MA3484 Methods of Mathematical Economics
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The Transportation Problem: Basic Framework

The Transportation Problem in the case where total supply equals total demand can be presented as follows:

determine $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$

so as minimize $\sum_{i,j} c_{i,j} x_{i,j}$

subject to the constraints

$x_{i,j} \geq 0$ *for all* i *and* j,

$\sum_{j=1}^{n} x_{i,j} = s_i$ *and* $\sum_{i=1}^{m} x_{i,j} = d_j$, *where*

$s_i \geq 0$ *and* $d_j \geq 0$ *for all* i *and* j, *and*

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

The Transportation Problem: Basic Framework (continued)

We commence the analysis of the Transportation Problem by studying the interrelationships between the various real vector spaces and linear transformations that arise naturally from the statement of the Transportation Problem.

The quantities $x_{i,j}$ to be determined are coefficients of an $m \times n$ matrix X. This matrix X is represented as an element of the real vector space $M_{m,n}(\mathbb{R})$ that consists of all $m \times n$ matrices with real coefficients.

The Transportation Problem: Basic Framework (continued)

The non-negative quantities $s_1, s_2, \ldots, s_m$ that specify the sums of the coefficients in the rows of the unknown matrix X are the components of a *supply vector* $\mathbf{s}$ belonging to the m-dimensional real vector space $\mathbb{R}^m$.

Similarly the non-negative quantities $d_1, d_2, \ldots, d_n$ that specify the sums of the coefficients in the columns of the unknown matrix X are the components of a *demand vector* $\mathbf{d}$ belonging to the n-dimensional space $\mathbb{R}^n$.

The requirement that total supply equals total demand translates into a requirement that the sum $\sum_{i=1}^{m} (\mathbf{s})_i$ of the components of the supply vector $\mathbf{s}$ must equal the sum $\sum_{j=1}^{n} (\mathbf{d})_j$ of the components of the demand vector $\mathbf{d}$.

Accordingly we introduce a real vector space W consisting of all ordered pairs $(\mathbf{s}, \mathbf{d})$ for which $\mathbf{s} \in \mathbb{R}^m$, $\mathbf{d} \in \mathbb{R}^n$ and $\sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j$.

The Transportation Problem: Basic Framework (continued)

It is straightforward to verify that the dimension of the real vector space W is m when $n = 1$. Suppose that $n > 1$. Given real numbers $s_1, s_2, \ldots, s_n$ and $d_1, d_2, \ldots, d_{n-1}$, there exists exactly one element $(\mathbf{s}, \mathbf{d})$ of W that satisfies $(\mathbf{s})_i = s_i$ for $i = 1, 2, \ldots, m$ and $(\mathbf{d})_j = d_j$ for $j = 1, 2, \ldots, n-1$. The remaining component $(\mathbf{d})_n$ of the n-dimensional vector $\mathbf{d}$ is then determined by the equation

$$(\mathbf{d})_n = \sum_{i=1}^{m} s_i - \sum_{j=1}^{m-1} d_j.$$

It follows from this that $\dim W = m + n - 1$.

The supply and demand constraints on the sums of the rows and columns of the unknown matrix X can then be specified by means of linear transformations

$$\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$$

and

$$\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n,$$

where, for each $X \in M_{m,n}(\mathbb{R})$, the components of the m-dimensional vector $\rho(X)$ are the sums of the coefficients along each row of X, and the components of the n-dimensional vector $\sigma(X)$ are the sums of the coefficients along each column of X.

The Transportation Problem: Basic Framework (continued)

Accordingly, for each $X \in M_{m,n}(\mathbb{R})$, the ith component $\rho(X)_i$ of the vector $\rho(X)$ is determined by the equation

$$\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j} \quad \text{for } i = 1, 2, \ldots, m,$$

for $i = 1, 2, \ldots, m$, and the jth component $\sigma(X)_j$ of $\sigma(X)$ is determined by the equation

$$\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j} \quad \text{for } j = 1, 2, \ldots, n.$$

for $j = 1, 2, \ldots, n$.

The Transportation Problem: Basic Framework (continued)

The costs $c_{i,j}$ are the components of an $m \times n$ matrix C, the *cost matrix*, that in turn determines a linear functional

$$f\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}$$

on the vector space $M_{m,n}(\mathbb{R})$ defined such that

$$f(X) = \mathrm{trace}(C^T X) = \sum_{i=1}^{m} \sum_{j=1}^{n} (C)_{i,j} X_{i,j}$$

for all $X \in M_{m,n}(\mathbb{R})$.

The Transportation Problem: Basic Framework (continued)

We now discuss the definitions of *feasible solutions* and *optimal solutions* of the Transportation Problem in the case where total supply equals total demand.

An instance of the problem is specified by specifying a supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C. The components of $\mathbf{s}$ and $\mathbf{d}$ are required to be non-negative real numbers. Moreover $(\mathbf{s}, \mathbf{d}) \in W$, where W is the real vector space consisting of all ordered pairs $(\mathbf{s}, \mathbf{d})$ with $\mathbf{s} \in \mathbb{R}^m$ and $\mathbf{d} \in \mathbb{R}^n$ for which the sum of the components of the vector $\mathbf{s}$ equals the sum of the components of the vector $\mathbf{d}$.

The Transportation Problem: Basic Framework (continued)

A *feasible solution* of the Transportation Problem with given supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C is represented by an $m \times n$ matrix X satisfying the following three conditions:—

- The coefficients of X are all non-negative;
- $\rho(X) = \mathbf{s}$;
- $\sigma(X) = \mathbf{d}$.

The cost functional $f\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}$ is defined so that $f(X) = \mathrm{trace}(C^T X)$ for all $X \in M_{m,n}(\mathbb{R})$. A feasible solution X of this Transportation Problem is said to be *optimal* if it minimizes $f(X)$ amongst all feasible solutions of the problem. Thus a feasible solution X is optimal if and only if $f(X) \leq f(\overline{X})$ for all feasible solutions $\overline{X}$ of the problem.

The Transportation Problem: Basic Framework (continued)

Let $X \in M_{m,n}(\mathbb{R})$. Then

$$\sum_{i=1}^{m} \rho(X)_i = \sum_{i=1}^{m} \sum_{j=1}^{n} (X)_{i,j} = \sum_{j=1}^{n} \sigma(X)_j.$$

It follows that $(\rho(X), \sigma(X)) \in W$ for all $X \in M_{m,n}(\mathbb{R})$. Thus the linear transformations $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ together determine a linear transformation

$$\theta\colon M_{m,n}(\mathbb{R}) \to W,$$

where $\theta(X) = (\rho(X), \sigma(X))$ for all $X \in M_{m,n}(\mathbb{R})$.

The Transportation Problem: Basic Framework (continued)

An $m \times n$ matrix X then represents a feasible solution of the transportation problem with supply vector $\mathbf{s}$ and demand vector $\mathbf{d}$ if and only if the following two conditions are satisfied:—

- The coefficients of X are all non-negative;
- $\theta(X) = (\mathbf{s}, \mathbf{d})$.

The real vector space $M_{m,n}(\mathbb{R})$ consisting of all $m \times n$ matrices with real coefficients has a natural basis consisting of the matrices $E^{(i,j)}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where, for each i and j, the coefficient of the matrix $E^{(i,j)}$ in the ith row and jth column has the value 1, and all other coefficients are zero. Indeed

$$X = \sum_{i=1}^{m} \sum_{j=1}^{n} (X)_{i,j} E^{(i,j)}$$

for all $X \in M_{m,n}(\mathbb{R})$.

The Transportation Problem: Basic Framework (continued)

Now $\rho(E^{(i,j)}) = \overline{\mathbf{b}}^{(i)}$ and $\sigma(E^{(i,j)}) = \mathbf{b}^{(j)}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where $\overline{\mathbf{b}}^{(i)}$ denotes the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero, and $\mathbf{b}^{(j)}$ denotes the vector in $\mathbb{R}^n$ whose jth component is equal to 1 and whose other components are zero. It follows that

$$\theta(E^{(i,j)}) = \beta^{(i,j)}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where

$$\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)}).$$

The Transportation Problem: Basic Framework (continued)

Let

$$I = \{1, 2, \ldots, m\} \quad \text{and} \quad J = \{1, 2, \ldots, n\}.$$

Then $I \times J$ is the set of ordered pairs (i, j) of indices where i and j are integers satisfying $1 \le i \le m$ and $1 \le j \le n$.

Each subset K of $I \times J$ determines corresponding vector subspace M_K of $M_{m,n}(\mathbb{R})$ where

$$M_K = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ when } (i, j) \notin K\}.$$

The Transportation Problem: Basic Framework (continued)

The real vector space M_K has a basis consisting of the elements $E^{(i,j)}$ for $(i,j) \in K$, where $E^{(i,j)}$ denotes the matrix whose coefficient in the ith row and jth column is equal to 1 and whose other coefficients are zero.

It follows that $\dim M_K = |K|$, where $|K|$ denotes the number of elements in the set K.

For each subset K of $I \times J$, let $\theta_K \colon M_K \to W$ denote the restriction of the linear transformation $\theta \colon M_{m,n}(\mathbb{R}) \to W$ to M_K. Then $\theta_K(E^{(i,j)}) = \beta^{(i,j)}$ for all $(i,j) \in K$.

It follows from basic linear algebra that

$$\theta_K \colon M_K \to W$$

is injective if and only if the elements $\beta^{(i,j)}$ of W determined by the ordered pairs (i,j) belonging to the set K are linearly independent.

Also

$$\theta_K \colon M_K \to W$$

is surjective if and only if the elements $\beta^{(i,j)}$ of W determined by the ordered pairs (i,j) belonging to the set K span the vector space W.

These results ensure that

$$\theta_K \colon M_K \to W$$

is an isomorphism if and only if the elements $\beta^{(i,j)}$ of W determined by the ordered pairs (i, j) belonging to the set K constitute a basis for the real vector space W.

The Transportation Problem: Basic Framework (continued)

Now $\dim M_K = |K|$ and $\dim W = m + n - 1$.

It follows that if $\theta_K \colon M_K \to W$ is injective then $\dim M_K \leq \dim W$, and therefore $|K| \leq m + n - 1$.

If $\theta_K \colon M_K \to W$ is surjective then $\dim M_K \geq \dim W$, and therefore $|K| \geq m + n - 1$.

If $\theta_K \colon M_K \to W$ is an isomorphism then $|K| = m + n - 1$.

We say that a subset B of $I \times J$ is a *basis* for the Transportation Problem if the elements $\beta^{(i,j)}$ determined by the ordered pairs (i, j) belonging to the set B constitute a basis for the real vector space W.

The results described previously ensure that a subset B of $I \times J$ is a basis for the Transportation Problem if and only if the associated linear transformation $\theta_B \colon M_B \to W$ is an isomorphism.

Thus a subset B of $I \times J$ is a basis for the Transportation Problem if and only if, given any element $(\mathbf{s}, \mathbf{d})$ of W, there exists a unique $m \times n$ matrix X for which $\rho(X) = \mathbf{s}$ and $\sigma(X) = \mathbf{d}$.