 6;$

$x_j \geq 0$ *for* $j = 1, 2, 3, 4, 5.$

Simplex Method Example (continued)

The constraints require that x_1, x_2, x_3, x_4, x_5 be non-negative real numbers satisfying the matrix equation

$$\begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Thus we are required to find a (column) vector $\mathbf{x}$ with components x_1, x_2, x_3, x_4 and x_5 satisfying the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Simplex Method Example (continued)

Let

$$\mathbf{a}^{(1)} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \quad \mathbf{a}^{(2)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{a}^{(3)} = \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

$$\mathbf{a}^{(4)} = \begin{pmatrix} 7 \\ 8 \end{pmatrix} \quad \text{and} \quad \mathbf{a}^{(5)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

For a feasible solution to the problem we must find non-negative real numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\mathbf{a}^{(1)} + x_2\mathbf{a}^{(2)} + x_3\mathbf{a}^{(3)} + x_4\mathbf{a}^{(4)} + x_5\mathbf{a}^{(5)} = \mathbf{b}.$$

An optimal solution to the problem is a feasible solution that minimizes

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5$$

amongst all feasible solutions to the problem, where $c_1 = 3$, $c_2 = 4$, $c_3 = 2$, $c_4 = 9$ and $c_5 = 5$.

Simplex Method Example (continued)

Let $\mathbf{c}$ denote the column vector whose ith component is c_i respectively. Then

$$\mathbf{c}^T = \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix},$$

and an optimal solution is a feasible solution that minimizes $\mathbf{c}^T\mathbf{x}$ amongst all feasible solutions to the problem. We refer to the quantity $\mathbf{c}^T\mathbf{x}$ as the *cost* of the feasible solution $\mathbf{x}$.

Let $I = \{1, 2, 3, 4, 5\}$. We define a *basis* for this optimization problem to be a subset $\{j_1, j_2\}$ of I, where $j_1 \neq j_2$, for which the corresponding vectors $\mathbf{a}^{j_1}, \mathbf{a}^{j_2}$ constitute a basis of $\mathbb{R}^2$. By inspection we see that each pair of vectors taken from the list $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}, \mathbf{a}^{(4)}, \mathbf{a}^{(5)}$ consists of linearly independent vectors, and therefore each pair of vectors from this list constitutes a basis of $\mathbb{R}^2$. It follows that every subset of I with exactly two elements is a basis for the optimization problem.

Simplex Method Example (continued)

A feasible solution $(x_1, x_2, x_3, x_4, x_5)$ to this optimization problem is said to be a *basic feasible solution* if there exists a basis B for the optimization problem such that $x_j = 0$ when $j \neq B$.
In the case of the present problem, all subsets of $\{1, 2, 3, 4, 5\}$ with exactly two elements are bases for the problem. It follows that a feasible solution to the problem is a *basic feasible solution* if and only if the number of non-zero components of the solution does not exceed 2.

We take as given the following initial basic feasible solution $x_1 = 1$, $x_2 = 2$, $x_3 = x_4 = x_5 = 0$. One can readily verify that $\mathbf{a}^{(1)} + 2\mathbf{a}^{(2)} = \mathbf{b}$. This initial basic feasible solution is associated with the basis $\{1, 2\}$. The cost of this solution is 11.

We apply the procedures of the *simplex method* to test whether or not this basic feasible solution is optimal, and, if not, determine how to improve it.

The basis $\{1, 2\}$ determines a 2×2 minor M_B of A consisting of the first two columns of A. Thus

$$M_B = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}.$$

We now determine the components of the vector $\mathbf{p} \in \mathbb{R}^2$ whose transpose $\begin{pmatrix} p_1 & p_2 \end{pmatrix}$ satisfies the matrix equation

$$\begin{pmatrix} c_1 & c_2 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} M_B.$$

Now

$$M_B^{-1} = -\frac{1}{7} \begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}.$$

Simplex Method Example (continued)

It follows that

$$
\begin{aligned}
\mathbf{p}^T &= \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \end{pmatrix} M_B^{-1} \\
&= -\frac{1}{7} \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix} \\
&= \begin{pmatrix} \frac{13}{7} & -\frac{11}{7} \end{pmatrix}.
\end{aligned}
$$

We next compute a vector $\mathbf{q} \in \mathbb{R}^5$, where $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$.

Simplex Method Example (continued)

Solving the equivalent matrix equation for the transpose $\mathbf{q}^T$ of the column vector $\mathbf{q}$, we find that

$$\begin{aligned}
\mathbf{q}^T &= \mathbf{c}^T - \mathbf{p}^T A \\
&= \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} - \begin{pmatrix} \frac{13}{7} & -\frac{11}{7} \end{pmatrix} \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 4 & \frac{19}{7} & \frac{3}{7} & -\frac{5}{7} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & -\frac{5}{7} & \frac{60}{7} & \frac{40}{7} \end{pmatrix}.
\end{aligned}$$

We denote the jth component of the vector j by q_j.

Now $q_3 < 0$. We show that this implies that the initial basic feasible solution is not optimal, and that it can be improved by bringing 3 (the index of the third column of A) into the basis.

Suppose that $\overline{\mathbf{x}}$ is a feasible solution of this optimization problem. Then $A\overline{\mathbf{x}} = \mathbf{b}$, and therefore

$$\mathbf{c}^T\overline{\mathbf{x}} = \mathbf{p}^T A\overline{\mathbf{x}} + \mathbf{q}^T\mathbf{x} = \mathbf{p}^T\mathbf{b} + \mathbf{q}^T\overline{\mathbf{x}}.$$

The initial basic feasible solution $\mathbf{x}$ satisfies

$$\mathbf{q}^T\mathbf{x} = \sum_{j=1}^{5} q_j x_j = 0,$$

because $q_1 = q_2 = 0$ and $x_3 = x_4 = x_5 = 0$. This comes about because the manner in which we determined first $\mathbf{p}$ then $\mathbf{q}$ ensures that $q_j = 0$ for all $j \in B$, whereas the components of the basic feasible solution $\mathbf{x}$ associated with the basis B satisfy $x_j = 0$ for $j \notin B$. We find therefore that $\mathbf{p}^T\mathbf{b}$ is the cost of the initial basic feasible solution.

The cost of the initial basic feasible solution is 11, and this is equal to the value of $\mathbf{p}^T\mathbf{b}$. The cost $\mathbf{c}^T\overline{\mathbf{x}}$ of any other basic feasible solution satisfies

$$\mathbf{c}^T\overline{\mathbf{x}} = 11 - \frac{5}{7}\overline{x}_3 + \frac{60}{7}\overline{x}_4 + \frac{40}{7}\overline{x}_5,$$

where $\overline{x}_j$ denotes the jth component of $\overline{\mathbf{x}}$.

We seek to determine a new basic feasible solution $\overline{\mathbf{x}}$ for which $\overline{x}_3 > 0$, $\overline{x}_4 = 0$ and $\overline{x}_5 = 0$. The cost of such a basic feasible solution will then be less than that of our initial basic feasible solution.

Simplex Method Example (continued)

In order to find our new basic feasible solution we determine the relationships between the coefficients of a feasible solution $\overline{\mathbf{x}}$ for which $\overline{x}_4 = 0$ and $\overline{x}_5 = 0$. Now such a feasible solution must satisfy

$$\overline{x}_1 \mathbf{a}^{(1)} + \overline{x}_2 \mathbf{a}^{(2)} + \overline{x}_3 \mathbf{a}^{(3)} = \mathbf{b} = x_1 \mathbf{a}^{(1)} + x_2 \mathbf{a}^{(2)},$$

where x_1 and x_2 are the non-zero coefficients of the initial basic feasible solution. Now the vectors $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ constitute a basis of the real vector space $\mathbb{R}^2$. It follows that there exist real numbers $t_{1,3}$ and $t_{2,3}$ such that $\mathbf{a}^{(3)} = t_{1,3}\mathbf{a}^{(1)} + t_{2,3}\mathbf{a}^{(2)}$. It follows that

$$(\overline{x}_1 + t_{1,3}\overline{x}_3)\mathbf{a}^{(1)} + (\overline{x}_2 + t_{2,3}\overline{x}_3)\mathbf{a}^{(2)} = x_1 \mathbf{a}^{(1)} + x_2 \mathbf{a}^{(2)}.$$

Simplex Method Example (continued)

The linear independence of $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ then ensures that $\overline{x}_1 + t_{1,3}\overline{x}_3 = x_1$ and $\overline{x}_2 + t_{2,3}\overline{x}_3 = x_2$. Thus if $\overline{x}_3 = \lambda$, where $\lambda \geq 0$ then

$$\overline{x}_1 = x_1 - \lambda t_{1,3}, \quad \overline{x}_2 = x_2 - \lambda t_{2,3}.$$

Thus, once $t_{1,3}$ and $t_{2,3}$ have been determined, we can determine the range of values of λ that ensure that $\overline{x}_1 \geq 0$ and $\overline{x}_2 \geq 0$.

Simplex Method Example (continued)

In order to determine the values of $t_{1,3}$ and $t_{2,3}$ we note that

$$\begin{aligned} \mathbf{a}^{(1)} &= \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathbf{a}^{(2)} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

and therefore

Simplex Method Example (continued)

$$\begin{aligned}\mathbf{a}^{(3)} &= t_{3,1}\mathbf{a}^{(1)} + t_{3,2}\mathbf{a}^{(2)} = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} t_{3,1} \\ t_{3,2} \end{pmatrix} \\ &= M_B \begin{pmatrix} t_{3,1} \\ t_{3,2} \end{pmatrix},\end{aligned}$$

where $M_B = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}$. It follows that

$$\begin{pmatrix} t_{3,1} \\ t_{3,2} \end{pmatrix} = M_B^{-1}\mathbf{a}^{(3)} = -\frac{1}{7} \begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} \\ \frac{1}{7} \end{pmatrix}.$$

Thus $t_{3,1} = \frac{5}{7}$ and $t_{3,2} = \frac{1}{7}$.

Simplex Method Example (continued)

We now determine the feasible solutions $\overline{\mathbf{x}}$ of this optimization problem that satisfy $\overline{x}_3 = \lambda$ and $\overline{x}_4 = \overline{x}_5 = 0$. we have already shown that

$$\overline{x}_1 = x_1 - \lambda t_{1,3}, \quad \overline{x}_2 = x_2 - \lambda t_{2,3}.$$

Now $x_1 = 1$, $x_2 = 2$, $t_{1,3} = \frac{5}{7}$ and $t_{2,3} = \frac{1}{7}$. It follows that $\overline{x}_1 = 1 - \frac{5}{7}\lambda$ and $\overline{x}_2 = 2 - \frac{1}{7}\lambda$. Now the components of a feasible solution must satisfy $\overline{x}_1 \geq 0$ and $\overline{x}_2 \geq 0$. it follows that $0 \leq \lambda \leq \frac{7}{5}$. Moreover on setting $\lambda = \frac{7}{5}$ we find that $\overline{x}_1 = 0$ and $\overline{x}_2 = \frac{9}{5}$. We thus obtain a new basic feasible solution $\overline{\mathbf{x}}$ associated to the basis $\{2, 3\}$, where

$$\overline{\mathbf{x}}^T = \begin{pmatrix} 0 & \frac{9}{5} & \frac{7}{5} & 0 & 0 \end{pmatrix}.$$

The cost of this new basic feasible solution is 10.

Simplex Method Example (continued)

We now let B and $\mathbf{x}$ denote the new basic and new associated basic feasible solution respectively, so that $B = \{2, 3\}$ and

$$\overline{\mathbf{x}}^T = \begin{pmatrix} 0 & \frac{9}{5} & \frac{7}{5} & 0 & 0 \end{pmatrix}.$$

We also let M_B be the 2×2 minor of the matrix A with columns indexed by the new basis B, so that

$$M_B = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad M_B^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}.$$

Simplex Method Example (continued)

We now determine the components of the vector $\mathbf{p} \in \mathbb{R}^2$ whose transpose $\begin{pmatrix} p_1 & p_2 \end{pmatrix}$ satisfies the matrix equation

$$\begin{pmatrix} c_2 & c_3 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} M_B.$$

We find that

$$\begin{aligned} \begin{pmatrix} p_1 & p_2 \end{pmatrix} &= \begin{pmatrix} c_2 & c_3 \end{pmatrix} M_B^{-1} \\ &= \frac{1}{5} \begin{pmatrix} 4 & 2 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 \end{pmatrix}. \end{aligned}$$

Simplex Method Example (continued)

We next compute the components of the vector $\mathbf{q} \in \mathbb{R}^5$ so as to ensure that

$$\begin{aligned}
\mathbf{q}^T &= \mathbf{c}^T - \mathbf{p}^T A \\
&= \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} - \begin{pmatrix} 2 & -2 \end{pmatrix} \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 2 & -2 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 11 & 7 \end{pmatrix}.
\end{aligned}$$

The components of the vector $\mathbf{q}$ determined using the new basis $\{2, 3\}$ are all non-negative. This ensures that the new basic feasible solution is an optimal solution.

Simplex Method Example (continued)

Indeed let $\overline{\mathbf{x}}$ be a feasible solution of this optimization problem. Then $A\overline{\mathbf{x}} = \mathbf{b}$, and therefore

$$\mathbf{c}^T \overline{\mathbf{x}} = \mathbf{p}^T A \overline{\mathbf{x}} + \mathbf{q}^T \mathbf{x} = \mathbf{p}^T \mathbf{b} + \mathbf{q}^T \overline{\mathbf{x}}.$$

Moreover $\mathbf{p}^T \mathbf{b} = 10$. It follows that

$$\mathbf{c}^T \overline{\mathbf{x}} = 10 + \overline{x}_1 + 11\overline{x}_4 + 7\overline{x}_5 \geq 10,$$

and thus the new basic feasible solution $\mathbf{x}$ is optimal.

Simplex Method Example (continued)

We summarize the result we have obtained. The optimization problem was the following:—

minimize

$$3x_1 + 4x_2 + 2x_3 + 9x_4 + 5x_5$$

subject to the following constraints:
$5x_1 + 3x_2 + 4x_3 + 7x_4 + 3x_5 = 11;$
$4x_1 + x_2 + 3x_3 + 8x_4 + 4x_5 = 6;$
$x_j \geq 0$ *for* $j = 1, 2, 3, 4, 5.$

We have found the following basic optimal solution to the problem:

$$x_1 = 0, \quad x_2 = \frac{9}{5}, \quad x_3 = \frac{7}{5}, \quad x_4 = 0, \quad x_5 = 0.$$

Simplex Method Example (continued)

We shall make a further study of this optimization problem, and then describe some tableaux that systematically record steps in the calculation.

First we find all basic feasible solutions to the problem. The problem is to find $\mathbf{x} \in \mathbb{R}^5$ that minimizes $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where

$$A = \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

and

$$\mathbf{c}^T = \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix}.$$

For each two-element subset B of $\{1, 2, 3, 4, 5\}$ we compute M_B, M_B^{-1} and $M_B^{-1}\mathbf{b}$, where M_B is the 2×2 minor of the matrix A whose columns are indexed by the elements of B. We find the following:—

B	M_B	M_B^{-1}	$M_B^{-1}\mathbf{b}$	$\mathbf{c}^T M_B^{-1}\mathbf{b}$
$\{1, 2\}$	$\begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}$	$-\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	11
$\{1, 3\}$	$\begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$	$-\begin{pmatrix} 3 & -4 \\ -4 & 5 \end{pmatrix}$	$\begin{pmatrix} -9 \\ 14 \end{pmatrix}$	1
$\{1, 4\}$	$\begin{pmatrix} 5 & 7 \\ 4 & 8 \end{pmatrix}$	$\frac{1}{12}\begin{pmatrix} 8 & -7 \\ -4 & 5 \end{pmatrix}$	$\begin{pmatrix} \frac{23}{6} \\ -\frac{7}{6} \end{pmatrix}$	1

Simplex Method Example (continued)

B	M_B	M_B^{-1}	$M_B^{-1}\mathbf{b}$	$\mathbf{c}^T M_B^{-1}\mathbf{b}$
$\{1,5\}$	$\begin{pmatrix} 5 & 3 \\ 4 & 4 \end{pmatrix}$	$\frac{1}{8}\begin{pmatrix} 4 & -3 \\ -4 & 5 \end{pmatrix}$	$\begin{pmatrix} \frac{13}{4} \\ -\frac{7}{4} \end{pmatrix}$	1
$\{2,3\}$	$\begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$	$\frac{1}{5}\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}$	$\begin{pmatrix} \frac{9}{5} \\ \frac{7}{5} \end{pmatrix}$	10
$\{2,4\}$	$\begin{pmatrix} 3 & 7 \\ 1 & 8 \end{pmatrix}$	$\frac{1}{17}\begin{pmatrix} 8 & -7 \\ -1 & 3 \end{pmatrix}$	$\begin{pmatrix} \frac{46}{17} \\ \frac{7}{17} \end{pmatrix}$	$\frac{247}{17}$
$\{2,5\}$	$\begin{pmatrix} 3 & 3 \\ 1 & 4 \end{pmatrix}$	$\frac{1}{9}\begin{pmatrix} 4 & -3 \\ -1 & 3 \end{pmatrix}$	$\begin{pmatrix} \frac{26}{9} \\ \frac{7}{9} \end{pmatrix}$	$\frac{139}{9}$
$\{3,4\}$	$\begin{pmatrix} 4 & 7 \\ 3 & 8 \end{pmatrix}$	$\frac{1}{11}\begin{pmatrix} 8 & -7 \\ -3 & 4 \end{pmatrix}$	$\begin{pmatrix} \frac{46}{11} \\ -\frac{9}{11} \end{pmatrix}$	1

Simplex Method Example (continued)

B	M_B	M_B^{-1}	$M_B^{-1}\mathbf{b}$	$\mathbf{c}^T M_B^{-1}\mathbf{b}$
$\{3,5\}$	$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$	$\frac{1}{7}\begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}$	$\begin{pmatrix} \frac{26}{7} \\ -\frac{9}{7} \end{pmatrix}$	1
$\{4,5\}$	$\begin{pmatrix} 7 & 3 \\ 8 & 4 \end{pmatrix}$	$\frac{1}{4}\begin{pmatrix} 4 & -3 \\ -8 & 7 \end{pmatrix}$	$\begin{pmatrix} \frac{13}{2} \\ -\frac{23}{2} \end{pmatrix}$	1

From this data, we see that there are four basic feasible solutions to the problem. We tabulate them below:—

B	$\mathbf{x}$	*Cost*
$\{1,2\}$	$(1,2,0,0,0)$	11
$\{2,3\}$	$(0,\frac{9}{5},\frac{7}{5},0,0)$	10
$\{2,4\}$	$(0,\frac{46}{17},0,\frac{7}{17},0)$	$\frac{247}{17} = 14.529\ldots$
$\{2,5\}$	$(0,\frac{26}{9},0,0,\frac{7}{9})$	$\frac{139}{9} = 15.444\ldots$