MA3484 Methods of Mathematical Economics
School of Mathematics, Trinity College
Hilary Term 2015
Lecture 11 (February 5, 2015)

David R. Wilkins

Note

Lecture 11 completed the task of finding basic optimal solutions to the instance of the Transportation Problem with 6 suppliers and 5 recipients discussed in Lecture 10.

The lecture began with a review of that problem, based on the slides for Lecture 10. The discussion continued as detailed in the following slides.

In order to determine whether or not the new basic feasible solution is optimal, and, if not, how to improve it, we determine u_i for $1 \leq i \leq 5$ and v_j for $1 \leq j \leq 6$ such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$, where B is now the current basis. We then calculate $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3, 4, 5, 6$ and $j = 1, 2, 3, 4, 5$.

Accordingly we determine the numbers u_i and v_j, setting $u_1 = 0$ and using the following tableau:—

The Transportation Problem: The Minimum Cost Method (continued)

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 ?	8 ?	9 ?	4 • 0	6 ?	0
2	5 ?	10 ?	8 ?	9 ?	5 • 0	?
3	6 ?	4 • 0	12 ?	12 ?	4 ?	?
4	5 • 0	7 • 0	12 ?	10 • 0	8 • 0	?
5	4 • 0	6 ?	8 • 0	10 ?	12 ?	?
6	7 ?	3 • 0	7 ?	12 ?	8 ?	?
v_j	?	?	?	?	?	

Now $(1,4) \in B$, $u_1 = 0$ and $c_{1,4} = 4$ force $v_4 = 4$. Then $(4,4) \in B$, $v_4 = 4$ and $c_{4,4} = 10$ force $u_4 = -6$. Then $u_4 = -6$ and $(4,1), (4,2), (4,5) \in B$ force $v_1 = -1$, $v_2 = 1$ and $v_5 = 2$. After recording these values the tableau is as follows:—

The Transportation Problem: The Minimum Cost Method (continued)

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 ?	8 ?	9 ?	4 • 0	6 ?	0
2	5 ?	10 ?	8 ?	9 ?	5 • 0	?
3	6 ?	4 • 0	12 ?	12 ?	4 ?	?
4	5 • 0	7 • 0	12 ?	10 • 0	8 • 0	-6
5	4 • 0	6 ?	8 • 0	10 ?	12 ?	?
6	7 ?	3 • 0	7 ?	12 ?	8 ?	?
v_j	-1	1	?	4	2	

Then $(2, 5) \in B$, $v_5 = 2$ and $c_{2,5} = 5$ force $u_2 = -3$.

Also $(3, 2) \in B$, $v_2 = 1$ and $c_{3,2} = 4$ force $u_3 = -3$.

Also $(5, 1) \in B$, $v_1 = -1$ and $c_{6,2} = 4$ force $u_5 = -5$.

Also $(6, 2) \in B$, $v_2 = 1$ and $c_{6,2} = 3$ force $u_6 = -2$.

Then $(5, 3) \in B$, $u_5 = -5$ and $c_{5,3} = 8$ force $v_3 = 3$.

Thus after recording the values of u_i and v_j for all i and j the tableau is as follows:—

The Transportation Problem: The Minimum Cost Method (continued)

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 ?	8 ?	9 ?	4 • 0	6 ?	0
2	5 ?	10 ?	8 ?	9 ?	5 • 0	−3
3	6 ?	4 • 0	12 ?	12 ?	4 ?	−3
4	5 • 0	7 • 0	12 ?	10 • 0	8 • 0	−6
5	4 • 0	6 ?	8 • 0	10 ?	12 ?	−5
6	7 ?	3 • 0	7 ?	12 ?	8 ?	−2
v_j	−1	1	3	4	2	

The next stage is to compute the values of $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3, 4, 5, 6$ and $j = 1, 2, 3, 4, 5$. The values of $q_{i,j}$ are accordingly recorded in the following tableau:—

The Transportation Problem: The Minimum Cost Method (continued)

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 / 13	8 / 7	9 / 6	4 • / 0	6 / 4	0
2	5 / 3	10 / 6	8 / 2	9 / 2	5 • / 0	-3
3	6 / 4	4 • / 0	12 / 6	12 / 5	4 / -1	-3
4	5 • / 0	7 • / 0	12 / 3	10 • / 0	8 • / 0	-6
5	4 • / 0	6 / 0	8 • / 0	10 / 1	12 / 5	-5
6	7 / 6	3 • / 0	7 / 2	12 / 6	8 / 4	-2
v_j	-1	1	3	4	2	

The Transportation Problem: The Minimum Cost Method (continued)

The fact that $q_{3,5} = -1$ shows that the current basic feasible solution is not optimal. We therefore seek to bring $(3, 5)$ into the basis, and, to achieve this, we calculate the coefficients $y_{i,j}$ of a 6×5 matrix Y satisfying the following conditions:—

- $y_{3,5} = 1$;
- $y_{i,j} = 0$ when $(i, j) \notin B \cup \{(3, 5)\}$;
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1	2	3	4	5	
1				? •		0
2					? •	0
3		? •			1 ∘	0
4	? •	? •		? •	? •	0
5	? •		? •			0
6		? •				0
	0	0	0	0	0	0

The constraints that the 1st, 2nd, 3rd and 6th rows of the body of the table sum to zero force $y_{1,4} = 0$, $y_{2,5} = 0$, $y_{3,2} = -1$ and $y_{6,2} = 0$. The constraint that the 3rd column of the body of the table sum to zero forces $y_{5,3} = 0$. After entering these values, the tableau is as follows:—

$y_{i,j}$	1		2		3		4		5		
1							0	•			0
2									0	•	0
3			−1	•					1	○	0
4	?	•	?	•			?	•	?	•	0
5	?	•			0	•					0
6			0	•							0
	0		0		0		0		0		0

The constraints that the 5th row and the 2nd, 4th and 5th column of the body of the table sum to zero force $y_{5,1} = 0$, $y_{4,2} = 1$, $y_{4,4} = 0$ and $y_{4,5} = -1$.

The then requirement that the first column of the body of the table, and the identity $y_{5,1} = 0$ together force $y_{4,1} = 0$. The completed tableau is thus as follows:—

$y_{i,j}$	1	2	3	4	5	
1				0 •		0
2					0 •	0
3		−1 •			1 ○	0
4	0 •	1 •		0 •	−1 •	0
5	0 •		0 •			0
6		0 •				0
	0	0	0	0	0	0

We now determine those values of λ for which $X + \lambda Y$ is a feasible solution, where

$$X + \lambda Y = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 5-\lambda & 0 & 0 & \lambda \\ 7 & 3+\lambda & 0 & 5 & 1-\lambda \\ 1 & 0 & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

From this matrix, it is clear that $X + \lambda Y$ is a feasible solution for $0 \leq \lambda \leq 1$. Moreover the next basis is obtained by adding $(3, 5)$ to the existing basis and removing $(4, 5)$. The new basic feasible solution corresponding to the new basis is obtained from $X + \lambda Y$ by setting $\lambda = 1$.

We now let B denote the new basis and let X denote the new basic feasible solution corresponding to the new basis. Accordingly

$$\begin{aligned} B &= \{(6,2),\ (1,4),\ (5,1),\ (3,2),\ (2,5),\ (4,1),\\ &\qquad (4,2),\ (3,5),\ (4,4)\ (5,3)\}, \end{aligned}$$

and

$$X = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 4 & 0 & 0 & 1 \\ 7 & 4 & 0 & 5 & 0 \\ 1 & 0 & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover

$$\text{Cost} = \text{Old Cost} + 1 * (-1) = 319 - 1 = 318.$$

The cost of the current feasible solution can also be obtained from the data recorded in the following tableau that represents the current feasible solution:—

The Transportation Problem: The Minimum Cost Method (continued)

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12 / 0	8 / 0	9 / 0	4 • / 9	6 / 0	9
2	5 / 0	10 / 0	8 / 0	9 / 0	5 • / 14	14
3	6 / 0	4 • / 4	12 / 0	12 / 0	4 • / 1	5
4	5 • / 7	7 • / 4	12 / 0	10 • / 5	8 / 0	16
5	4 • / 1	6 / 0	8 • / 6	10 / 0	12 / 0	7
6	7 / 0	3 • / 9	7 / 0	12 / 0	8 / 0	9
d_j	8	17	6	14	15	60

In order to determine whether or not the new basic feasible solution is optimal, and, if not, how to improve it, we determine u_i for $1 \leq i \leq 5$ and v_j for $1 \leq j \leq 6$ such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$, where B is now the current basis. We then calculate $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3, 4, 5, 6$ and $j = 1, 2, 3, 4, 5$.

Accordingly we determine the numbers u_i and v_j, setting $u_1 = 0$ and using the following tableau:—

The Transportation Problem: The Minimum Cost Method (continued)

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 ?	8 ?	9 ?	4 • 0	6 ?	0
2	5 ?	10 ?	8 ?	9 ?	5 • 0	?
3	6 ?	4 • 0	12 ?	12 ?	4 • 0	?
4	5 • 0	7 • 0	12 ?	10 • 0	8 ?	?
5	4 • 0	6 ?	8 • 0	10 ?	12 ?	?
6	7 ?	3 • 0	7 ?	12 ?	8 ?	?
v_j	?	?	?	?	?	

The Transportation Problem: The Minimum Cost Method (continued)

Solving the equations determining u_i and v_j, we find, successively, $u_1 = 0$, $v_4 = 4$, $u_4 = -6$, $v_1 = -1$, $v_2 = 1$, $u_5 = -5$, $v_3 = 3$, $u_3 = -3$, $u_6 = -2$, $v_5 = 1$ and $u_2 = -4$.

The Transportation Problem: The Minimum Cost Method (continued)

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 ?	8 ?	9 ?	4 • 0	6 ?	0
2	5 ?	10 ?	8 ?	9 ?	5 • 0	-4
3	6 ?	4 • 0	12 ?	12 ?	4 • 0	-3
4	5 • 0	7 • 0	12 ?	10 • 0	8 ?	-6
5	4 • 0	6 ?	8 • 0	10 ?	12 ?	-5
6	7 ?	3 • 0	7 ?	12 ?	8 ?	-2
v_j	-1	1	3	4	1	

The next stage is to compute the values of $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3, 4, 5, 6$ and $j = 1, 2, 3, 4, 5$. The values of $q_{i,j}$ are accordingly recorded in the following tableau:—

The Transportation Problem: The Minimum Cost Method (continued)

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 13	8 7	9 6	4 • 0	6 5	0
2	5 2	10 5	8 1	9 1	5 • 0	-4
3	6 4	4 • 0	12 6	12 5	4 • 0	-3
4	5 • 0	7 • 0	12 3	10 • 0	8 1	-6
5	4 • 0	6 0	8 • 0	10 1	12 6	-5
6	7 6	3 • 0	7 2	12 6	8 5	-2
v_j	-1	1	3	4	1	

We now summarize what has been achieved. The problem was to find a basic optimal solution to a transportation problem with 6 suppliers and 5 recipients.

The supply vector is $(9, 14, 5, 16, 7, 9)$ and the demand vector is $(8, 17, 6, 14, 15)$. The components of both the supply vector and the demand vector add up to 60.

The costs are as specified in the following cost matrix:

$$\begin{pmatrix} 12 & 8 & 9 & 4 & 6 \\ 5 & 10 & 8 & 9 & 5 \\ 6 & 4 & 12 & 12 & 4 \\ 5 & 7 & 12 & 10 & 8 \\ 4 & 6 & 8 & 10 & 12 \\ 7 & 3 & 7 & 12 & 8 \end{pmatrix}.$$

The solution is provided by the following matrix, whose coefficient in the ith row and jth column represents the quantity of the commodity to be transported from the ith supplier to the jth recipient:

$$X = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 4 & 0 & 0 & 1 \\ 7 & 4 & 0 & 5 & 0 \\ 1 & 0 & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

This solution is a basic solution associated with the following basis:

$$\begin{aligned} B \;=\; & \{(1,4),\ (2,5),\ (3,2),\ (3,5),\ (4,1),\ (4,2), \\ & (4,4),\ (5,1),\ (5,3),\ (6,2)\}. \end{aligned}$$

The cost of this basic optimal solution is 318. We have determined values of $u_1, u_2, u_3, u_4, u_5, u_6$ and v_1, v_2, v_3, v_4, v_5 such that the cost $c_{i,j}$ of transporting the commoditity from the ith supplier to the jth recipient satisfies $c_{i,j} = v_j - u_i$ whenever $(i, j) \in B$. These numbers have the following values:—

$$u_1 = 0, \;\; u_2 = -4, \;\; u_3 = -3, \;\; u_4 = -6, \;\; u_5 = -5, \;\; u_6 = -2.$$

$$v_1 = -1, \;\; , v_1 = 1, \;\; v_2 = 3, \;\; v_4 = 4, \;\; v_5 = 1.$$

Moreover we have determined numbers $q_{i,j}$ such that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3, 4$. Let Q be the matrix with $(Q)_{i,j} = q_{i,j}$ for all i and j. Then

$$Q = \begin{pmatrix} 13 & 7 & 6 & 0 & 5 \\ 2 & 5 & 1 & 1 & 0 \\ 4 & 0 & 6 & 5 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \\ 6 & 0 & 2 & 6 & 5 \end{pmatrix}.$$

The Transportation Problem: The Minimum Cost Method (continued)

The manner in which the matrix Q has been constructed ensures that its component $q_{i,j}$ in the ith row and jth column satisfies $q_{i,j} = 0$ whenever $(i, j) \in B$. Thus the matrix Q must have at least ten coefficients equal to zero. In fact there is an eleventh coefficient equal to zero, since $q_{5,2} = 0$, though $(5, 2) \notin B$. The significance of this is that this particular transportation problem has a second basic optimal solution.

To find this second optimal solution, we determine a 6×5 matrix Y, with coefficient $y_{i,j}$ in the ith row and jth column, where this matrix Y satisfies the following conditions:—

- $y_{5,2} = 1$;
- $y_{i,j} = 0$ when $(i,j) \notin B \cup \{(5,2)\}$;
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the $\bullet$ symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1		2		3		4		5		
1							?	$\bullet$			0
2									?	$\bullet$	0
3			?	$\bullet$					?	$\bullet$	0
4	?	$\bullet$	?	$\bullet$			?	$\bullet$			0
5	?	$\bullet$	1	$\circ$	?	$\bullet$					0
6			?	$\bullet$							0
	0		0		0		0		0		0

The completed tableau is as follows:—

$y_{i,j}$	1		2		3		4		5		
1							0	$\bullet$			0
2									0	$\bullet$	0
3			0	$\bullet$					0	$\bullet$	0
4	1	$\bullet$	-1	$\bullet$			0	$\bullet$			0
5	-1	$\bullet$	1	$\circ$	0	$\bullet$					0
6			0	$\bullet$							0
	0		0		0		0		0		0

Then $X + \lambda Y$ is an optimal solution for $0 \leq \lambda \leq 1$, where

$$X + \lambda Y = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 4 & 0 & 0 & 1 \\ 7+\lambda & 4-\lambda & 0 & 5 & 0 \\ 1-\lambda & \lambda & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

The fact that all these solutions are optimal stems from the fact that the costs satisfy $c_{4,1} + c_{5,2} = c_{5,1} + c_{4,2}$. Indeed $c_{4,1} + c_{5,2} = 5 + 6 = 11$ and $c_{5,1} + c_{4,2} = 4 + 7 = 11$.