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The Transportation Problem: the General Algorithm

We now describe in more generality the method for solving the Transportation Problem, in the case where total supply equals total demand.

Thus we suppose that we have m suppliers and n recipients. The ith supplier can provide at most s_i units of a commodity, and the jth recipient requires d_j units, where $s_i \geq 0$ for all i, $d_j \geq 0$ for all j and $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. The cost of transporting the commodity from the ith supplier to the jth recipient is $c_{i,j}$. It is required to determine the amount $x_{i,j}$ of the commodity that should be transported from the ith supplier to the jth recipient, consistent with the constraints that $x_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$.

A set of quantities

$$(x_{i,j} : i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n)$$

satisfying these constraints is a *feasible solution* to the problem. The *cost* of a feasible solution $(x_{i,j})$ is $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$. An *optimal solution* to this transportation problem is a feasible solution to the problem whose cost does not exceed that of any other feasible solution.

As usual, we denote by $M_{m,n}(\mathbb{R})$ the set of all $m \times n$ matrices with real coefficients, we denote by $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ the linear transformations defined such that $\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$, we denote by W the set of all ordered pairs $(\mathbf{s}, \mathbf{d})$, where $\mathbf{s} \in \mathbb{R}^m$, $\mathbf{d} \in \mathbb{R}^n$ and $\sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j$, and we denote by $\theta\colon M_{m,n}(\mathbb{R}) \to W$ the linear transformation from $M_{m,n}(\mathbb{R})$ to W defined such that $\theta(X) = (\rho(X), \sigma(X))$ for all $X \in M_{m,n}(\mathbb{R})$. Then

$$\theta(X) = \left(\sum_{j=1}^{n} (X)_{i,j}, \sum_{i=1}^{m} (X)_{i,j} \right).$$

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. Each subset K of $I \times J$ determines a corresponding subspace M_K of the real vector space $M_{m,n}(\mathbb{R})$, where

$$M_K = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ when } (i,j) \notin K\}.$$

The real vector space M_K is of dimension $|K|$, where $|K|$ denotes the number of elements in the subset K of $I \times J$. The linear transformation $\theta\colon M_{m,n}(\mathbb{R}) \to W$ restricts to a linear transformation $\theta_K\colon M_K \to W$.

A subset B of $I \times J$ is a *basis* for the transportation problem if and only if the corresponding linear transformation

$$\theta_B\colon M_B(\mathbb{R}) \to W$$

is an isomorphism.

Proposition

Proposition TP-G01 *Let X, C and Q be $m \times n$ matrices with real coefficients, and let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be real numbers. Suppose that*

$$(C)_{i,j} = v_j - u_i + (Q)_{i,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then

$$\operatorname{trace}(C^T X) = \sum_{i=1}^{m} v_j \sigma(X)_j - \sum_{j=1}^{n} u_i \rho(X)_i + \operatorname{trace}(Q^T X),$$

where $\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$.

Proof

Let $x_{i,j} = (X)_{i,j}$, $c_{i,j} = (C)_{i,j}$ and $q_{i,j} = (Q)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then $c_{i,j} = v_j - u_i + q_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and therefore

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$$
\begin{aligned}
\operatorname{trace}(C^T X) &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \sum_{i=1}^{m}\sum_{j=1}^{n}(v_j - u_i + q_{i,j})x_{i,j} \\
&= \sum_{j=1}^{n}\left(v_j \sum_{i=1}^{m} x_{i,j}\right) - \sum_{i=1}^{m}\left(u_i \sum_{j=1}^{n} x_{i,j}\right) \\
&\qquad + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}x_{i,j} \\
&= \sum_{i=1}^{m} v_j \sigma(X)_j - \sum_{j=1}^{n} u_i \rho(X)_i + \operatorname{trace}(Q^T X),
\end{aligned}
$$

as required. ∎

Corollary

Corollary TP-G0X *Let m and n be integers, and let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. Let X and C be $m \times n$ matrices, and let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be real numbers. Suppose that $(C)_{i,j} = v_j - u_i$ for all $(i, j) \in I \times J$ for which $(X)_{i,j} \neq 0$. Then*

$$\mathrm{trace}(C^T X) = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

where $s_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $d_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$.

Proof

Let Q be the $m \times n$ matrix defined such that $(Q)_{i,j} = (C)_{i,j} + u_i - v_j$ for all $i \in I$ and $j \in J$. Then $(C)_{i,j} = v_j - v_i + (Q)_{i,j}$ for all $i \in I$ and $j \in J$, and $Q_{i,j} = 0$ whenever $(X)_{i,j} \neq 0$. It follows from this that

$$\mathrm{trace}(Q^T X) = \sum_{i=1}^{m} \sum_{j=1}^{n} (Q)_{i,j} (X)_{i,j} = 0.$$

It then follows from Proposition TP-G01 that

$$\mathrm{trace}(C^T X) = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

as required. ∎

Proposition

Proposition TP-G0Y *Let m and n be integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let B be a subset of $I \times J$ that is a basis for the transportation problem with m suppliers and n recipients. For each $(i, j) \in B$ let $c_{i,j}$ be a corresponding real number. Then there exist real numbers u_i for $i \in I$ and v_j for $j \in J$ such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. Moreover if $\overline{u}_i$ and $\overline{v}_j$ are real numbers for $i \in I$ and $j \in J$ that satisfy the equations $c_{i,j} = \overline{v}_j - \overline{u}_i$ for all $(i, j) \in B$, then there exists some real number k such that $\overline{u}_i = u_i + k$ for all $i \in I$ and $\overline{v}_j = v_j + k$ for all $j \in J$.*

Proof

Let

$$W = \left\{ (\mathbf{s}, \mathbf{d}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j \right\},$$

let

$$M_B = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ when } (i,j) \notin B\},$$

let $\theta\colon M_{m,n}(\mathbb{R}) \to W$ be defined so that $\theta(X) = (\rho(X), \sigma(X))$, where $\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$, and let $\theta_B\colon M_B \to W$ be the restriction of $\theta\colon M_{m,n}(\mathbb{R}) \to W$ to M_B.

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Let $f\colon M_B \to \mathbb{R}$ be the linear transformation from M_B to $\mathbb{R}$ defined such that

$$f(X) = \sum_{(i,j)\in B} c_{i,j}(X)_{i,j}$$

for all $X \in M_B$. The requirement that B be a basis for the transportation problem ensures that $\theta_K\colon M_B \to W$ is an isomorphism. It follows that there is a well-defined linear transformation $g\colon W \to \mathbb{R}$ from W to $\mathbb{R}$ that satisfies $f(X) = g(\theta_B(X))$ for all $X \in M_B$. Indeed $g = f \circ \theta_B^{-1}$.

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For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ let $E^{(i,j)}$ denote the matrix whose coefficient in the in the ith row and jth column has value 1 and whose other coefficients are zero, let $\overline{\mathbf{b}}^{(i)}$ denote the vector in $\mathbb{R}^m$ whose ith component has the value 1 and whose other components are zero, and let $\mathbf{b}^{(j)}$ denote the vector in $\mathbb{R}^n$ whose jth component has the value 1 and whose other components are zero. Then $\theta_B(E^{(i,j)}) = \beta^{(i,j)}$ for all $(i,j) \in B$, where $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$. It follows that

$$g(\beta^{(i,j)}) = g(\theta(E^{(i,j)})) = f(E^{(i,j)}) = c_{i,j}$$

for all $(i,j) \in B$.

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Let $(\mathbf{s}, \mathbf{d}) \in W$, where

$$\mathbf{s} = (s_1, s_2, \ldots, s_m) \quad \text{and} \quad \mathbf{d} = (d_1, d_2, \ldots, d_n).$$

Then

$$s_1 = \begin{cases} \sum_{j=1}^{n} d_j & \text{if } m = 1, \\ \sum_{j=1}^{n} d_j - \sum_{i=2}^{m} s_i & \text{if } m > 1, \end{cases}$$

and therefore

$$(\mathbf{s}, \mathbf{d}) = \sum_{j=1}^{n} d_j(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) - \sum_{s=1}^{m} s_i(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0}).$$

Moreover $(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0}) \in W$ for $i = 1, 2, \ldots, m$, and $(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) \in W$ for $j = 1, 2, \ldots, n$.

Let $u_i = g(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0})$ for $i = 1, 2, \ldots, m$ and $v_j = g(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)})$ for $j = 1, 2, \ldots, n$. Then

$$\begin{aligned} g(\mathbf{s}, \mathbf{d}) &= \sum_{j=1}^{n} d_j g(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) - \sum_{s=1}^{m} s_i g(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0}) \\ &= \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i. \end{aligned}$$

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Moreover $u_1 = 0$, and

$$\begin{aligned} v_j - u_i &= g(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) - g(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0}) \\ &= g\left((\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) - (\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0})\right) \\ &= g(\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)}) = g(\beta^{(i,j)}) \\ &= c_{i,j} \end{aligned}$$

for all $(i, j) \in B$.

Now let $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_m$ and $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$ be real numbers with the property that $c_{i,j} = \overline{v}_j - \overline{u}_i$ for all $(i,j) \in B$, let $X \in M_B$, and let $(\mathbf{s}, \mathbf{d}) = \theta_B(X)$. Then X is an $m \times n$ matrix that satisfies $(X)_{i,j} = 0$ when $(i,j) \notin B$. Let C be a $m \times n$ matrix with $C_{i,j} = c_{i,j}$ for all $(i,j) \in B$. It then follows from Corollary TP-G0X that

$$g(\mathbf{s}, \mathbf{d}) = f(X) = \mathrm{trace}(C^T X) = \sum_{i=1}^{m} d_j \overline{v}_j - \sum_{j=1}^{n} s_i \overline{u}_i,$$

where $s_i = (\mathbf{s})_i$ for $i = 1, 2, \ldots, m$ and $d_j = (\mathbf{d})_j$ for $j = 1, 2, \ldots, n$.

Now $\theta_K \colon M_B \to W$ is surjective, because B is a basis of the transportation problem. It therefore follows that

$$\sum_{i=1}^{m} d_j \overline{v}_j - \sum_{j=1}^{n} s_i \overline{u}_i = g(\mathbf{s}, \mathbf{d}) = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i$$

for all $(\mathbf{s}, \mathbf{d}) \in W$.

Thus

$$\sum_{j=1}^{n} \overline{v}_j d_j - \sum_{i=1}^{m} \overline{u}_i s_i = \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i$$

for all real numbers $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$ that satisfy $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.

On applying this result with appropriate choices of numbers s_i and d_j satisfying $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$, we find that $\overline{u}_i - \overline{u}_1 = u_i - u_1$ for $2 \leq i \leq m$ and $\overline{v}_j - \overline{u}_1 = v_j - u_1$ for $j = 1, 2, \ldots, n$. Thus $\overline{u}_i = u_i + k$ for $i = 1, 2, \ldots, m$ and $\overline{v}_j = v_j + k$ for $j = 1, 2, \ldots, n$, where $k = \overline{u}_1 - u_1$, as required. $\blacksquare$