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The Transportation Problem in the case where total supply equals total demand can be presented as follows:

determine $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$

so as minimize $\sum_{i,j} c_{i,j} x_{i,j}$

subject to the constraints

$x_{i,j} \geq 0$ *for all* i *and* j,

$\sum_{j=1}^{n} x_{i,j} = s_i$ *and* $\sum_{i=1}^{m} x_{i,j} = d_j$, *where*

$s_i \geq 0$ *and* $d_j \geq 0$ *for all* i *and* j, *and*

$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.

Let

$$I = \{1, 2, \ldots, m\} \quad \text{and} \quad J = \{1, 2, \ldots, n\}.$$

Then $I \times J$ is the set of ordered pairs (i, j) of indices where i and j are integers satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$.

Let $M_{m,n}(\mathbb{R})$ denote the real vector space that consists of all $m \times n$ matrices with real coefficients, let K be a subset of $I \times J$, and let

$$M_K = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ when } (i, j) \notin K\}.$$

Then M_K is a vector subspace of the space $M_{m,n}(\mathbb{R})$.

The real vector space M_K has a basis consisting of the elements $E^{(i,j)}$ for $(i,j) \in K$, where $E^{(i,j)}$ denotes the matrix whose coefficient in the ith row and jth column is equal to 1 and whose other coefficients are zero.

It follows that $\dim M_K = |K|$, where $|K|$ denotes the number of elements in the set K.

The Transportation Problem: Row and Column Sums (continued)

Let

$$\overline{\mathbf{b}}^{(1)}, \overline{\mathbf{b}}^{(2)}, \ldots \overline{\mathbf{b}}^{(m)}$$

denote the standard basis of $\mathbb{R}^m$ where, for each integer i between 1 and m, $\overline{\mathbf{b}}^{(i)}$ is the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero. Similarly let

$$\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(n)}$$

denote the standard basis of $\mathbb{R}^n$ where, for each integer j between 1 and n, $\overline{\mathbf{b}}^{(j)}$ is the vector in $\mathbb{R}^n$ whose jth component is equal to 1 and whose other components are zero. Then $\mathbf{s} = \sum_{i=1}^{m} (\mathbf{s})_i \overline{\mathbf{b}}^{(i)}$ for all $\mathbf{s} \in \mathbb{R}^m$, where, for each i, $(\mathbf{s})_i$ denotes the ith component of $\mathbf{s}$, and $\mathbf{d} = \sum_{j=1}^{n} (\mathbf{d})_j \mathbf{b}^{(j)}$, for all $\mathbf{d} \in \mathbb{R}^n$, where, for each j, $(\mathbf{d})_j$ denotes the jth component of $\mathbf{d}$.

The Transportation Problem: Row and Column Sums (continued)

We denote by W the real vector space consisting of all ordered pairs $(\mathbf{s}, \mathbf{d})$ for which $\mathbf{s} \in \mathbb{R}^m$, $\mathbf{d} \in \mathbb{R}^n$ and $\sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j$.

Let $\mathbf{s} \in \mathbb{R}^m$ and $\mathbf{d} \in \mathbb{R}^n$, and let $s_i = (\mathbf{s})_i$ for $i = 1, 2, \ldots, m$ and $d_j = (\mathbf{d})_j$ for $j = 1, 2, \ldots, n$. Then $(\mathbf{s}, \mathbf{d}) \in W$ if and only if

$$d_n = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n-1} d_j.$$

It follows that, given any element $\mathbf{w}$ of W, there exist uniquely-determined real numbers $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_{n-1}$ such that

$$\mathbf{w} = \sum_{i=1}^{m} s_i(\overline{\mathbf{b}}_i, \mathbf{b}_n) + \sum_{j=1}^{n-1} d_j(\mathbf{0}, \mathbf{b}_j - \mathbf{b}_n).$$

It follows that the real vector space W is of dimension $m + n - 1$, and that W has a basis consisting of the elements $(\overline{\mathbf{b}}_i, \mathbf{b}_n)$ for $i = 1, 2, \ldots, m$, together with the elements $(\mathbf{0}, \mathbf{b}_j - \mathbf{b}_n)$ for $j = 1, 2, \ldots, n - 1$.

The Transportation Problem: Row and Column Sums (continued)

Given an $m \times n$ matrix X with real coefficients, let $\rho(X)$ denote the m-dimensional vector whose ith component $\rho(X)_i$ is the sum of the elements of the ith row of X, and let $\sigma(X)$ denote the n-dimensional vector whose jth component is the sum of the elements of the jth column of X. Then $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ are linear transformations. Moreover

$$\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j} \quad \text{for } i = 1, 2, \ldots, m,$$

and

$$\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j} \quad \text{for } j = 1, 2, \ldots, n.$$

The Transportation Problem: Row and Column Sums (continued)

The linear transformations $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ then satisfy

$$\rho(E^{(i,j)}) = \overline{\mathbf{b}}^{(i)} \quad \text{and} \quad \sigma(E^{(i,j)}) = \mathbf{b}^{(j)}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where $E^{(i,j)}$ denotes the $m \times n$ matrix whose coefficient in the ith row and jth column is equal to 1 and whose other coefficients are zero, $\overline{\mathbf{b}}^{(i)}$ denotes the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero, and where $\mathbf{b}^{(j)}$ denotes the vector in $\mathbb{R}^n$ whose jth component is equal to 1 and whose other components are zero.

The Transportation Problem: Row and Column Sums (continued)

Let $\theta\colon M_{m,n}(\mathbb{R}) \to W$ denote the linear transformation from the space $M_{m,n}(\mathbb{R})$ of $m \times n$ matrices with real coefficients to the vector space W, defined such that $\theta(X) = (\rho(X), \sigma(X))$ for all $X \in M_{m,n}(\mathbb{R})$, where $\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$.

Then

$$\theta(E^{(i,j)}) = \beta^{(i,j)}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where

$$\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)}).$$

For each subset K of $I \times J$, where

$$I = \{1, 2, \ldots, m\} \quad \text{and} \quad J = \{1, 2, \ldots, n\},$$

the linear transformation $\theta\colon M_{m,n}(\mathbb{R}) \to W$ restricts to a linear transformation $\theta_K\colon M_K \to W$, where

$$M_K = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ whenever } (i,j) \notin K\}.$$

The *rank* $\operatorname{rank}(\theta_K)$ of θ_K is the dimension of the image $\theta_K(M_K)$ of θ_K, where

$$\theta_K(M_K) = \{\theta(X) : X \in M_K\}.$$

The *nullity* $\operatorname{nullity}(\theta_K)$ of θ_K is the dimension of the kernel $\ker\theta_K$ of θ_K, where

$$\ker\theta_K = \{X \in M_K : \theta_K(X) = (\mathbf{0}, \mathbf{0})\}.$$

The Transportation Problem: Row and Column Sums (continued)

A basic theorem of linear algebra guarantees that the sum of the rank and nullity of a linear transformation is equal to the dimension of the domain of that transformation. Now the domain of θ_K is M_K, and this vector space is of dimension $|K|$, where $|K|$ denotes the number of elements in the finite set K. Therefore

$$\mathrm{rank}(\theta_K) + \mathrm{nullity}(\theta_K) = |K|.$$

Now $\theta_K(M_K)$ is a subspace of W, and therefore

$$\mathrm{rank}(\theta_K) = \dim \theta_K(M_K) \leq \dim W = m + n - 1.$$

Thus $\mathrm{rank}(\theta_K) \leq m + n - 1$. Moreover $\mathrm{rank}(\theta_K) = m + n - 1$ if and only if $\theta_K \colon M_K \to W$ is surjective.

If $\theta_K\colon M_K \to W$ is injective then $\mathrm{nullity}(\theta_K) = 0$, and therefore

$$|K| = \mathrm{rank}(\theta_K) \le m + n - 1.$$

If $\theta_K\colon M_K \to W$ is surjective then $\mathrm{rank}(\theta_K) = m + n - 1$, and therefore

$$|K| = \mathrm{rank}(\theta_K) + \mathrm{nullity}(\theta_K) \ge m + n - 1.$$

It follows that if $\theta_K\colon M_K \to W$ is an isomorphism then $|K| = m + n - 1$.

Definition

We say that a subset B of $I \times J$ is a *basis* for the Transportation Problem if the corresponding linear transformation

$$\theta_B \colon M_B(\mathbb{R}) \to W$$

is an isomorphism.

If B is a basis for the Transportation Problem then the linear transformation

$$\theta_B \colon M_B(\mathbb{R}) \to W$$

must be both injective and surjective, and therefore the number $|B|$ of elements of that basis must satisfy

$$|B| = \dim W = m + n - 1.$$

The Transportation Problem: Row and Column Sums (continued)

It then follows from this definition that a subset B of $I \times J$ is a *basis* (in the context of the Transportation Problem) if and only if, given any real numbers $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$ satisfying

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

there exist uniquely-determined real numbers $x_{i,j}$ such that

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{i,j} = 0 \quad \text{whenever } (i,j) \notin B.$$

Here $E^{(i,j)}$ denotes the $m \times n$ matrix whose coefficient in the ith row and jth column is equal to 1 and whose other coefficients are zero, W is the real vector space consisting of all ordered pairs $(\mathbf{s}, \mathbf{d})$ in $\mathbb{R}^m \times \mathbb{R}^n$ for which $\sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j$, and $\theta\colon M_{m,n}(\mathbb{R}) \to W$ is defined such that

$$\theta(X) = \left(\sum_{j=1}^{n} (X)_{i,j}, \sum_{i=1}^{m} (X)_{i,j} \right)$$

for all $X \in M_{m,n}(\mathbb{R})$.

Let $\beta^{(i,j)} \in W$ be defined for all $(i,j) \in I \times J$ such that $\beta^{(i,j)} = \theta(E^{(i,j)})$. Then $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$ where $(\overline{\mathbf{b}}^{(i)})_i = 1$, $(\overline{\mathbf{b}}^{(i)})_{i'} = 0$ for all $i' \in I \setminus \{i\}$, $(\mathbf{b}^{(j)})_j = 1$ and $(\mathbf{b}^{(j)})_{j'} = 0$ for all $j' \in J \setminus \{j\}$.

A subset B of $I \times J$ is a basis for the Transportation Problem if and only if the elements $\beta^{(i,j)}$ for which $(i,j) \in B$ consistitute a basis of the vector space W.

We now prove that if K and L are subsets of $I \times J$, if the elements $\beta^{(i,j)}$ of W for which $(i,j) \in K$ are linearly independent, and if the elements $\beta^{(i,j)}$ for which $(i,j) \in L$ span the vector space W, then there exists a basis B of the Transportation Problem satisfying $K \subset B \subset L$.

The Transportation Problem: Row and Column Sums (continued)

Proposition

Proposition TP-01 *Let m and n be positive integers, and let*

$$W = \left\{ (\mathbf{s}, \mathbf{d}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j \right\}.$$

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, m\}$, and, for each $(i, j) \in I \times J$, let $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$ where $\overline{\mathbf{b}}^{(i)}$ is the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero, and $\mathbf{b}^{(j)}$ is the vector in $\mathbb{R}^n$ whose jth component is equal to 1 and whose other components are zero. Let K and L be subsets of $I \times J$ satisfying $K \subset L$. Suppose that the elements $\beta^{(i,j)}$ of W for which $(i, j) \in K$ are linearly independent, and that the elements $\beta^{(i,j)}$ for which $(i, j) \in L$ span the vector space W. Then there exists a basis B of the Transportation Problem satisfying $K \subset B \subset L$.

Proof

Let $K_0 = K$, and let W_0 be the subspace of W spanned by the elements $\beta^{(i,j)}$ for which $(i,j) \in K_0$. Suppose that W_0 is a proper subspace of W. Then there exists $(i_1, j_1) \in L$ such that $\beta^{(i_1,j_1)} \notin W_0$. Let $K_1 = K_0 \cup \{(i_1, j_1)\}$. Then the elements $\beta^{(i,j)}$ for which $(i,j) \in K_1$ are also linearly independent, and they span a subspace W_1 of W for which $\dim W_1 > \dim W_0$.

Successive iterations of this process will eventually generate a subset B of L for which the elements of the set

$$\{\theta(E^{(i,j)}) : (i,j) \in B\}$$

are linearly independent and also span W. Then B is a basis for the Transportation Problem, and $K \subset B \subset L$, as required. ∎

Corollary

Corollary TP-02 *Let m and n be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let K be a subset of $I \times J$. Suppose that there is no basis of the Transportation Problem for which $K \subset B$. Then there exist real numbers $y_{i,j}$ for $(i, j) \in I \times J$, not all zero, which satisfy the following conditions:—*

- $\displaystyle\sum_{j=1}^{n} y_{i,j} = 0$ *for* $i = 1, 2, \ldots, m$;
- $\displaystyle\sum_{i=1}^{m} y_{i,j} = 0$ *for* $j = 1, 2, \ldots, n$;
- $y_{i,j} = 0$ *when* $(i, j) \notin K$.

Proof

Let the real vector space W and the elements $\beta^{(i,j)}$ of W corresponding to ordered pairs (i,j) belonging to $I \times J$ be defined as in the statement of Proposition TP-01. Also let $E^{(i,j)}$ denote the $m \times n$ matrix whose coefficient in the ith row and jth column is equal to 1 and whose remaining coefficients are zero. Then $\beta^{(i,j)} = \theta(E^{(i,j)})$ for all $(i,j) \in I \times J$, where

$$\theta(X) = \left(\sum_{j=1}^{n} (X)_{i,j}, \sum_{i=1}^{m} (X)_{i,j} \right)$$

for all $X \in M_{I \times J}$.

If it were the case that the elements $\beta^{(i,j)}$ for which $(i,j) \in K$ were linearly independent elements of the vector space W then it would follow from an application of Proposition TP-01 (with $L = I \times J$) that there would exist a basis B for the transportation problem satisfying $K \subset B \subset I \times J$.

However there is no basis B for the Transportation Problem for which $K \subset B$. It follows that the elements $\beta^{(i,j)}$ for which $(i,j) \in K$ must be linearly dependent elements of W, and therefore there must exist real numbers $y_{i,j}$ for all $(i,j) \in I \times J$, not all zero, such that $y_{i,j} = 0$ when $(i,j) \notin K$ and $\sum_{(i,j)\in K} y_{i,j}\beta^{(i,j)} = 0_W$, where 0_W denotes the zero element of W.

Now $\beta^{(i,j)} = \theta(E^{(i,j)})$ for all $(i,j) \in I \times J$, where $E^{(i,j)}$ denotes the $m \times n$ matrix whose coefficient in the ith row and jth column is equal to 1 and whose other coefficients are zero, W is the real vector space consisting of all ordered pairs $(\mathbf{s}, \mathbf{d})$ in $\mathbb{R}^m \times \mathbb{R}^n$ for which $\sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j$, and $\theta\colon M_{m,n}(\mathbb{R}) \to W$ is defined such that

The Transportation Problem: Row and Column Sums (continued)

$$\theta(X) = \left(\sum_{j=1}^{n} (X)_{i,j}, \sum_{i=1}^{m} (X)_{i,j} \right)$$

for all $X \in M_{m,n}(\mathbb{R})$. It follows that

$$\theta\left(\sum_{(i,j)\in K} y_{i,j} E^{(i,j)} \right) = \sum_{(i,j)\in K} y_{i,j} \theta(E^{(i,j)}) = \sum_{(i,j)\in K} y_{i,j} \beta^{(i,j)} = 0_W.$$

It then follows that $\sum_{j=1}^{n} y_{i,j} = 0$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} y_{i,j} = 0$ for $j = 1, 2, \ldots, n$. Moreover $y_{i,j} = 0$ when $(i,j) \notin K$, as required. ∎

Corollary

Corollary TP-03 *Let K be a subset of $I \times J$, where $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. Suppose that the number $|K|$ of elements of K satisfies $|K| > m + n - 1$. Then there exist real numbers $y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, not all zero, such that*

$$\begin{aligned} \sum_{j=1}^{n} y_{i,j} &= 0 \quad \text{for } i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} y_{i,j} &= 0 \quad \text{for } j = 1, 2, \ldots, n. \end{aligned}$$

and $y_{i,j} = 0$ whenever $i, j \notin K$.

Proof

Let W be the real vector space consisting of ordered pairs $(\mathbf{s}, \mathbf{d})$ of vectors where $\mathbf{s}$ has components $s_1, s_2, \ldots, s_m$, $\mathbf{d}$ has components $d_1, d_2, \ldots, d_n$ and

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

Then $\dim W = m + n - 1$.

Now every basis for the Transportation Problem contains $m + n - 1$ elements. But $|K| > m + n - 1$, and thus that there cannot exist any basis B for the Transportation Problem that satisfies $K \subset B$. It therefore follows from Corollary TP-02 that there must exist real numbers $y_{i,j}$ for $(i,j) \in I \times J$ that satisfy the required conditions. ∎

Remark

Corollary TP-03 follows from the basic principle that, in a system of simultaneous linear equations, if the number of unknowns exceeds the number of independent equations, then solutions to the system are not uniquely determined.

The Transportation Problem: Row and Column Sums (continued)

In the context of Lemma TP-03 we consider a system of simultanous linear equations of the form

$$\begin{aligned} \sum_{j=1}^{n} x_{i,j} &= s_i \quad \text{for } i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} x_{i,j} &= d_i \quad \text{for } j = 1, 2, \ldots, n, \end{aligned}$$

where $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$ are given real numbers that satisfy

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

The Transportation Problem: Row and Column Sums (continued)

We also have a subset K of $I \times J$ with $|K|$ elements, where

$$I = \{1, 2, \ldots, m\} \quad \text{and} \quad J = \{1, 2, \ldots, n\},$$

we require that $x_{i,j} = 0$ when $(i, j) \notin K$, and the system of simultaneous equations is to be solved to determine the unknowns $x_{i,j}$ for $(i, j) \in K$.

We thus have a system of $n + m$ simultaneous linear equations to determine $|K|$ unknowns.

The Transportation Problem: Row and Column Sums (continued)

However the linear equations in the system are not independent. Indeed suppose that $n > 1$ and that

$$\begin{aligned} \sum_{j=1}^{n} x_{i,j} &= s_i \quad \text{for } i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} x_{i,j} &= d_i \quad \text{for } j = 1, 2, \ldots, n-1, \end{aligned}$$

Then

$$x_{i,n} = s_i - \sum_{j=1}^{n-1} x_{i,j},$$

and therefore

The Transportation Problem: Row and Column Sums (continued)

$$\begin{aligned}
\sum_{i=1}^{m} x_{i,n} &= \sum_{i=1}^{m} s_i - \sum_{i=1}^{m}\sum_{j=1}^{n-1} x_{i,j} \\
&= \sum_{j=1}^{n} d_j - \sum_{j=1}^{n-1}\left(\sum_{i=1}^{m} x_{i,j}\right) \\
&= \sum_{j=1}^{n} d_j - \sum_{j=1}^{n-1} d_j \\
&= d_n.
\end{aligned}$$

Thus the system of simultaneous linear equations has at most $n + m - 1$ independent equations.

It follows that if the system is to be solved to determine $|K|$ unknowns, then those unknowns are not uniquely determined.

In particular the solution of the set of simultanous linear equations is not unique in the case where $s_i = 0$ for all i and $d_j = 0$ for all j. It follows that there must exist real numbers $y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, not all zero, such that

$$\begin{aligned} \sum_{j=1}^{n} y_{i,j} &= 0 \quad \text{for } i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} y_{i,j} &= 0 \quad \text{for } j = 1, 2, \ldots, n. \end{aligned}$$

and $y_{i,j} = 0$ whenever $i, j \notin K$.

Thus the conclusions of Lemma TP-03 thus follow from the basic principle that solutions to systems of simultaneous linear equations are not uniquely determined when the number of unknowns exceeds the number of independent equations in the system.

Feasible, Basic and Optimal Solutions of the Transportation Problem

Consider the Transportation Problem with m suppliers and n recipients, where the ith supplier can provide at most s_i units of some given commodity, where $s_i \geq 0$, and the jth recipient requires at least d_j units of that commodity, where $d_j \geq 0$. We suppose also that total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

The cost of transporting the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

Definition

A *feasible* solution to the Transportation Problem takes the form of real numbers $x_{i,j}$, where

- $x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;
- $\sum_{j=1}^{n} x_{i,j} = s_i$;
- $\sum_{i=1}^{m} x_{i,j} = d_j$.

Definition

A feasible solution $(x_{i,j})$ of a Transportation Problem is said to be *basic* if there exists a basis B for that Transportation Problem such that $x_{i,j} = 0$ whenever $(i, j) \notin B$.

The cost associated with a feasible solution $(x_{i,j})$ to the Transportation Problem is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

Definition

A feasible solution $(x_{i,j})$ is said to be *optimal* if it minimizes cost amongst all feasible solutions of the Transportation Problem.

Thus a feasible solution $(x_{i,j})$ is optimal if and only if

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} \overline{x}_{i,j}$$

for all feasible solutions $(\overline{x}_{i,j})$ of the Transportation Problem.

Definition

An optimal solution $(x_{i,j})$ of a Transportation Problem is said to be *basic* if there exists a basis B for that Transportation Problem such that $x_{i,j} = 0$ whenever $(i,j) \notin B$.

Example

Consider the instance of the Transportation Problem where $m = n = 2$, $s_1 = 8$, $s_2 = 3$, $d_1 = 2$, $d_2 = 9$, $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$.

A feasible solution takes the form of a 2×2 matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

with non-negative components which satisfies the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}.$$

A basic feasible solution will have at least one component equal to zero. There are four matrices with at least one zero component which satisfy the required equations. They are the following:—

$$\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 8 & 0 \\ -6 & 9 \end{pmatrix}, \quad \begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 9 \\ 3 & 0 \end{pmatrix}.$$

The first and third of these matrices have non-negative components. These two matrices represent basic feasible solutions to the problem, and moreover they are the only basic feasible solutions.

The Transportation Problem: Optimal Solutions (continued)

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$.

The cost of the basic feasible solution $\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}$ is

$$8c_{1,2} + 2c_{2,1} + c_{2,2} = 24 + 8 + 1 = 33.$$

The cost of the basic feasible solution $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is

$$2c_{1,1} + 6c_{1,2} + 3c_{2,2} = 4 + 18 + 3 = 25.$$

Now any 2×2 matrix $\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ satisfying the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}$$

must be of the form

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} \lambda & 8 - \lambda \\ 2 - \lambda & 1 + \lambda \end{pmatrix}$$

for some real number λ.

But the matrix $\begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$ has non-negative components if and only if $0 \leq \lambda \leq 2$. It follows that the set of feasible solutions of this instance of the transportation problem is

$$\left\{ \begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix} : \lambda \in \mathbb{R} \text{ and } 0 \leq \lambda \leq 2 \right\}.$$

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$. Therefore, for each real number λ satisfying $0 \leq \lambda \leq 2$, the cost $f(\lambda)$ of the feasible solution $\begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$ is given by

$$f(\lambda) = 2\lambda + 3(8-\lambda) + 4(2-\lambda) + (1+\lambda) = 33 - 4\lambda.$$

Cost is minimized when $\lambda = 2$, and thus $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is the optimal solution of this instance of the Transportation Problem. The cost of this optimal solution is 25.

Proposition

Proposition TP-04 *Given any feasible solution of the Transportation Problem, there exists a basic feasible solution with whose cost does not exceed that of the given solution.*

Proof

Let m and n be positive integers, and let $\mathbf{s}$ and $\mathbf{d}$ be elements of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively that satisfy $(\mathbf{s})_i \geq 0$ for $i = 1, 2, \ldots, m$, $(\mathbf{d})_j \geq 0$ for $j = 1, 2, \ldots, n$ and $\sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j$, let C be an $m \times n$ matrix whose components are non-negative real numbers, and let X be a feasible solution of the resulting instance of the Transportation Problem with cost matrix C.

Let $s_i = (\mathbf{s})_i$, $d_j = (\mathbf{d})_j$, $x_{i,j} = (X)_{i,j}$ and $c_{i,j} = (C)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then $x_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$.

The cost of the feasible solution X is then $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$.

If the feasible solution X is itself basic then there is nothing to prove. Suppose therefore that X is not a basic solution. We show that there then exists a feasible solution $\overline{X}$ with fewer non-zero components than the given feasible solution.

The Transportation Problem: Optimal Solutions (continued)

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let

$$K = \{(i, j) \in I \times J : x_{i,j} > 0\}.$$

Because X is not a basic solution to the Transportation Problem, there does not exist any basis B for the Transportation Problem satisfying $K \subset B$. It therefore follows from Corollary TP-02 that there exist real numbers $y_{i,j}$ for $(i, j) \in I \times J$, not all zero, which satisfy the following conditions:—

- $\displaystyle\sum_{j=1}^{n} y_{i,j} = 0$ for $i = 1, 2, \ldots, m$;
- $\displaystyle\sum_{i=1}^{m} y_{i,j} = 0$ for $j = 1, 2, \ldots, n$;
- $y_{i,j} = 0$ when $(i, j) \notin K$.

The Transportation Problem: Optimal Solutions (continued)

We can assume without loss of generality that $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} y_{i,j} \geq 0$, because otherwise we can replace $y_{i,j}$ with $-y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Let Y be the $m \times n$ matrix satisfying $(Y)_{i,j} = y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and let $Z_\lambda = X - \lambda Y$ for all real numbers λ. Then $(Z_\lambda)_{i,j} = x_{i,j} - \lambda y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Moreover the matrix Z_λ has the following properties:—

- $\displaystyle\sum_{j=1}^{n} (Z_\lambda)_{i,j} = s_i$;
- $\displaystyle\sum_{i=1}^{m} (Z_\lambda)_{i,j} = d_j$;
- $(Z_\lambda)_{i,j} = 0$ whenever $(i,j) \notin K$;
- $\displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} (Z_\lambda)_{i,j} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} (X)_{i,j}$ whenever $\lambda \geq 0$.

Now the matrix Y is a non-zero matrix whose rows and columns all sum to zero. It follows that at least one of its coefficients must be strictly positive. Thus there exists at least one ordered pair (i,j) belonging to the set K for which $y_{i,j} > 0$. Let

$$\lambda_0 = \text{minimum} \left\{ \frac{x_{i,j}}{y_{i,j}} : (i,j) \in K \text{ and } y_{i,j} > 0 \right\}.$$

Then $\lambda_0 > 0$. Moreover if $0 \leq \lambda < \lambda_0$ then $x_{i,j} - \lambda y_{i,j} > 0$ for all $(i,j) \in K$, and if $\lambda > \lambda_0$ then there exists at least one element (i_0, j_0) of K for which $x_{i_0,j_0} - \lambda y_{i_0,j_0} < 0$. It follows that $x_{i,j} - \lambda_0 y_{i,j} \geq 0$ for all $(i,j) \in K$, and $x_{i_0,j_0} - \lambda_0 y_{i_0,j_0} = 0$.

Thus Z_{λ_0} is a feasible solution of the given Transportation Problem whose cost does not exceed that of the given feasible solution X. Moreover Z_{λ_0} has fewer non-zero components than the given feasible solution X.

If Z_{λ_0} is itself a basic feasible solution, then we have found the required basic feasible solution whose cost does not exceed that of the given feasible solution. Otherwise we can iterate the process until we arrive at the required basic feasible solution whose cost does not exceed that of the given feasible solution. $\blacksquare$

The Transportation Problem: Optimal Solutions (continued)

A given instance of the Transportation Problem has only finitely many basic feasible solutions. Indeed there are only finitely many bases for the problem, and any basis is associated with at most one basic feasible solution. Therefore there exists a basic feasible solution whose cost does not exceed the cost of any other basic feasible solution. It then follows from Proposition TP-04 that the cost of this basic feasible solution cannot exceed the cost of any other feasible solution of the given instance of the Transportation Problem. This basic feasible solution is thus a basic optimal solution of the Transportation Problem.