MA3484 Methods of Mathematical Economics
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The Transportation Problem: Basic Framework (continued)

We now discuss *basic feasible solutions* of the Transportation Problem with supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C.

Let X be a feasible solution of this problem. Then the components $(X)_{i,j}$ of X are all non-negative, $\rho(X) = \mathbf{s}$ and $\sigma(X) = \mathbf{d}$, where the ith component of $\rho(X)$ is the sum of the coefficients occuring in the ith row of the matrix X and the jth component of $\sigma(X)$ is the sum of the coefficients occuring in the jth column of X.
Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let K be the subset of $I \times J$ defined such that

$$K = \{(i, j) \in I \times J : (X)_{i,j} > 0\}.$$

Each element (i, j) of K determines a corresponding element $\beta^{(i,j)}$ of W defined such that

$$\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)}),$$

where $\overline{\mathbf{b}}^{(i)}$ denotes the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero, and where $\mathbf{b}^{(j)}$ denotes the vector in $\mathbb{R}^n$ whose jth component is equal to 1 and whose other components are zero.

We now prove that if K and L are subsets of $I \times J$, if the elements $\beta^{(i,j)}$ of W for which $(i, j) \in K$ are linearly independent, and if the elements $\beta^{(i,j)}$ for which $(i, j) \in L$ span the vector space W, then there exists a basis B of the Transportation Problem satisfying $K \subset B \subset L$.

Proposition

Proposition TP-01 *Let m and n be positive integers, and let*

$$W = \left\{ (\mathbf{s}, \mathbf{d}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j \right\}.$$

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, m\}$, and, for each $(i, j) \in I \times J$, let $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$ where $\overline{\mathbf{b}}^{(i)}$ is the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero, and $\mathbf{b}^{(j)}$ is the vector in $\mathbb{R}^n$ whose jth component is equal to 1 and whose other components are zero. Let K and L be subsets of $I \times J$ satisfying $K \subset L$. Suppose that the elements $\beta^{(i,j)}$ of W for which $(i, j) \in K$ are linearly independent, and that the elements $\beta^{(i,j)}$ for which $(i, j) \in L$ span the vector space W. Then there exists a basis B of the Transportation Problem satisfying $K \subset B \subset L$.

Proof

Let $K_0 = K$, and let W_0 be the subspace of W spanned by the elements $\beta^{(i,j)}$ for which $(i,j) \in K_0$. Suppose that W_0 is a proper subspace of W. Then there exists $(i_1, j_1) \in L$ such that $\beta^{(i_1,j_1)} \notin W_0$. Let $K_1 = K_0 \cup \{(i_1, j_1)\}$. Then the elements $\beta^{(i,j)}$ for which $(i,j) \in K_1$ are also linearly independent, and they span a subspace W_1 of W for which $\dim W_1 > \dim W_0$.

Successive iterations of this process will eventually generate a subset B of L for which the elements of the set

$$\{\theta(E^{(i,j)}) : (i,j) \in B\}$$

are linearly independent and also span W. Then B is a basis for the Transportation Problem, and $K \subset B \subset L$, as required. ∎

The feasible solution X is said to be *basic* if the elements $\beta^{(i,j)}$ of W determined by the ordered pairs (i,j) belonging to the set K are linearly independent. It follows from Proposition TP-01 that this is the case if and only if $K \subset B$ for some basis B for the Transportation Problem.

Given any basis B contained in $I \times J$, there can exist at most one basic feasible solution X that satisfies $(X)_{i,j} = 0$ whenever $(i,j) \notin B$. Indeed the linear transformation $\theta_B \colon M_B \to W$ that sends $X \in M_B$ to $(\rho(X), \sigma(X))$ is an isomorphism, and $\theta_B(X) = (\mathbf{s}, \mathbf{d})$, and therefore if X is a feasible solution that satisfies $(X)_{i,j} = 0$ whenever $(i,j) \notin B$, then $X = \theta_B^{-1}(\mathbf{s}, \mathbf{d})$.

The Transportation Problem: Basic Framework (continued)

The $m \times n$ matrix $\theta_B^{-1}(\mathbf{s}, \mathbf{d})$ can be computed for any basis B contained in $I \times J$. However this matrix will often have negative coefficients, in which case it does not represent a feasible solution. The Transportation Problem determined by the supply vector, demand vector and cost matrix has only finitely many basic feasible solutions, because there are only finitely many bases for the problem, and each basis can determine at most one basic feasible solution. Nevertheless the number of basic feasible solutions may be quite large.

The Transportation Problem: Basic Framework (continued)

But it can be shown that the Transportation Problem always has a basic optimal solution. It can be found using an algorithm that implements the Simplex Method devised by George B. Dantzig in the 1940s. This algorithm involves passing from one basis to another, lowering the cost at each stage, until one eventually finds a basis that can be shown to determine a basic optimal solution of the Transportation Problem.

The Numerical Example Revisited

We return to the discussion of the solution of the particular example of the Transportation Problem with 4 suppliers and 5 recipients, where the supply and demand vectors and the cost matrix are as follows:—

$$s^T = (9, 11, 4, 5), \quad d^T = (6, 7, 5, 3, 8).$$

$$C = \begin{pmatrix} 2 & 4 & 3 & 7 & 5 \\ 4 & 8 & 5 & 1 & 8 \\ 5 & 9 & 4 & 4 & 2 \\ 7 & 2 & 5 & 5 & 3 \end{pmatrix}.$$

We found a basic feasible solution to this problem using the Northwest Corner Method. The values $x_{i,j}$ that constitute this basic feasible solution are as set out in the following tableau:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	3	4
4	0	0	0	0	5	5
d_j	6	7	5	3	8	29

This basic feasible solution is associated with the basis B, where

$$B = \{(1,1), (1,2), (2,2), (2,3), (2,4), (3,4), (3,5), (4,5)\}.$$

The Transportation Problem: A Numerical Example (continued)

The rows of the tableau are labelled on the left by the indices that represent the suppliers. The columns of the tableau are labelled on the top by the indices that represent the recipients. The basic feasible solution is represented as an array of real numbers $x_{i,j}$, where $1 \leq i \leq 4$ and $1 \leq j \leq 5$. These real numbers must be non-negative and must satisfy $\sum_{j=1}^{5} x_{i,j} = s_i$ for $i = 1, 2, 3, 4$ and $\sum_{i=1}^{4} x_{i,j} = d_j$ for $j = 1, 2, 3, 4, 5$, where

$$s_1 = 9, \quad s_2 = 11, \quad s_3 = 4, \quad s_4 = 5,$$

$$d_1 = 6, \quad d_2 = 7, \quad d_3 = 5, \quad d_4 = 3, \quad d_5 = 8.$$

The values of s_1, s_2, s_3 and s_4 associated with the rows of the tableau are listed to the right of the tableau, and the values of d_1, d_2, d_3, d_4 and d_5 associated with the columns of the tableau are listed at the bottom of the tableau. The rows of the array presented in the body of the tableau must sum up to the values listed to the right, and the columns of this array must sum up to the values listed along the bottom.

It should be noted that the values $x_{i,j}$ presented in the table satisfy $x_{i,j} = 0$ when $(i, j) \notin B$. This ensures that the feasible solution $(x_{i,j})$ is a basic feasible solution associated with the basis B.

The vector space $\mathbb{R}^4$ has basis $\overline{\mathbf{b}}^{(1)}, \overline{\mathbf{b}}^{(2)}, \overline{\mathbf{b}}^{(3)}, \overline{\mathbf{b}}^{(4)}$, where

$$\overline{\mathbf{b}}^{(1)} = (1, 0, 0, 0), \quad \overline{\mathbf{b}}^{(2)} = (0, 1, 0, 0),$$

$$\overline{\mathbf{b}}^{(3)} = (0, 0, 1, 0), \quad \overline{\mathbf{b}}^{(4)} = (0, 0, 0, 1).$$

Similarly the vector space $\mathbb{R}^5$ has basis $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \mathbf{b}^{(3)}, \mathbf{b}^{(4)}, \mathbf{b}^{(5)}$, where

$$\mathbf{b}^{(1)} = (1, 0, 0, 0, 0), \quad \mathbf{b}^{(2)} = (0, 1, 0, 0, 0),$$

$$\mathbf{b}^{(3)} = (0, 0, 1, 0, 0), \quad \mathbf{b}^{(4)} = (0, 0, 0, 1, 0),$$

$$\mathbf{b}^{(5)} = (0, 0, 0, 0, 1).$$

The Transportation Problem: A Numerical Example (continued)

Let W be the 8-dimensional real vector space defined such that

$$W = \left\{ (\mathbf{s}, \mathbf{d}) \in \mathbb{R}^4 \times \mathbb{R}^5 : \sum_{i=1}^{4} (\mathbf{s})_i = \sum_{j=1}^{5} (\mathbf{d})_j \right\}.$$

Then each ordered pair (i, j) with $1 \leq i \leq 4$ and $1 \leq j \leq 5$ determines a corresponding element $\beta^{(i,j)}$ of W, where $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$.

Let B be a subset of the set of ordered pairs (i, j) of integers for which $1 \leq i \leq 4$ and $1 \leq j \leq 5$. The set B is said to be a *basis* for the Transportation Problem (with 4 suppliers and 5 recipients) if the elements $\beta^{(i,j)}$ of W for which $(i, j) \in B$ constitute a basis of the real vector space W.

The Transportation Problem: A Numerical Example (continued)

In the particular case of the Transportation Problem in which

$$s_1 = 9, \quad s_2 = 11, \quad s_3 = 4, \quad s_4 = 5,$$

$$d_1 = 6, \quad d_2 = 7, \quad d_3 = 5, \quad d_4 = 3, \quad d_5 = 8,$$

the Northwest Corner Method determines a basis B for the Transportation Problem, where

$$B = \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)\}.$$

The basis vectors in W determined by the ordered pairs

$$(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)$$

belonging to the the basis B are as follows:—

$$\begin{aligned}
\beta^{(1,1)} &= ((1,0,0,0),(1,0,0,0,0)),\\
\beta^{(1,2)} &= ((1,0,0,0),(0,1,0,0,0)),\\
\beta^{(2,2)} &= ((0,1,0,0),(0,1,0,0,0)),\\
\beta^{(2,3)} &= ((0,1,0,0),(0,0,1,0,0)),\\
\beta^{(2,4)} &= ((0,1,0,0),(0,0,0,1,0)),\\
\beta^{(3,4)} &= ((0,0,1,0),(0,0,0,1,0)),\\
\beta^{(3,5)} &= ((0,0,1,0),(0,0,0,0,1)),\\
\beta^{(4,5)} &= ((0,0,0,1),(0,0,0,0,1)).
\end{aligned}$$

Previous calculations have established that if $\sum_{j=1}^{5} x_{i,j} = s_i$ for $i = 1, 2, 3, 4$, $\sum_{i=1}^{4} x_{i,j} = d_j$ for $j = 1, 2, 3, 4, 5$, and if $x_{i,j} = 0$ when $(i, j) \notin B$, then

$$\begin{aligned}
x_{1,1} &= d_1, \\
x_{1,2} &= s_1 - d_1, \\
x_{2,2} &= d_2 - s_1 + d_1, \\
x_{2,3} &= d_3, \\
x_{2,4} &= s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{3,4} &= d_4 - s_2 + d_3 + d_2 - s_1 + d_1, \\
x_{3,5} &= s_3 - d_4 + s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{4,5} &= d_5 - s_3 + d_4 - s_2 + d_3 + d_2 - s_1 + d_1.
\end{aligned}$$

The Transportation Problem: A Numerical Example (continued)

It follows from this that

$$
\begin{aligned}
&((s_1, s_2, s_3, s_4), (d_1, d_2, d_3, d_4, d_5)) \\
&\quad = \quad d_1\beta^{(1,1)} + (s_1 - d_1)\beta^{(1,2)} \\
&\qquad\qquad + (d_2 - s_1 + d_1)\beta^{(2,2)} + d_3\beta^{(2,3)} \\
&\qquad\qquad + (s_2 - d_3 - d_2 + s_1 - d_1))\beta^{(2,4)} \\
&\qquad\qquad + (d_4 - s_2 + d_3 + d_2 - s_1 + d_1)\beta^{(3,4)} \\
&\qquad\qquad + (s_3 - d_4 + s_2 - d_3 - d_2 + s_1 - d_1)\beta^{(3,5)} \\
&\qquad\qquad + (d_5 - s_3 + d_4 - s_2 + d_3 + d_2 - s_1 + d_1)\beta^{(4,5)}
\end{aligned}
$$

for all $((s_1, s_2, s_3, s_4), (d_1, d_2, d_3, d_4, d_5)) \in W$.

We can then express the elements of $\beta^{(i,j)}$ of W corresponding to ordered pairs (i,j) not belonging to B as linear combinations of elements of the basis

$$\beta^{(1,1)}, \beta^{(1,2)}, \beta^{(2,2)}, \beta^{(2,3)}, \beta^{(2,4)}, \beta^{(3,4)}, \beta^{(3,5)}, \beta^{(4,5)}$$

of W. For example, to determine $\beta^{(3,1)}$ we set $s_3 = 1$, $s_i = 0$ for $i \neq 3$, $d_1 = 1$ and $d_j = 0$ for $j \neq 1$. We then find that

$$\beta^{(3,1)} = \beta^{(1,1)} - \beta^{(1,2)} + \beta^{(2,2)} - \beta^{(2,4)} + \beta^{(3,4)}.$$

Similarly to determine $\beta^{(4,2)}$ we set $s_4 = 1$ $s_i = 0$ for $i \neq 4$, $d_2 = 1$, and $d_j = 0$ for $j \neq 2$. We then find that

$$\beta^{(4,2)} = \beta^{(2,2)} - \beta^{(2,4)} + \beta^{(3,4)} - \beta^{(3,5)} + \beta^{(4,5)}.$$

The Transportation Problem: A Numerical Example (continued)

These formulae determining $\beta^{(i,j)}$ in terms of the elements of the basis of W can readily be checked. Indeed $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. It follows that

$$\begin{aligned}
&\beta^{(1,1)} - \beta^{(1,2)} + \beta^{(2,2)} - \beta^{(2,4)} + \beta^{(3,4)} \\
&\qquad = (\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(1)}) - (\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(2)}) + (\overline{\mathbf{b}}^{(2)}, \mathbf{b}^{(2)}) \\
&\qquad\quad - (\overline{\mathbf{b}}^{(2)}, \mathbf{b}^{(4)}) + (\overline{\mathbf{b}}^{(3)}, \mathbf{b}^{(4)}) \\
&\qquad = (\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(1)} + \overline{\mathbf{b}}^{(2)} - \overline{\mathbf{b}}^{(2)} + \overline{\mathbf{b}}^{(3)}, \\
&\qquad\quad \mathbf{b}^{(1)} - \mathbf{b}^{(2)} + \mathbf{b}^{(2)} - \mathbf{b}^{(4)} + \mathbf{b}^{(4)}) \\
&\qquad = (\overline{\mathbf{b}}^{(3)}, \mathbf{b}^{(1)}) \\
&\qquad = \beta^{(3,1)}.
\end{aligned}$$