MA3484 Methods of Mathematical Economics
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Dual Spaces

Definition

Let V be a real vector space. A *linear functional* $\varphi\colon V \to \mathbb{R}$ on V is a linear transformation from the vector space V to the field $\mathbb{R}$ of real numbers.

Given linear functionals $\varphi\colon V \to \mathbb{R}$ and $\psi\colon V \to \mathbb{R}$ on a real vector space V, and given any real number λ, we define $\varphi + \psi$ and $\lambda\varphi$ to be the linear functionals on V defined such that $(\varphi + \psi)(\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v})$ and $(\lambda\varphi)(\mathbf{v}) = \lambda\,\varphi(\mathbf{v})$ for all $\mathbf{v} \in V$.

The set V^* of linear functionals on a real vector space V is itself a real vector space with respect to the algebraic operations of addition and multiplication-by-scalars defined above.

Definition

Let V be a real vector space. The *dual space* V^* of V is the vector space whose elements are the linear functionals on the vector space V.

Dual Spaces (continued)

Now suppose that the real vector space V is finite-dimensional. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, where $n = \dim V$. Given any $\mathbf{v} \in V$ there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j$. It follows that there are well-defined functions $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ from V to the field $\mathbb{R}$ defined such that

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. These functions are linear transformations, and are thus linear functionals on V.

Lemma

Lemma LA-13 *Let V be a finite-dimensional real vector space, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the linear functionals on V defined such that*

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ constitute a basis of the dual space V^* of V. Moreover

$$\varphi = \sum_{i=1}^{n} \varphi(\mathbf{u_i}) \varepsilon_i$$

for all $\varphi \in V^*$.

Proof

Let $\mu_1, \mu_2, \ldots, \mu_n$ be real numbers with the property that $\sum_{i=1}^{n} \mu_i \varepsilon_i = \mathbf{0}_{V^*}$. Then

$$0 = \left(\sum_{i=1}^{n} \mu_i \varepsilon_i \right) (\mathbf{u}_j) = \sum_{i=1}^{n} \mu_i \varepsilon_i (\mathbf{u}_j) = \mu_j$$

for $j = 1, 2, \ldots, n$. Thus the linear functionals $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ on V are linearly independent elements of the dual space V^*.

Dual Spaces (continued)

Now let $\varphi\colon V \to \mathbb{R}$ be a linear functional on V, and let $\mu_i = \varphi(\mathbf{u}_i)$ for $i = 1, 2, \ldots, n$. Now

$$\varepsilon_i(\mathbf{u}_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

It follows that

$$\begin{aligned} \left(\sum_{i=1}^{n} \mu_i \varepsilon_i\right)\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i \lambda_j \varepsilon_i(\mathbf{u}_j) = \sum_{j=1}^{n} \mu_j \lambda_j \\ &= \sum_{j=1}^{n} \lambda_j \varphi(\mathbf{u}_j) = \varphi\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right) \end{aligned}$$

for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

It follows that

$$\varphi = \sum_{i=1}^{n} \mu_i \varepsilon_i = \sum_{i=1}^{n} \varphi(\mathbf{u}_i)\varepsilon_i.$$

We conclude from this that every linear functional on V can be expressed as a linear combination of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Thus these linear functionals span V^*. We have previously shown that they are linearly independent. It follows that they constitute a basis of V^*. Moreover we have verified that $\varphi = \sum_{i=1}^{n} \varphi(\mathbf{u}_i)\varepsilon_i$ for all $\varphi \in V^*$, as required. ∎

Definition

Let V be a finite-dimensional real vector space, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V. The corresponding *dual basis* of the dual space V^* of V consists of the linear functionals $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ on V, where

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Corollary

Corollary LA-14 *Let V be a finite-dimensional real vector space, and let V^* be the dual space of V. Then* $\dim V^* = \dim V$.

Proof

We have shown that any basis of V gives rise to a dual basis of V^*, where the dual basis of V has the same number of elements as the basis of V to which it corresponds. The result follows immediately from the fact that the dimension of a finite-dimensional real vector space is the number of elements in any basis of that vector space. ∎

Dual Spaces (continued)

Let V be a real-vector space, and let V^* be the dual space of V. Then V^* is itself a real vector space, and therefore has a dual space V^{**}. Now each element $\mathbf{v}$ of V determines a corresponding linear functional $E_{\mathbf{v}}\colon V^* \to \mathbb{R}$ on V^*, where $E_{\mathbf{v}}(\varphi) = \varphi(\mathbf{v})$ for all $\varphi \in V^*$. It follows that there exists a function $\iota\colon V \to V^{**}$ defined so that $\iota(\mathbf{v}) = E_{\mathbf{v}}$ for all $\mathbf{v} \in V$. Then $\iota(\mathbf{v})(\varphi) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in V$ and $\varphi \in V^*$.

Dual Spaces (continued)

Now

$$\iota(\mathbf{v}+\mathbf{w})(\varphi) = \varphi(\mathbf{v}+\mathbf{w}) = \varphi(\mathbf{v}) + \varphi(\mathbf{w}) = (\iota(\mathbf{v}) + \iota(\mathbf{w}))(\varphi)$$

and

$$\iota(\lambda\mathbf{v})(\varphi) = \varphi(\lambda\mathbf{v}) = \lambda\varphi(\mathbf{v}) = (\lambda\iota(\mathbf{v}))(\varphi)$$

for all $\mathbf{v}, \mathbf{w} \in V$ and $\varphi \in V^*$ and for all real numbers λ. It follows that $\iota(\mathbf{v}+\mathbf{w}) = \iota(\mathbf{v}) + \iota(\mathbf{w})$ and $\iota(\lambda\mathbf{v}) = \lambda\iota(\mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$ and for all real numbers λ. Thus $\iota\colon V \to V^{**}$ is a linear transformation.

Proposition

Proposition LA-15 *Let V be a finite-dimensional real vector space, and let $\iota\colon V \to V^{**}$ be the linear transformation defined such that $\iota(\mathbf{v})(\varphi) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in V$ and $\varphi \in V^*$. Then $\iota\colon V \to V^{**}$ is an isomorphism of real vector spaces.*

Proof

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the dual basis of V^*, where

$$\varepsilon_i(\mathbf{u}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and let $\mathbf{v} \in V$. Then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$.

Suppose that $\iota(\mathbf{v}) = \mathbf{0}_{V^{**}}$. Then $\varphi(\mathbf{v}) = E_{\mathbf{v}}(\varphi) = 0$ for all $\varphi \in V^*$. In particular $\lambda_i = \varepsilon_i(\mathbf{v}) = 0$ for $i = 1, 2, \ldots, n$, and therefore $\mathbf{v} = \mathbf{0}_V$. We conclude that $\iota\colon V \to V^{**}$ is injective.

Now let $F\colon V^* \to \mathbb{R}$ be a linear functional on V^*, let $\lambda_i = F(\varepsilon_i)$ for $i = 1, 2, \ldots, n$, let $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$, and let $\varphi \in V^*$. Then $\varphi = \sum_{i=1}^{n} \varphi(\mathbf{u}_i)\varepsilon_i$ (see Lemma LA-13), and therefore

$$\begin{aligned} \iota(\mathbf{v})(\varphi) &= \varphi(\mathbf{v}) = \sum_{i=1}^{n} \lambda_i \varphi(\mathbf{u}_i) = \sum_{i=1}^{n} F(\varepsilon_i)\varphi(\mathbf{u}_i) \\ &= F\left(\sum_{i=1}^{n} \varphi(\mathbf{u}_i)\varepsilon_i\right) = F(\varphi). \end{aligned}$$

Thus $\iota(\mathbf{v}) = F$. We conclude that the linear transformation $\iota\colon V \to V^{**}$ is surjective. We have previously shown that this linear transformation is injective. There $\iota\colon V \to V^{**}$ is an isomorphism between the real vector spaces V and V^{**} as required. ■

The following corollary is an immediate consequence of Proposition LA-15.

Corollary

Corollary LA-16 *Let V be a finite-dimensional real vector space, and let V^* be the dual space of V. Then, given any linear functional $F\colon V^* \to \mathbb{R}$, there exists some $\mathbf{v} \in V$ such that $F(\varphi) = \varphi(\mathbf{v})$ for all $\varphi \in V^*$.*

Definition

Let V and W be real vector spaces, and let $\theta\colon V \to W$ be a linear transformation from V to W. The *adjoint* $\theta^*\colon W^* \to V^*$ of the linear transformation $\theta\colon V \to W$ is the linear transformation from the dual space W^* of W to the dual space V^* of V defined such that $(\theta^*\eta)(\mathbf{v}) = \eta(\theta(\mathbf{v}))$ for all $\mathbf{v} \in V$ and $\eta \in W^*$.

Linear Transformations and Matrices

Let V and V' be finite-dimensional vector spaces, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, and let $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_{n'}$ be a basis of V'. Then every linear transformation $\theta\colon V \to V'$ can be represented with respect to these bases by an $n' \times n$ matrix, where $n = \dim V$ and $n' = \dim V'$. The basic formulae are presented in the following proposition.

Proposition

Proposition LA-17 *Let V and V' be finite-dimensional vector spaces, and let $\theta\colon V \to V'$ be a linear transformation from V to V'. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, and let $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_{n'}$ be a basis of V'. Let A be the $n' \times n$ matrix whose coefficients $(A)_{k,j}$ are determined such that $\theta(\mathbf{u}_j) = \sum_{k=1}^{m} (A)_{k,j}\mathbf{u}'_k$ for $k = 1, 2, \ldots, n'$. Then*

$$\theta\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right) = \sum_{k=1}^{m} \mu_k \mathbf{u}'_k,$$

where $\mu_k = \sum_{j=1}^{n} (A)_{k,j}\lambda_j$ for $k = 1, 2, \ldots, n'$.

Proof

This result is a straightforward calculation, using the linearity of $\theta\colon V \to V'$. Indeed

$$\begin{aligned} \theta\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right) &= \sum_{j=1}^{n} \lambda_j \theta(\mathbf{u}_j) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n'} (A)_{k,j} \lambda_j \mathbf{u}'_k. \end{aligned}$$

It follows that $\theta\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right) = \sum_{k=1}^{n'} \mu_k \mathbf{u}'_k$, where $\mu_k = \sum_{j=1}^{n} (A)_{k,j} \lambda_j$ for $k = 1, 2, \ldots, n'$, as required. ∎

Corollary

Corollary LA-18 *Let V, V' and V'' be finite-dimensional vector spaces, and let $\theta\colon V \to V'$ be a linear transformation from V to V' and let $\psi\colon V' \to V''$ be a linear transformation from V' to V''. Let A and B be the matrices representing the linear transformations θ and ψ respectively with respect to chosen bases of V, V' and V''. Then the matrix representing the composition $\psi \circ \theta$ of the linear transformations θ and ψ is the product BA of the matrices representing those linear transformations.*

Proof

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, let $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_{n'}$ be a basis of V', and let $\mathbf{u}''_1, \mathbf{u}''_2, \ldots, \mathbf{u}''_{n''}$ be a basis of V''. Let A and B be the matrices whose coefficients $(A)_{k,j}$ and $(B)_{i,k}$ are determined such that $\theta(\mathbf{u}_j) = \sum_{k=1}^{n'} (A)_{k,j} \mathbf{u}'_k$ for $k = 1, 2, \ldots, n'$ and $\psi(\mathbf{u}'_k) = \sum_{i=1}^{p} (B)_{i,k} \mathbf{u}''_i$. Then

$$\psi\left(\theta\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right)\right) = \sum_{i=1}^{p} \nu_i \mathbf{u}''_i,$$

where

Linear Transformations and Matrices (continued)

$$\nu_i = \sum_{j=1}^{n} \left(\sum_{k=1}^{n'} (B)_{l,k} (A)_{k,j} \right) \lambda_j$$

for $l = 1, 2, \ldots, p$. Thus the composition $\psi \circ \theta$ of the linear transformations $\theta \colon V \to V'$ and $\psi \colon V' \to V''$ is represented by the product BA of the matrix B representing ψ and the matrix A representing A with respect to the chosen bases of V, V' and V'', as required. ∎

Lemma

Lemma LA-19 *Let V and W be finite-dimensional real vector spaces, and let $\theta\colon V \to W$ be a linear transformation from V to W. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the corresponding dual basis of the dual space V^* of V, let $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_{n'}$ be a basis of W, and let $\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_n$ be the corresponding dual basis of the dual space W^* of W. Then the matrix representing the adjoint $\theta^*\colon W^* \to V^*$ of $\theta\colon V \to W$ with respect to the dual bases of W^* and V^* is the transpose of the matrix representing $\theta\colon V \to W$ with respect to the chosen bases of V and W.*

Proof

Let A be the $n' \times n$ matrix representing the linear transformation $\theta\colon V \to W$ with respect to the chosen bases. Then $\varphi(\mathbf{u}_j) = \sum_{i=1}^{n'} (A)_{i,j} \mathbf{u}'_i$ for $j = 1, 2, \ldots, n$. Let $\mathbf{v} \in V$ and $\eta \in W^*$, let $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$, let $\eta = \sum_{j=1}^{n'} c_i \varepsilon'_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $c_1, c_2, \ldots, c_{n'}$ are real numbers. Then

Linear Transformations and Matrices (continued)

$$\begin{aligned}
(\theta^*\eta)(\mathbf{v}) &= \eta(\theta(\mathbf{v})) = \eta\left(\sum_{j=1}^{n} \lambda_j \theta(\mathbf{u}_j)\right) \\
&= \sum_{j=1}^{n} \lambda_j \eta((\theta(\mathbf{u}_j)) = \sum_{j=1}^{n} \lambda_j \eta\left(\sum_{i=1}^{n'} (A)_{i,j} \mathbf{u}'_i\right) \\
&= \sum_{i=1}^{n'} \sum_{j=1}^{n} (A)_{i,j} \lambda_j \eta(\mathbf{u}'_i) = \sum_{i=1}^{n'} \sum_{j=1}^{n} (A)_{i,j} \lambda_j c_i.
\end{aligned}$$

Linear Transformations and Matrices (continued)

Thus if $\eta = \sum_{j=1}^{n'} c_i \varepsilon'_i$, where $c_1, c_2, \ldots, c_n$ are real numbers, then $\theta^* \eta = \sum_{i=1}^{n} h_j \varepsilon_j$, where where

$$h_j = \sum_{i=1}^{n'} (A)_{i,j} c_i = \sum_{i=1}^{n'} (A^T)_{j,i} c_i$$

for $j = 1, 2, \ldots, n$, and where A^T is the transpose of the matrix A, defined so that $(A^T)_{j,i} = A_{i,j}$ for $i = 1, 2, \ldots, n'$ and $j = 1, 2, \ldots, n$. It follows from this that the matrix that represents the adjoint θ^* with respect to the dual bases on W^* and V^* is the transpose of the matrix A that represents θ with respect to the chosen bases on V and W, as required. ∎