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Definition
A *real vector space* consists of a set V on which there is defined an operation of vector addition, yielding an element $\mathbf{v} + \mathbf{w}$ of V for each pair $\mathbf{v}, \mathbf{w}$ of elements of V, and an operation of multiplication-by-scalars that yields an element $\lambda\mathbf{v}$ of V for each $\mathbf{v} \in V$ and for each real number λ. The operation of vector addition is required to be commutative and associative. There must exist a zero element $\mathbf{0}_V$ of V that satisfies $\mathbf{v} + \mathbf{0}_V = \mathbf{v}$ for all $\mathbf{v} \in V$, and, for each $\mathbf{v} \in V$ there must exist an element $-\mathbf{v}$ of V for which $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}_V$. The following identities must also be satisfied for all $\mathbf{v}, \mathbf{w} \in V$ and for all real numbers λ and μ:

$$(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}, \quad \lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w},$$

$$\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}, \quad 1\mathbf{v} = \mathbf{v}.$$

Let n be a positive integer. The set $\mathbb{R}^n$ consisting of all n-tuples of real numbers is then a real vector space, with addition and multiplication-by-scalars defined such that

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$$

and

$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}$ and for all real numbers λ.

The set $M_{m,n}(\mathbb{R})$ of all $m \times n$ matrices is a real vector space with respect to the usual operations of matrix addition and multiplication of matrices by real numbers.

Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ of a real vector space V are said to be *linearly dependent* if there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$, not all zero, such that

$$\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_m \mathbf{u}_m = \mathbf{0}_V.$$

If elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ of real vector space V are not linearly dependent, then they are said to be *linearly independent*.

Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of a real vector space V are said to *span* V if, given any element $\mathbf{v}$ of V, there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n$.

A vector space is said to be *finite-dimensional* if there exists a finite subset of V whose members span V.

Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of a finite-dimensional real vector space V are said to constitute a *basis* of V if they are linearly independent and span V.

Lemma

Lemma LA-01   *Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of a real vector space V constitute a basis of V if and only if, given any element $\mathbf{v}$ of V, there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that*

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n.$$

Proof

Suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a basis of V. Let $\mathbf{v}$ be an element V. The requirement that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span V ensures that there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$v = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n.$$

If $\mu_1, \mu_2, \ldots, \mu_n$ are real numbers for which

$$v = \mu_1 \mathbf{u}_1 + \mu_2 \mathbf{u}_2 + \cdots + \mu_n \mathbf{u}_n,$$

then

$$(\mu_1 - \lambda_1)\mathbf{u}_1 + (\mu_2 - \lambda_2)\mathbf{u}_2 + \cdots + (\mu_n - \lambda_n)\mathbf{u}_n = \mathbf{0}_V.$$

It then follows from the linear independence of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ that $\mu_i - \lambda_i = 0$ for $i = 1, 2, \ldots, n$, and thus $\mu_i = \lambda_i$ for $i = 1, 2, \ldots, n$. This proves that the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are uniquely-determined.

Conversely suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a list of elements of V with the property that, given any element $\mathbf{v}$ of V, there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$v = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n.$$

Then $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span V. Moreover we can apply this criterion when $\mathbf{v} = 0$. The uniqueness of the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ then ensures that if

$$\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n = \mathbf{0}_V$$

then $\lambda_i = 0$ for $i = 1, 2, \ldots, n$. Thus $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent. This proves that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a basis of V, as required. ∎

Proposition

Proposition LA-02   *Let V be a finite-dimensional vector space, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be elements of V that span the vector space V, and let K be a subset of $\{1, 2, \ldots, n\}$. Suppose either that $K = \emptyset$ or else that those elements $\mathbf{u}_i$ for which $i \in K$ are linearly independent. Then there exists a basis of V whose members belong to the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ which includes all the vectors $\mathbf{u}_i$ for which $i \in K$.*

Proof

We prove the result by induction on the number of elements in the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of vectors that span V. The result is clearly true when $n = 1$. Thus suppose, as the induction hypothesis, that the result is true for all lists of elements of V that span V and that have fewer than n members.

If the elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent, then they constitute the required basis. If not, then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n = \mathbf{0}_V.$$

Now there cannot exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all zero, such that both $\sum_{i=1}^{n} \lambda_i \mathbf{u}_i = \mathbf{0}_V$ and also $\lambda_i = 0$ whenever $i \neq K$. Indeed, in the case where $K = \emptyset$, this conclusion follows from the requirement that the real numbers λ_i cannot all be zero, and, in the case where $K \neq \emptyset$, the conclusion follows from the linear independence of those $\mathbf{u}_i$ for which $i \in K$. Therefore there must exist some integer i satisfying $1 \leq i \leq n$ for which $\lambda_i \neq 0$ and $i \notin K$. Without loss of generality, we may suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are ordered so that $n \notin K$ and $\lambda_n \neq 0$. Then

$$\mathbf{u}_n = -\sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda_n} \mathbf{u}_i.$$

Let $\mathbf{v}$ be an element of V. Then there exist real numbers $\mu_1, \mu_2, \ldots, \mu_n$ such that $\mathbf{v} = \sum\limits_{i=1}^{n} \mu_i \mathbf{u}_i$, because $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span V. But then

$$\mathbf{v} = \sum_{i=1}^{n-1} \left(\mu_i - \frac{\mu_n \lambda_i}{\lambda_n} \right) \mathbf{u}_i.$$

We conclude that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$ span the vector space V. The induction hypothesis then ensures that there exists a basis of V consisting of members of this list that includes the linearly independent elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$, as required. ∎

Corollary

Corollary LA-03   *Let V be a finite-dimensional vector space, and let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be elements of V that span the vector space V. Then there exists a basis of V whose elements are members of the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.*

Proof

This result is a restatement of Proposition LA-02 in the special case where the set K in the statement of that proposition is the empty set. ∎

Proposition

Proposition LA-04 *Let V be a finite-dimensional real vector space, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, let $\mathbf{w}$ be an element of V, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the unique real numbers for which $\mathbf{w} = \sum\limits_{i=1}^{n} \lambda_i \mathbf{u}_i$. Suppose that $\lambda_j \neq 0$ for some integer j between 1 and n. Then the element $\mathbf{w}$ of V and those elements $\mathbf{u}_i$ of the given basis for which $i \neq j$ together constitute a basis of V.*

Proof

We result follows directly when $n = 1$. Thus it suffices to prove the result when $n > 1$. We may suppose, without loss of generality, that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are ordered so that $j = n$. We must then show that $\mathbf{w}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$ is a basis of V. Now

$$\mathbf{w} = \sum_{i=1}^{n-1} \lambda_i \mathbf{u}_i + \lambda_n \mathbf{u}_n,$$

where $\lambda_n \neq 0$, and therefore

$$\mathbf{u}_n = \frac{1}{\lambda_n} \mathbf{w} - \sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda_n} \mathbf{u}_{n-1}.$$

Let $\mathbf{v}$ be an element of V. Then there exist real numbers $\mu_1, \mu_2, \ldots, \mu_n$ such that $\mathbf{v} = \sum\limits_{i=1}^{n} \mu_i \mathbf{u}_i$. Then

$$\mathbf{v} = \frac{\mu_n}{\lambda_n} \, \mathbf{w} + \sum_{i=1}^{n-1} \left(\mu_i - \frac{\lambda_i \mu_n}{\lambda_n} \right) \, \mathbf{u}_i.$$

We conclude from this that the vectors $\mathbf{w}, \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$ span the vector space V.

Now let $\rho_0, \rho_1, \ldots, \rho_{n-1}$ be real numbers with the property that

$$\rho_0 \mathbf{w} + \sum_{i=1}^{n-1} \rho_i \mathbf{u}_i = \mathbf{0}_V.$$

Then

$$\sum_{i=1}^{n-1} (\rho_i + \rho_0 \lambda_i) \mathbf{u}_i + \rho_0 \lambda_n \mathbf{u}_n = \mathbf{0}_V.$$

It then follows from the linear independence of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ that $\rho_i + \rho_0 \lambda_i = 0$ for $i = 1, 2, \ldots, n-1$ and $\rho_0 \lambda_n = 0$. But $\lambda_n \neq 0$. It follows that $\rho_0 = 0$. But then $\rho_i = -\rho_0 \lambda_i = 0$ for $i = 1, 2, \ldots, n-1$. This proves that $\mathbf{w}, \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$ are linearly independent. These vectors therefore constitute a basis of the vector space V, as required. ■

Proposition

Proposition LA-05 *Let V be a finite-dimensional real vector space, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be elements of V that span the vector space V and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be linearly independent elements of V. Then $m \leq n$, and there exists a basis of V consisting of the elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ together with not more than $n - m$ elements belonging to the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.*

Proof

If the elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ spanning V are not linearly independent then it follows from Corollary LA-03 that we may remove elements from this list so as to obtain a basis for the vector space V. We may therefore assume, without loss of generality, that the elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ constitute a basis of V with n elements.

Suppose that $m \geq 1$. It then follows from Proposition LA-04 that there exists a basis of V consisting of $\mathbf{w}_1$ together with $n - 1$ members of the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Suppose that, for some integer k satisfying $1 \leq k < m$ and $k < n$, there exist distinct integers $j_1, j_2, \ldots, j_k$ between 1 and n such that the elements $\mathbf{w}_i$ for $1 \leq i \leq k$ together with the elements $\mathbf{u}_i$ for $i \notin \{j_1, j_2, \ldots, j_k\}$ together constitute a basis of the vector space V. Then there exist real numbers $\rho_1, \rho_2, \ldots, \rho_k$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\mathbf{w}_{k+1} = \sum_{s=1}^{k} \rho_s \mathbf{w}_s + \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$$

and

$$\lambda_i = 0 \text{ for } i = j_1, j_2, \ldots, j_k.$$

If it were the case that $\lambda_i = 0$ for all integers i satisfying $1 \leq i \leq n$ then $\mathbf{w}_{k+1}$ would be expressible as a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$, and therefore the elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k+1}$ of V would be linearly dependent. But these elements are linearly independent. It follows that $\lambda_{j_{k+1}} \neq 0$ for some integer j_{k+1} between 1 and n. Moreover the integers $j_1, j_2, \ldots, j_{k+1}$ are then distinct, and it follows from Proposition LA-04 that the elements $\mathbf{w}_i$ for $1 \leq i \leq k+1$ together with the elements $\mathbf{u}_i$ for $i \notin \{j_1, j_2, \ldots, j_{k+1}\}$ together constitute a basis of the vector space V.

It then follows by repeated applications of this result that if m_0 is the minimum of m and n then there exists a basis of V consisting of the elements $\mathbf{w}_i$ for $1 \leq i \leq m_0$ together with $n - m_0$ members of the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

If it were the case that $n < m$ then the n elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ would be a basis of V, and thus the elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ would not be linearly independent. Therefore $n \geq m$, and there exists a basis of V consisting of the elements $\mathbf{w}_i$ for $1 \leq i \leq m$ together with $n - m$ members of the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, as required. ∎

Corollary

Corollary LA-06 *Any two bases of a finite-dimensional real vector space contain the same number of elements.*

Proof
It follows from Proposition LA-05 that the number of members in a list of linearly independent elements of a finite-dimensional real vector space V cannot exceed the number of members in a list of elements of V that spans V. The members of a basis of V are linearly independent and also span V. Therefore the number of members of one basis of V cannot exceed the number of members of another. The result follows. ∎

Definition
The *dimension* of a finite-dimensional real vector space V is the number of members of any basis of V.

The dimension of a real vector space V is denoted by $\dim V$.

It follows from Corollary LA-03 that every finite-dimensional real vector space V has a basis. It follows from Corollary LA-06 that any two bases of that vector space have the same number of elements. These results ensure that every finite-dimensional real vector space has a well-defined dimension that is equal to the number of members of any basis of that vector space.

Definition
Let V be a finite-dimensional vector space. A subset U of V is said to be a *subspace* of V if the following two conditions are satisfied:—

- $\mathbf{v} + \mathbf{w} \in U$ for all $\mathbf{v}, \mathbf{w} \in U$;
- $\lambda\mathbf{v} \in U$ for all $\mathbf{v} \in U$ and for all real numbers λ.

Every subspace of a real vector space is itself a real vector space.

Proposition

Proposition LA-07   *Let V be a finite-dimensional vector space, and let U be a subspace of V. Then U is itself a finite-dimensional vector space, and $\dim U \leq \dim V$.*

Proof
It follows from Proposition LA-05 the number of members of any list of linearly independent elements of U cannot exceed the dimension $\dim V$ of the real vector space V. Let m be the maximum number of members in any list of linearly independent elements of U, and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be a list consisting of m linearly independent elements of U. We claim that this list constitutes a basis of U.

Let $\mathbf{v} \in U$. Then the maximality of m ensures that the elements $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ must be linearly dependent. Therefore there exist a real number ρ and real numbers $\lambda_1, \ldots, \lambda_m$, where these real numbers ρ and λ_i are not all zero, such that

$$\rho\mathbf{v} + \sum_{i=1}^{m} \lambda_i\mathbf{w}_i = \mathbf{0}_V.$$

But then $\rho \neq 0$, because otherwise the elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ would be linearly dependent. It then follows that

$$\mathbf{v} = -\sum_{i=1}^{m} \frac{\lambda_i}{\rho}\, \mathbf{w}_i.$$

This shows that the linearly independent elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ of U span U and therefore constitute a basis of U. Thus U is a finite-dimensional vector space, and $\dim U = m$. But $m \leq n$. It follows that $\dim U \leq \dim V$, as required. ∎

Definition

Let V and W be real vector spaces. A function $\theta\colon V \to W$ from V to W is said to be a *linear transformation* if it satisfies the following two conditions:—

- $\theta(\mathbf{v} + \mathbf{w}) = \theta(\mathbf{v}) + \theta(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V$;
- $\theta(\lambda\mathbf{v}) = \lambda\theta(\mathbf{v})$ for all $\mathbf{v} \in V$ and for all real numbers λ.

Definition
The *image* of a linear transformation $\theta \colon V \to W$ between real vector spaces V and W is the subspace $\theta(V)$ of W defined such that

$$\theta(V) = \{\theta(\mathbf{v}) : \mathbf{v} \in V\}.$$

Definition
The *rank* of a linear transformation $\theta \colon V \to W$ between real vector spaces V and W is the dimension of the image $\theta(V)$ of θ.

A linear transformation $\theta \colon V \to W$ is *surjective* if and only if $\theta(V) = W$. Thus the linear transformation $\theta \colon V \to W$ is surjective if and only if its rank is equal to the dimension $\dim W$ of the codomain W.

Definition
The *kernel* of a linear transformation $\theta\colon V \to W$ between real vector spaces V and W is the subspace $\ker\theta$ of V defined such that

$$\ker\theta = \{\mathbf{v} \in V : \theta(\mathbf{v}) = 0\}.$$

Definition
The *nullity* of a linear transformation $\theta\colon V \to W$ between real vector spaces V and W is the dimension of the kernel $\ker\theta$ of θ.

A linear transformation $\theta\colon V \to W$ is *injective* if and only if $\ker\theta = \{\mathbf{0}_V\}$. Indeed let $\mathbf{v}$ and $\mathbf{v}'$ be elements of V satisfying $\theta(\mathbf{v}) = \theta(\mathbf{v}')$. Tnen

$$\theta(\mathbf{v} - \mathbf{v}') = \theta(\mathbf{v}) - \theta(\mathbf{v}') = \mathbf{0}_W,$$

and therefore $\mathbf{v} - \mathbf{v}' \in \ker\theta$. It follows that if $\ker\theta = \{\mathbf{0}_W\}$ and if elements $\mathbf{v}$ and $\mathbf{v}'$ of V satisfy $\theta(\mathbf{v}) = \theta(\mathbf{v}')$ then $\mathbf{v} - \mathbf{v}' = \mathbf{0}_V$, and therefore $\mathbf{v} = \mathbf{v}'$. Thus if $\ker\theta = \{\mathbf{0}_V\}$ then the linear transformation $\theta\colon V \to W$ is injective. The converse is immediate. It follows that $\theta\colon V \to W$ is injective if and only if $\ker\theta = \{\mathbf{0}_W\}$.

A linear transformation $\theta\colon V \to W$ between vector spaces V and W is an *isomorphism* if and only if it is both injective and surjective.

Proposition

Proposition LA-08 Let V and W be finite-dimensional real vector spaces, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V, let $\theta\colon V \to W$ be a linear transformation from V to W, and let $\theta(V)$ be the image of this linear transformation. Let $I = \{1, 2, \ldots, n\}$, and let K and L be a subsets of I satisfying $K \subset L$ that satisfy the following properties:

- the elements $\theta(\mathbf{u}_i)$ for which $i \in K$ are linearly independent;
- the elements $\theta(\mathbf{u}_i)$ for which $i \in L$ span the vector space $\theta(V)$.

Then there exists a subset B of I satisfying $K \subset B \subset L$ such that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis for the vector space $\theta(V)$.

Proof
The elements $\theta(\mathbf{u}_i)$ for which $i \in L$ span the real vector space $\theta(V)$. The result therefore follows immediately on applying Proposition LA-02. ∎

Lemma

Lemma LA-09 *Let V and W be finite-dimensional real vector spaces, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V, let $\theta \colon V \to W$ be a linear transformation from V to W, let $I = \{1, 2, \ldots, n\}$, and let B be a subset of I. Suppose that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis of the image $\theta(V)$ of the linear transformation θ. Then, for each $j \in I \setminus B$, there exist uniquely-determined real numbers $\kappa_{i,j}$ for all $i \in B$ such that*

$$\mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i \in \ker \theta.$$

Proof

The elements $\theta(\mathbf{u}_i)$ of $\theta(V)$ for which $i \in B$ constitute a basis of $\theta(V)$. Therefore, for each $j \in I \setminus B$, the element $\theta(\mathbf{u}_j)$ may be expressed as a linear combination $\sum\limits_{i \in B} \kappa_{i,j}\theta(\mathbf{u}_i)$ of the basis elements. Moreover the linear independence of the basis elements ensures that the real numbers $\kappa_{i,j}$ that occur as coefficients in this expression of $\theta(\mathbf{u}_j)$ as a linear combination of basis elements are uniquely determined. But then

$$\theta\left(\mathbf{u}_j - \sum_{i \in B} \kappa_{i,j}\mathbf{u}_i\right) = \theta(\mathbf{u}_j) - \sum_{i \in B} \kappa_{i,j}\theta(\mathbf{u}_i) = \mathbf{0}_W,$$

and thus $\mathbf{u}_j - \sum\limits_{i \in B} \kappa_{i,j}\mathbf{u}_i \in \ker\theta$, as required. ∎

Proposition

Proposition LA-10 *Let V and W be finite-dimensional real vector spaces, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V, let $\theta \colon V \to W$ be a linear transformation from V to W, let B be a subset of I, where $I = \{1, 2, \ldots, n\}$, with the property that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis of the image $\theta(V)$ of the linear transformation θ, and let*

$$\mathbf{g}_j = \mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i,$$

for all $j \in I \setminus B$, where $\kappa_{i,j}$ are the unique real numbers for which $\mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i \in \ker \theta$. Then the elements $\mathbf{u}_i$ for $i \in B$ and $\mathbf{g}_j$ for $j \in I \setminus B$ together constitute a basis for the vector space V.

Proof
Let λ_i for $i \in B$ and μ_j for $j \in I \setminus B$ are real numbers with the property that

$$\sum_{i \in B} \lambda_i \mathbf{u}_i + \sum_{j \in I \setminus B} \mu_j \mathbf{g}_j = \mathbf{0}_V.$$

Then

$$\sum_{i \in B} \left(\lambda_i - \sum_{j \in I \setminus B} \kappa_{i,j} \mu_j \right) \mathbf{u}_i + \sum_{j \in I \setminus B} \mu_j \mathbf{u}_j = \mathbf{0}_V.$$

It then follows from the linear independence of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ that $\lambda_i - \sum_{j \in I \setminus B} \kappa_{i,j} \mu_j = 0$ for all $i \in B$ and $\mu_j = 0$ for all $j \in I \setminus B$. But then $\lambda_i = 0$ for all $i \in B$. This shows that the elements $\mathbf{u}_i$ for $i \in B$ and $\mathbf{g}_j$ for $j \in I \setminus B$ are linearly independent.

Let $\mathbf{v} \in V$. Then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \sum\limits_{i=1}^{n} \lambda_i \mathbf{u}_i$. But then

$$\mathbf{v} = \sum_{i \in B} \left(\lambda_i + \sum_{j \in I \setminus B} \kappa_{i,j} \lambda_i \right) \mathbf{u}_i + \sum_{j \in I \setminus B} \lambda_j \mathbf{g}_j.$$

It follows that the elements $\mathbf{u}_i$ for $i \in B$ and $\mathbf{g}_j$ for $j \in I \setminus B$ span the vector space V. We have shown that these elements are linearly independent. It follows that they constitute a basis for the vector space V, as required. ∎

Corollary

Corollary LA-11 *Let V and W be finite-dimensional real vector spaces, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V, let $\theta \colon V \to W$ be a linear transformation from V to W, let B be a subset of I, where $I = \{1, 2, \ldots, n\}$, with the property that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis of the image $\theta(V)$ of the linear transformation θ, and let*

$$\mathbf{g}_j = \mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i,$$

for all $j \in I \setminus B$, where $\kappa_{i,j}$ are the unique real numbers for which $\mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i \in \ker \theta$. Then the elements $\mathbf{g}_j$ for $j \in I \setminus B$ constitute a basis for $\ker \theta$.

Proof

We have shown that the elements $\mathbf{u}_i$ for $i \in B$ and $\mathbf{g}_j$ for $j \in I \setminus B$ together constitute a basis for the vector space B (Proposition LA-10). It follows that the elements $\mathbf{g}_j$ for which $j \in I \setminus B$ are linearly independent.

Let $\mathbf{v} \in \ker \theta$. Then there exist real numbers λ_i for $i \in B$ and μ_j for $j \in I \setminus B$ such that

$$\mathbf{v} = \sum_{i \in B} \lambda_i \mathbf{u}_i + \sum_{j \in I \setminus B} \mu_i \mathbf{g}_j.$$

Now $\theta(\mathbf{g}_j) = \mathbf{0}_W$ for all $j \in I \setminus B$, because $\mathbf{g}_j \in \ker \theta$. Also $\theta(\mathbf{v}) = \mathbf{0}_W$, because $\mathbf{v} \in \ker \theta$. It follows that

$$\mathbf{0}_W = \theta(\mathbf{v}) = \sum_{i \in B} \lambda_i \theta(\mathbf{u}_i).$$

However the subset B of I has the property that the elements $\theta(\mathbf{u}_i)$ for $i \in B$ constitute a basis of the vector space $\theta(V)$. It follows that $\lambda_i = 0$ for all $i \in B$. Thus

$$\mathbf{v} = \sum_{j \in I \setminus B} \mu_i \mathbf{g}_j.$$

This proves that the elements $\mathbf{g}_j$ for $j \in I \setminus B$ span the kernel $\ker \theta$ of the linear transformation $\theta \colon V \to W$. This elements have been shown to be linearly independent. It follows that they constitute a basis for $\ker \theta$, as required. ∎

Corollary

Corollary LA-12 *Let V and W be finite-dimensional vector spaces, let $\theta\colon V \to W$ be a linear transformation from V to W, and let $\mathrm{rank}(\theta)$ and $\mathrm{nullity}(\theta)$ denote the rank and nullity respectively of the linear transformation θ. Then*

$$\mathrm{rank}(\theta) + \mathrm{nullity}(\theta) = \dim V.$$

Proof
Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V. Then there exists a subset B of I, where $I = \{1, 2, \ldots, n\}$, with the property that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis of the image $\theta(V)$ of the linear transformation θ (see Proposition LA-08). Let

$$\mathbf{g}_j = \mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i,$$

for all $j \in I \setminus B$, where $\kappa_{i,j}$ are the unique real numbers for which $\mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i \in \ker \theta$. Then the elements $\mathbf{g}_j$ for $j \in I \setminus B$ constitute a basis for $\ker \theta$.

Now the rank of the linear transformation θ is by definition the dimension of the real vector space $\theta(V)$, and is thus equal to the number of elements in any basis of that vector space. The elements $\theta(\mathbf{u}_i)$ for $i \in B$ constitute a basis of that vector space. Therefore $\mathrm{rank}(\theta) = |B|$, where $|B|$ denotes the number of integers belonging to the finite set B. Similarly the nullity of θ is by definition the dimension of the kernel $\ker \theta$ of θ. The elements $\mathbf{g}_j$ for $j \in I \setminus B$ constitute a basis of $\ker \theta$. Therefore $\mathrm{nullity}(\theta) = |I \setminus B|$, where $|I \setminus B|$ denotes the number of integers belonging to the finite set $I \setminus B$.

Now $|B| + |I \setminus B| = n$. It follows that

$$\mathrm{rank}(\theta) + \mathrm{nullity}(\theta) = n = \dim V,$$

as required. ∎