MA3484 Methods of Mathematical Economics
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A Numerical Example

We discuss in detail how to solve a particular example of the Transportation Problem with 4 suppliers and 5 recipients, where the supply and demand vectors and the cost matrix are as follows:—

$$s^T = (9, 11, 4, 5), \quad d^T = (6, 7, 5, 3, 8).$$

$$C = \begin{pmatrix} 2 & 4 & 3 & 7 & 5 \\ 4 & 8 & 5 & 1 & 8 \\ 5 & 9 & 4 & 4 & 2 \\ 7 & 2 & 5 & 5 & 3 \end{pmatrix}.$$

Finding a Basic Feasible Solution using the Northwest Corner Method

We find a basic feasible solution to the Transportation Problem example with 4 suppliers and 5 recipients, where the supply and demand vectors are as follows:—

$$s^T = (9, 11, 4, 5), \quad d^T = (6, 7, 5, 3, 8).$$

We apply an method known as the *Northwest Corner Method*. We need to fill in the entries in a tableau of the form

$x_{i,j}$	1	2	3	4	5	s_i
1	·	·	·	·	·	9
2	·	·	·	·	·	11
3	·	·	·	·	·	4
4	·	·	·	·	·	5
d_j	6	7	5	3	8	29

The Transportation Problem: Northwest Corner Method (continued)

In the tableau just presented the labels on the left hand side identify the suppliers, the labels at the top identify the recipients, the numbers on the right hand side list the number of units that the relevant supplier must provide, and the numbers at the bottom identify the number of units that the relevant recipient must obtain. Number in the bottom right hand corner gives the common value of the total supply and the total demand.

The values in the individual cells must be non-zero, the rows must sum to the value on the right, and the columns must sum to the value on the bottom.

The Transportation Problem: Northwest Corner Method (continued)

The Northwest Corner Method is applied recursively. At each stage the undetermined cell in at the top left (the northwest corner) is given the maximum possible value allowable with the constraints. The remainder of either the first row or the first column must then be completed with zeros. This leads to a reduced tableau to be determined with either one fewer row or else one fewer column. One continues in this fashion, as exemplified in the solution of this particular problem, until the entire tableau has been completed.

The method also determines a basis associated with the basic feasible solution determined by the Northwest Corner Method. This basis lists the cells that play the role of northwest corner at each stage of the method.

The Transportation Problem: Northwest Corner Method (continued)

At the first stage, the northwest corner cell is associated with supplier 1 and recipient 1. This cell is assigned a value equal to the mimimum of the corresponding column and row sums.d Thus, this example, the northwest corner cell, is given the value 6, which is the desired column sum. The remaining cells in that row are given the value 0.

The tableau then takes the following form:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	·	·	·	·	9
2	0	·	·	·	·	11
3	0	·	·	·	·	4
4	0	·	·	·	·	5
d_j	6	7	5	3	8	29

The ordered pair $(1, 1)$ commences the list of elements making up the associated basis.

The Transportation Problem: Northwest Corner Method (continued)

At the second stage, one applies the Northwest Corner Method to the following reduced tableau:—

$x_{i,j}$	2	3	4	5	s_i
1	·	·	·	·	3
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	7	5	3	8	23

The required value for the first row sum of the reduced tableau has been reduced to reflect the fact that the values in the remaining undetermined cells of the first row must sum to the value 3.

At the second stage, one applies the Northwest Corner Method to the following reduced tableau:—
The value 3 is then assigned to the northwest corner cell of the reduced tableau (as 3 is the maximum possible value for this cell subject to the constraints on row and column sums). The reduced tableau therefore takes the following form after the second stage:—

$x_{i,j}$	2	3	4	5	s_i
1	3	0	0	0	3
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	7	5	3	8	23

The Transportation Problem: Northwest Corner Method (continued)

The main tableau at the completion of the second stage then stands as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	·	·	·	·	11
3	0	·	·	·	·	4
4	0	·	·	·	·	5
d_j	6	7	5	3	8	29

The list of ordered pairs representing the basis elements determined at the second stage then stands as follows:—

Basis: $(1,1), (1,2), \ldots$.

The reduced tableau for the third stage then stands as follows:—

$x_{i,j}$	2	3	4	5	s_i
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	4	5	3	8	20

Accordingly the northwest corner of the reduced tableau should be assigned the value 4, and the remaining elements of the first column should be assigned the value 0.

The reduced tableau at the completion of the third stage stands as follows:—

$x_{i,j}$	2	3	4	5	s_i
2	4	·	·	·	11
3	0	·	·	·	4
4	0	·	·	·	5
d_j	4	5	3	8	20

The main tableau and list of basis elements at the completion of the third stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	·	·	·	11
3	0	0	·	·	·	4
4	0	0	·	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1), (1,2), (2,2), \ldots$.

The reduced tableau at the completion of the fourth stage is as follows:—

$x_{i,j}$	3	4	5	s_i
2	5	·	·	7
3	0	·	·	4
4	0	·	·	5
d_j	5	3	8	16

The Transportation Problem: Northwest Corner Method (continued)

The main tableau and list of basis elements at the completion of the fourth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	·	·	11
3	0	0	0	·	·	4
4	0	0	0	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(1,2),(2,2),(2,3),\ldots$.

The Transportation Problem: Northwest Corner Method (continued)

At the fifth stage the sum of the undetermined cells for the 2nd supplier must sum to 2. Therefore the main tableau and list of basis elements at the completion of the fifth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	·	·	4
4	0	0	0	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(1,2),(2,2),(2,3),(2,4),\ldots$.

The Transportation Problem: Northwest Corner Method (continued)

At the sixth stage the sum of the undetermined cells for the 4th recipient must sum to 1. Therefore the main tableau and list of basis elements at the completion of the sixth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	·	4
4	0	0	0	0	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),\ldots$.

Two further stages suffice to complete the tableau. Moreover, at the completion of the eighth and final stage the main tableau and list of basis elements stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	3	4
4	0	0	0	0	5	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)$.

We now check that we have indeed obtained a basis B, where

$$B = \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)\}.$$

If B is indeed a basis, then arbitrary values s_1, s_2, s_3, s_4 and d_1, d_2, d_3, d_4, d_5 should determine corresponding values of $x_{i,j}$ for $(i,j) \in B$, as indicated in the following tableau:—

$x_{i,j}$	1	2	3	4	5	
1	$x_{1,1}$	$x_{1,2}$				s_1
2		$x_{2,2}$	$x_{2,3}$	$x_{2,4}$		s_2
3				$x_{3,4}$	$x_{3,5}$	s_3
4					$x_{4,5}$	s_4
	d_1	d_2	d_3	d_4	d_5	

Now analysis of the Northwest Corner Method shows that, when successive elements of the set B are ordered by the stage of the method at which they are determined. Then the value of $x_{i',j'}$ for a given ordered pair $(i', j') \in B$ is determined by the values of the row sums s_i, the column sums d_j, together with the values $x_{i,j}$ for the ordered pairs (i, j) in the set B determined at earlier stages of the method.

The Transportation Problem: Northwest Corner Method (continued)

In the specific numerical example that we have just considered, we find that the values of $x_{i,j}$ for ordered pairs (i,j) in the set B, where

$$B = \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)\},$$

are determined by solving, successively, the following equations:—

$$x_{1,1} = d_1, \quad x_{1,2} = s_1 - x_{1,1}, \quad x_{2,2} = d_2 - x_{1,2},$$

$$x_{2,3} = d_3, \quad x_{2,4} = s_2 - x_{2,3} - x_{2,2}, \quad x_{3,4} = d_4 - x_{2,4},$$

$$x_{3,5} = s_3 - x_{3,4}, \quad x_{4,5} = d_5 - x_{3,5},$$

It follows that the values of $x_{i,j}$ for $(i,j) \in B$ are indeed determined by s_1, s_2, s_3, s_4 and d_1, d_2, d_3, d_4, d_5.

Indeed we find that

$$
\begin{aligned}
x_{1,1} &= d_1, \\
x_{1,2} &= s_1 - d_1, \\
x_{2,2} &= d_2 - s_1 + d_1, \\
x_{2,3} &= d_3, \\
x_{2,4} &= s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{3,4} &= d_4 - s_2 + d_3 + d_2 - s_1 + d_1, \\
x_{3,5} &= s_3 - d_4 + s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{4,5} &= d_5 - s_3 + d_4 - s_2 + d_3 + d_2 - s_1 + d_1.
\end{aligned}
$$

Note that, in this specific example, the values of $x_{i,j}$ for ordered pairs (i, j) in the basis B are expressed as sums of terms of the form $\pm s_i$ and $\pm d_j$. Moreover the summands s_i all have the same sign, the summands d_j all have the same sign, and the sign of the terms s_i is opposite to the sign of the terms d_j. Thus, for example

$$x_{4,5} = (d_1 + d_2 + d_3 + d_4 + d_5) - (s_1 + s_2 + s_3).$$

This pattern is in fact a manifestation of a general result applicable to all instances of the Transportation Problem.

The Transportation Problem: Cost of the Basic Feasable Solution

(i,j)	$x_{i,j}$	$c_{i,j}$	$c_{i,j}x_{i,j}$
$(1,1)$	6	2	12
$(1,2)$	3	4	12
$(2,2)$	4	8	32
$(2,3)$	5	5	25
$(2,4)$	2	1	2
$(3,4)$	1	4	4
$(3,5)$	3	2	6
$(4,5)$	5	3	15
Total			108

Now the basic feasible solution produced by applying the Northwest Corner Method is just one amongst many basic feasible solutions. There are many others. Some of these may be obtained on applying the Northwest Corner Method after reordering the rows and columns (thus renumbering the suppliers and recipients). It would take significant work to calculate all basic feasible solutions and then calculate the cost associated with each one.

However there is a method for passing from one feasible solution to another so as to progressively lower the cost until the feasible solution have been found and verified to be the optimal solution to the problem.

Let B be the basis consisting of the ordered pairs

$$(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5).$$

The following tableau records the costs $c_{i,j}$ associated with those ordered pairs (i,j) that belong to the basis B:

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				u_1
2		8	5	1		u_2
3				4	2	u_3
4					3	u_4
v_j	v_1	v_2	v_3	v_4	v_5	

We determine real numbers u_i for $i = 1, 2, 3, 4$ and v_j for $j = 1, 2, 3, 4, 5$ such that $v_j - u_i = c_{i,j}$ for all $(i, j) \in B$. These real numbers u_i and v_j are not required to be non-negative: they may be positive, negative or zero.

(We postpone till later an explanation as to why finding values u_i and v_j satisfying the above equation actually helps us in solving the problem.)

Now if the real numbers u_i and v_j provide a solution to these equations, then another solution is obtained on replacing u_i and v_j by $u_i + k$ and $v_j + k$, where k is some fixed constant. It follows that one of the required values can be set to an arbitrary value. Accordingly we seek a solution with $u_1 = 0$. Then we must have $v_1 = 2$ and $v_2 = 4$ in order to satisfy the equations determined by the costs associated with the basis elements with $i = 1$.

After setting $u_1 = 0$, and then determining the values of v_1 and v_2, the tableau for finding the numbers u_i and v_j takes the following form:—

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				0
2		8	5	1		u_2
3				4	2	u_3
4					3	u_4
v_j	2	4	v_3	v_4	v_5	

The equations $v_2 = 4$ and $c_{2,2} = 8$ then force $u_2 = -4$, which in turn forces $v_3 = 1$ and $v_4 = -3$.

After setting $u_1 = 0$, and then successively determining the values of v_1 v_2, u_2, v_3 and v_4, the tableau takes the following form:—

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				0
2		8	5	1		-4
3				4	2	u_3
4					3	u_4
v_j	2	4	1	-3	v_5	

The equations $v_4 - u_3 = c_{3,4}$, $v_5 - u_3 = c_{3,5}$ and $v_5 - u_4 = c_{4,5}$ then successively force $u_3 = -7$, $v_5 = -5$ and $u_4 = -8$.

The completed tableau for determining the values of u_i and v_j thus takes the following form:—

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				0
2		8	5	1		-4
3				4	2	-7
4					3	-8
v_j	2	4	1	-3	-5	