MA3484 Methods of Mathematical Economics
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We return to the discussion of the solution of the particular example of the Transportation Problem with 4 suppliers and 5 recipients, where the supply and demand vectors and the cost matrix are as follows:—

$$s^T = (9, 11, 4, 5), \quad d^T = (6, 7, 5, 3, 8).$$

$$C = \begin{pmatrix} 2 & 4 & 3 & 7 & 5 \\ 4 & 8 & 5 & 1 & 8 \\ 5 & 9 & 4 & 4 & 2 \\ 7 & 2 & 5 & 5 & 3 \end{pmatrix}.$$

We have already determined a basic feasible solution associated with the basis B, where

$$B = \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)\}.$$

The basic feasible solution and the basis B were determined using the Northwest Corner Method. The non-zero coefficients of the basic feasible solution are as follows:—

$$x_{1,1} = 6, \quad x_{1,2} = 3, \quad x_{2,2} = 4, \quad x_{2,3} = 5,$$

$$x_{2,4} = 2, \quad x_{3,4} = 1, \quad x_{3,5} = 3, \quad x_{4,5} = 5,$$

The components of the basic feasible solution corresponding to the basis B are this recorded in the following tableau:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	3	4
4	0	0	0	0	5	5
d_j	6	7	5	3	8	29

We have also determined real numbers u_i and v_j such that $c_{i,j} = v_j - u_i$ for $(i,j) \in B$. These values are recorded in the following tableau, which records the costs in the body of the tableau, the values u_i to the right and the values v_j along the bottom:—

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				0
2		8	5	1		-4
3				4	2	-7
4					3	-8
v_j	2	4	1	-3	-5	

Thus if

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = -7, \quad , u_4 = -8,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = -5,$$

then $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$.

However the values of $v_j - u_i$ can differ from $c_{i,j}$ when $(i,j) \notin B$. We can construct a tableau which records, in the top left of every cell of the body, the cost $c_{i,j}$ in the top left and the slackness $q_{i,j}$ in the bottom right, where $q_{i,j} = c_{i,j} + u_i - v_j$. The symbol $\bullet$ in the top right of a cell indicates that the cell represents an element of the basis. This tableau is then as follows:—

The Transportation Problem: Finding the Optimal Solution

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 •	4 •	3	7	5	0
	0	0	2	10	10	
2	4	8 •	5 •	1 •	8	-4
	-2	0	0	0	9	
3	5	9	4	4 •	2 •	-7
	-4	-2	-4	0	0	
4	7	2	5	5	3 •	-8
	-3	-10	-4	0	0	
v_j	2	4	1	-3	-5	

Let

$$Q = \begin{pmatrix} 0 & 0 & 2 & 10 & 10 \\ -2 & 0 & 0 & 0 & 9 \\ -4 & -2 & -4 & 0 & 0 \\ -3 & -10 & -4 & 0 & 0 \end{pmatrix}.$$

Then the costs $c_{i,j}$ satisfy $c_{i,j} = v_j - u_i + q_{i,j}$, where

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = -7, \quad , u_4 = -8,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = -5,$$

and $q_{i,j} = (Q)_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$.

Note that $q_{i,j} = 0$ for all $(i, j) \in B$.

Now let $\overline{X}$ be any feasible solution of the Transportation Problem under consideration, and let $\overline{x}_{i,j} = (\overline{X})_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. Then $\overline{x}_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{5} \overline{x}_{i,j} = s_i$ for $i = 1, 2, 3, 4$ and $\sum_{i=1}^{4} \overline{x}_{i,j} = d_j$ for $j = 1, 2, 3, 4, 5$. Then

The Transportation Problem: Finding the Optimal Solution

$$\begin{aligned}
\mathrm{trace}(C^T\overline{X}) &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} (v_j - u_i + q_{i,j})\overline{x}_{i,j} \\
&= \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{m} s_i u_i + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j} \\
&= \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{m} s_i u_i + \sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}.
\end{aligned}$$

Note that the summand $\sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}$. is determined by the values of $q_{i,j}$ and $\overline{x}_{i,j}$ for those ordered pairs (i,j) that do not belong to the basis B. Thus if we apply this formula to the basic feasible solution $(x_{i,j})$ determined by the basis B, we find that

$$\mathrm{trace}(C^T X) = \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{m} s_i u_i,$$

because $x_{i,j} = 0$ when $(i,j) \notin B$. It follows that

$$\mathrm{trace}(C^T \overline{X}) = \mathrm{trace}(C^T X) + \sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}$$

for any feasible solution $\overline{X}$ of the specified transportation problem.

The Transportation Problem: Finding the Optimal Solution

If it were the case that all the coefficients $q_{i,j}$ were non-negative then it would follow that $\sum_{(i,j) \notin B} q_{i,j} \overline{x}_{i,j} \geq 0$ for all feasible solutions $\overline{X}$, and therefore X would be a basic optimal solution. However the matrix Q has negative entries, and indeed $q_{4,2} = -10$. It turns out that we can construct a new basis, including the ordered pair $(4, 2)$ so as to ensure that this new basis determines a basic feasible solution whose cost is lower than that of the original basic feasible solution.

We complete the following tableau so that coefficients are determined only for those cells determined by the • symbol so as to ensure that all rows and columns of the body of the tableau add up to zero:—

$y_{i,j}$	1	2	3	4	5	
1	? •	? •				0
2		? •	? •	? •		0
3				? •	1 •	0
4		1 ∘			−1 •	0
	0	0	0	0	0	0

Examining the bottom row, we find that $y_{4,5} = -1$. Then, from the rightmost column we find that $y_{3,5} = 1$. Proceeding in this fashion, we successively determine $y_{3,4}$ and $y_{2,4}$. At this stage we see that

$$y_{4,5} = -1, \quad y_{3,5} = 1, \quad , y_{3,4} = -1 \quad \text{and} \quad y_{2,4} = 1.$$

To proceed further we note that $y_{1,1} = 0$, because the first column must sum to zero. This gives $y_{1,2} = 0$ and $y_{2,2} = -1$. This then forces $y_{2,3} = 0$. Thus the components $(Y)_{i,j}$ for which $(i,j) \in B$ are determined as in the following tableau:—

The Transportation Problem: Finding the Optimal Solution

$y_{i,j}$	1		2		3		4		5		
1	0	•	0	•							0
2			-1	•	0	•	1	•			0
3							-1	•	1	•	0
4			1	○					-1	•	0
	0		0		0		0		0		0

The remaining coefficients of the matrix Y must equal zero, and thus

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

Then

$$X + \lambda Y = \begin{pmatrix} 6 & 3 & 0 & 0 & 0 \\ 0 & 4-\lambda & 5 & 2+\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda & 3+\lambda \\ 0 & \lambda & 0 & 0 & 5-\lambda \end{pmatrix}.$$

An earlier calculation established that $\mathrm{trace}(C^T X) = 108$. Therefore

$$\begin{aligned} \mathrm{trace}(C^T(X+\lambda Y)) &= \mathrm{trace}(C^T X) + \lambda q_{4,2}(Y)_{4,2} \\ &= 108 - 10\lambda. \end{aligned}$$

The matrix $X + \lambda Y$ will represent a feasible solution when $0 \leq \lambda \leq 1$. But examination of the coefficient in the 3rd row and 4th column of $X + \lambda Y$ shows that $X + \lambda Y$ will not be a feasible solution for $\lambda > 1$. Thus taking $\lambda = 1$ determines a new basic feasible solution $\overline{X}$ determined by the basis obtained on adding $(4, 2)$ and removing $(3, 4)$ from the basis B.

Now let X and B denote the new feasible solution and the new basis associated with that feasible solution. Then

$$X = \begin{pmatrix} 6 & 3 & 0 & 0 & 0 \\ 0 & 3 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 4 \end{pmatrix}$$

and

$$B = \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,5),(4,2),(4,5)\}.$$

The cost associated with this new basic solution is 98.

The Transportation Problem: Cost of the Basic Feasable Solution

(i,j)	$x_{i,j}$	$c_{i,j}$	$c_{i,j}x_{i,j}$
$(1,1)$	6	2	12
$(1,2)$	3	4	12
$(2,2)$	3	8	24
$(2,3)$	5	5	25
$(2,4)$	3	1	3
$(3,5)$	4	2	8
$(4,2)$	1	2	2
$(4,5)$	4	3	12
Total			98

We now let X denote the new feasible solution and let B denote the new basis. We determine new values of the associated quantities u_i for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$, and calculate the new matrix Q so that $c_{i,j} = v_j - u_i + q_{i,j}$, where $q_{i,j} = (Q)_{i,j}$ for all i and j.

To determine u_i, v_j and $q_{i,j}$ we complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 ?	7 ?	5 ?	0
2	4 ?	8 • 0	5 • 0	1 • 0	8 ?	?
3	5 ?	9 ?	4 ?	4 ?	2 • 0	?
4	7 ?	2 • 0	5 ?	5 ?	3 • 0	?
v_j	?	?	?	?	?	

Now we choose $u_1 = 0$. Then, in order to ensure that $c_{i,j} = v_j - u_i$ for the two basis elements $(1, 1)$ and $(1, 2)$ we must take $v_1 = 2$ and $v_2 = 4$. Then $v_2 = 4$ and $c_{2,2} = 8$ force $u_2 = -4$. Then $u_2 = -4$, $c_{2,3} = 5$ and $c_{2,4} = 1$ force $v_3 = 1$ and $v_4 = -3$. Also $v_2 = 4$ and $c_{2,4} = 2$ force $u_4 = 2$. Then $u_4 = 2$ and $c_{4,5} = 3$ force $v_5 = 5$. Then $v_5 = 5$ and $c_{3,5} = 2$ force $u_3 = 3$.

Thus

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = 3, \quad u_4 = 2,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = 5.$$

The Transportation Problem: Finding the Optimal Solution

After determining u_i for $i = 1, 2, 3, 4$ and v_j for $j = 1, 2, 3, 4, 5$, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 ?	7 ?	5 ?	0
2	4 ?	8 • 0	5 • 0	1 • 0	8 ?	-4
3	5 ?	9 ?	4 ?	4 ?	2 • 0	3
4	7 ?	2 • 0	5 ?	5 ?	3 • 0	2
v_j	2	4	1	-3	5	

The Transportation Problem: Finding the Optimal Solution

Let $q_{i,j} = c_{i,j} + u_i - v_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. Then $q_{i,j}$ when $(i, j) \in B$. We calculate and record in the tableau the values of $q_{i,j}$ when $(i, j) \notin B$. We find that

$$q_{1,3} = 2, \quad q_{1,4} = 10, \quad q_{1,5} = 0, \quad q_{2,1} = -2,$$

$$q_{2,5} = -1, \quad q_{3,1} = 6, \quad q_{3,2} = 8, \quad q_{3,3} = 6,$$

$$q_{3,4} = 10, \quad q_{4,1} = 7, \quad q_{4,3} = 6, \quad q_{4,4} = 10.$$

The Transportation Problem: Finding the Optimal Solution

After calculating $q_{i,j}$ so that such that $q_{i,j} = c_{i,j} + u_i - v_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 2	7 10	5 0	0
2	4 -2	8 • 0	5 • 0	1 • 0	8 -1	-4
3	5 6	9 8	4 6	4 10	2 • 0	3
4	7 7	2 • 0	5 6	5 10	3 • 0	2
v_j	2	4	1	-3	5	

Let

$$Q = \begin{pmatrix} 0 & 0 & 2 & 10 & 0 \\ -2 & 0 & 0 & 0 & -1 \\ 6 & 8 & 6 & 10 & 0 \\ 7 & 0 & 6 & 10 & 0 \end{pmatrix}.$$

Then the costs $c_{i,j}$ satisfy $c_{i,j} = v_j - u_i + q_{i,j}$, where

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = 3, \quad u_4 = 2,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = 5.$$

and $q_{i,j} = (Q)_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$.

The matrix Q has some negative coefficients, and therefore the current basic feasible solution is not optimal. Because $q_{2,1}$ is the most negative coefficient of the matrix Q, we determine a new basis, consisting of the ordered pair $(2, 1)$ together with all but one ordered pair in the existing basis, so that the new basis corresponds to a new basic feasible solution with lower cost.

In order to determine the new basis, we complete the tableau below, (in which elements of the current basis are marked with the • symbol) in order that the rows and columns sum to zero. The tableau to be completed is as follows:—

$y_{i,j}$	1		2		3		4		5		
1	?	•	?	•							0
2	1	○	?	•	?	•	?	•			0
3									?	•	0
4			?	•					?	•	0
	0		0		0		0		0		0

The completed tableau is as follows:—

$y_{i,j}$	1		2		3		4		5		
1	-1	$\bullet$	1	$\bullet$							0
2	1	$\circ$	-1	$\bullet$	0	$\bullet$	0	$\bullet$			0
3									0	$\bullet$	0
4			0	$\bullet$					0	$\bullet$	0
	0		0		0		0		0		0

The remaining coefficients of the matrix Y must equal zero, and thus

$$Y = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we add λX to the current feasible solution X. We find that the coefficients of the resulting matrix $X + \lambda Y$ are as follows:—

$$X + \lambda Y = \begin{pmatrix} 6-\lambda & 3+\lambda & 0 & 0 & 0 \\ \lambda & 3-\lambda & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 4 \end{pmatrix}.$$

Then $X + \lambda Y$ determines a feasible solution with cost $98 - 2\lambda$ provided that $0 \leq \lambda < 3$. But the matrix does not represent a feasible solution for $\lambda > 3$. Accordingly we take $\lambda = 3$ to determine a new basic feasible solution. The ordered pair $(2, 1)$ enters the basis, and the ordered pair $(2, 2)$ leaves the basis.

We now let X denote the new feasible solution and let B denote the new basis associated with that feasible solution. We find that

$$X = \begin{pmatrix} 3 & 6 & 0 & 0 & 0 \\ 3 & 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 4 \end{pmatrix}.$$

and

$$B = \{(1,1), (1,2), (2,1), (2,3), (2,4), (3,5), (4,2), (4,5)\}.$$

The cost of this new feasible solution should be 92.

The following tableau verifies that the cost of the new feasible solution is indeed 92:—

(i, j)	$x_{i,j}$	$c_{i,j}$	$c_{i,j}x_{i,j}$
$(1, 1)$	3	2	6
$(1, 2)$	6	4	24
$(2, 1)$	3	4	12
$(2, 3)$	5	5	25
$(2, 4)$	3	1	3
$(3, 5)$	4	2	8
$(4, 2)$	1	2	2
$(4, 5)$	4	3	12
Total			92

To determine u_i, v_j and $q_{i,j}$ we complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 ?	7 ?	5 ?	0
2	4 • 0	8 ?	5 • 0	1 • 0	8 ?	?
3	5 ?	9 ?	4 ?	4 ?	2 • 0	?
4	7 ?	2 • 0	5 ?	5 ?	3 • 0	?
v_j	?	?	?	?	?	

To complete this tableau we first need to find values of u_i for $i = 1, 2, 3, 4$ and v_j for $j = 1, 2, 3, 4, 5$ such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. (The elements of the basis B have been marked with the $\bullet$ symbol.)
We set $u_1 = 0$. Then the $u_1 = 0$, $c_{1,1} = 2$ and $c_{1,2} = 4$ force $v_1 = 2$ and $v_2 = 4$. Then $v_1 = 2$ and $c_{1,2} = 4$ force $u_2 = -2$. Also $v_2 = 4$ and $c_{4,2} = 2$ force $u_4 = 2$. Then $u_2 = -3$, $c_{2,3} = 5$ and $c_{2,4} = 1$ force $v_3 = 3$ and $v_4 = -1$. Also $u_4 = 2$ and $c_{4,5} = 3$ forces $v_5 = 5$, which in turn forces $u_3 = 3$.

After determining u_i for $i = 1, 2, 3, 4$ and v_j for $j = 1, 2, 3, 4, 5$, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 ?	7 ?	5 ?	0
2	4 • 0	8 ?	5 • 0	1 • 0	8 ?	-2
3	5 ?	9 ?	4 ?	4 ?	2 • 0	3
4	7 ?	2 • 0	5 ?	5 ?	3 • 0	2
v_j	2	4	3	-1	5	

After calculating $q_{i,j}$ so that such that $q_{i,j} = c_{i,j} + u_i - v_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 0	7 8	5 0	0
2	4 • 0	8 2	5 • 0	1 • 0	8 1	-2
3	5 6	9 8	4 4	4 8	2 • 0	3
4	7 7	2 • 0	5 4	5 8	3 • 0	2
v_j	2	4	3	-1	5	

Let

$$Q = \begin{pmatrix} 0 & 0 & 0 & 8 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 6 & 8 & 4 & 8 & 0 \\ 7 & 0 & 4 & 8 & 0 \end{pmatrix}.$$

Then the costs $c_{i,j}$ satisfy $c_{i,j} = v_j - u_i + q_{i,j}$, where

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = 3, \quad u_4 = 2,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = 5.$$

and $q_{i,j} = (Q)_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$.

Note that $q_{i,j} \geq 0$ for all integers i and j satisfying $1 \leq i \leq 4$ and $1 \leq j \leq 5$, and that $q_{i,j} = 0$ for all $(i, j) \in B$.

The current basic feasible solution is an optimal solution to the Transportation problem with the given supply and demand vectors and the given cost matrix.

Indeed let $\overline{X}$ be any feasible solution of the Transportation Problem under consideration, and let $\overline{x}_{i,j} = (\overline{X})_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. Then $\overline{x}_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{5} \overline{x}_{i,j} = s_i$ for $i = 1, 2, 3, 4$ and $\sum_{i=1}^{4} \overline{x}_{i,j} = d_j$ for $j = 1, 2, 3, 4, 5$. Then, as we showed earlier, the cost $\mathrm{trace}(C^T \overline{X})$ of the feasible solution $\overline{X}$ satisfies

The Transportation Problem: Finding the Optimal Solution

$$\begin{aligned}
\mathrm{trace}(C^T\overline{X}) &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} (v_j - u_i + q_{i,j})\overline{x}_{i,j} \\
&= \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{m} s_i u_i + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j} \\
&= \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{m} s_i u_i + \sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}.
\end{aligned}$$

Note that the summand $\sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}$. is determined by the values of $q_{i,j}$ and $\overline{x}_{i,j}$ for those ordered pairs (i,j) that do not belong to the basis B. Thus if we apply this formula to the basic feasible solution $(x_{i,j})$ determined by the basis B, we find that

$$\mathrm{trace}(C^T X) = \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{m} s_i u_i,$$

because $x_{i,j} = 0$ when $(i,j) \notin B$. It follows that

$$\mathrm{trace}(C^T \overline{X}) = \mathrm{trace}(C^T X) + \sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}$$

for any feasible solution $\overline{X}$ of the specified transportation problem.

But we have calculed a basic feasible solution X of the transportation problem with the property that $q_{i,j} \geq 0$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. It follows that every feasible solution $\overline{X}$ of this transportation problem satisfies $\mathrm{trace}(C^T \overline{X}) \geq \mathrm{trace}(C^T X)$. Therefore the current basic feasible solution is an optimal feasible solution minimizing transportation costs.

We now summarize the conclusions of our numerical example. The problem was to determine a basic optimal solution of the transportation problem with given supply vector **s**, demand vector **d** and cost matrix C, where

$$s^T = (9, 11, 4, 5), \quad d^T = (6, 7, 5, 3, 8)$$

and

$$C = \begin{pmatrix} 2 & 4 & 3 & 7 & 5 \\ 4 & 8 & 5 & 1 & 8 \\ 5 & 9 & 4 & 4 & 2 \\ 7 & 2 & 5 & 5 & 3 \end{pmatrix}.$$

We have determined a basic optimal solution

$$X = \begin{pmatrix} 3 & 6 & 0 & 0 & 0 \\ 3 & 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 4 \end{pmatrix}.$$

associated to a basis B, where

$$B = \{(1,1),(1,2),(2,1),(2,3),(2,4),(3,5),(4,2),(4,5)\}.$$

The cost of this new feasible solution is 92.

Moreover we have also determined quantities u_i for $i = 1, 2, 3, 4$ and v_j for $j = 1, 2, 3, 4, 5$ satisfying the following two conditions:—

- $c_{i,j} \geq v_j - u_i$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$;
- $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$.