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David R. Wilkins

The Transportation Problem: Supplies, Demands and Costs

The Transportation Problem can be expressed generally in the following form. Some commodity is supplied by m suppliers and is transported from those suppliers to n recipients. The ith supplier can supply at most to s_i units of the commodity, and the jth recipient requires at least d_j units of the commodity. The cost of transporting a unit of the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

The total transport cost is then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

where $x_{i,j}$ denote the number of units of the commodity transported from the ith supplier to the jth recipient.

The Transportation Problem: Objective Function and Constraints

The Transportation Problem can then be presented as follows:

determine $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$

so as minimize $\sum_{i,j} c_{i,j} x_{i,j}$

subject to the constraints

$x_{i,j} \geq 0$ *for all* i *and* j,

$\sum_{j=1}^{n} x_{i,j} \leq s_i$ *and* $\sum_{i=1}^{m} x_{i,j} \geq d_j$, *where*

$s_i \geq 0$ *for all* i, $d_j \geq 0$ *for all* i, *and*

$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$.

The Transportation Problem: Matrix Notation

Let $M_{m,n}(\mathbb{R})$ denote the real vector space consisting of all $m \times n$ matrices with real coefficients.

For each pair of integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$, let $E^{(i,j)}$ denote the $m \times n$ matrix defined whose entry in the ith row and jth column and whose remaining entries are equal to zero.

If X is an $m \times n$ matrix with coefficients $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ then

$$X = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} E^{(i,j)}.$$

The matrices $E^{(i,j)}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ therefore constitute a basis for the real vector space $M_{m,n}(\mathbb{R})$.

The Transportation Problem: Matrix Notation

Given an $m \times n$ matrix X with real coefficients, we denote by $(X)_{i,j}$ the component of X in the ith row and jth column of X. Then

$$X = \sum_{i=1}^{m} \sum_{j=1}^{n} (X)_{i,j} E^{(i,j)}.$$

The Transportation Problem: Matrix Notation (continued)

Given $m \times n$ matrices X and Y, we write $X \leq Y$ (and $Y \geq X$) if and only if $(X)_{i,j} \leq (Y)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Thus if $x_{i,j}$ and $y_{i,j}$ are the components of X and Y respectively in the ith row and jth column, then $X \leq Y$ if and only if $x_{i,j} \leq y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Given any real number t, let us denote by $(t)_{[m,n]}$ the $m \times n$ matrix whose components are all equal to t.

In particular, $X \geq (0)_{[m,n]}$, where $(0)_{[m,n]}$ denotes the zero matrix with m rows and n columns, if and only if all the components of the matrix X are non-negative.

The Transportation Problem: Matrix Notation (continued)

Let $e = (1)_{[n,1]}$, so that e is the $n \times 1$ column vector whose coefficients all have the value 1. Then

$$\begin{aligned} Xe &= \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} x_{1,1} + x_{1,2} + \cdots + x_{1,n} \\ x_{2,1} + x_{2,2} + \cdots + x_{2,n} \\ \vdots \\ x_{m,1} + x_{m,2} + \cdots + x_{m,n} \end{pmatrix}. \end{aligned}$$

The Transportation Problem: Matrix Notation (continued)

Thus $\sum_{j=1}^{n} x_{i,j} \leq s_i$ for $i = 1, 2, \ldots, m$ if and only if $Xe \leq s$, where

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}.$$

Also $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ if and only if $Xe = s$.

Moreover standard properties of matrix multiplication ensure that $Xe = s$ if and only if $e^T X^T = s^T$.

The Transportation Problem: Matrix Notation (continued)

Also let $\overline{e} = (1)_{[m,1]}$ so that $\overline{e}$ is the $m \times 1$ column vector $(1)_{[m,1]}$ whose components are all equal to 1. Then

$$
\begin{aligned}
X^T \overline{e} &= \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{n,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,m} & x_{2,m} & \cdots & x_{n,m} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x_{1,1} + x_{2,1} + \cdots + x_{n,1} \\ x_{1,2} + x_{2,2} + \cdots + x_{n,2} \\ \vdots \\ x_{1,m} + x_{2,m} + \cdots + x_{n,m} \end{pmatrix}.
\end{aligned}
$$

The Transportation Problem: Matrix Notation (continued)

Thus $\sum_{i=1}^{m} x_{i,j} \geq d_j$ for $j = 1, 2, \ldots, n$ if and only if $X^T \overline{e} \geq d$, where

$$d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}.$$

Also $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$ if and only if $X^T \overline{e} = d$.

Moreover standard properties of matrix multiplication ensure that $X^T \overline{e} = d$ if and only if $\overline{e}^T X = d^T$.

The Transportation Problem: Matrix Notation (continued)

We have thus shown that the coefficients $x_{i,j}$ of a matrix X satisfy the conditions

$$\sum_{j=1}^{m} x_{i,j} \leq s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} \geq d_j \quad \text{for } j = 1, 2, \ldots, n.$$

if and only if $Xe \leq s$ and $\overline{e}^T X \geq d^T$, where s is the column vector with components $s_1, s_2, \ldots, s_m$ and d is the column vector with components $d_1, d_2, \ldots, d_n$.

The Transportation Problem: Matrix Notation (continued)

Let $s = (s_1, s_2, \ldots, s_m)$ and $d = (d_1, d_2, \ldots, d_n)$ where the quantities $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$ are non-negative real numbers. Suppose that there exists some $m \times n$ matrix X whose coefficients are non-negative real numbers which satisfies the constraints $Xe \leq s$ and $\overline{e}^T X \geq d^T$.Then the components of the column matrix $s - Xe$ are non-negative. It follows that $\overline{e}^T(s - Xe) \geq 0$. Moreover $\overline{e}^T(s - Xe) = 0$ if and only if $s = Xe$.

Similarly the components of the row matrix $\overline{e}^T X - d^T$ are non-negative. It follows that $(\overline{e}^T X - d^T)e \geq 0$. Moreover $(\overline{e}^T X - d^T)e = 0$ if and only if $\overline{e}^T X = d^T$.

It follows from these results that

$$\overline{e}^T s - d^T e = \overline{e}^T (s - Xe) + (\overline{e}^T X - d^T)e \geq 0,$$

and thus $\overline{e}^T s \geq d^T e$.

Moreover $\overline{e}^T s = d^T e$ if and only if both $Xe = s$ and $\overline{e}^T X = d^T$.

The Transportation Problem: Matrix Notation (continued)

The inequality $\overline{e}^T s \geq d^T e$ encodes the observation that if there is to be a feasible solution to the Transportation Problem then total supply must equal or exceed total demand.

Moreover $e^T X e$ represents the total amount of the commodity that is transported from suppliers to recipients. Thus the result that if $\overline{e}^T s = d^T e$ then $Xe = s$ and $\overline{e}^T X = d^T$ encodes the observation that if total supply equals total demand then the ith supplier must provide its full supply s_i of the commodity to recipients, and jth recipient will not receive more than the demand d_j.

The Transportation Problem: Matrix Notation (continued)

Transport costs determine an $m \times n$ matrix C with component $c_{i,j}$ in the ith row and jth column, where $c_{i,j}$ is the cost of transporting the commodity from the ith supplier to the jth recipient. Now the matrix product $C^T X$ has component $\sum_{i=1}^{m} c_{i,k} x_{i,j}$ in the kth row and jth column. The trace of a matrix is the sum of the components of that matrix along the leading diagonal. Therefore

$$\mathrm{trace}(C^T X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j},$$

and thus the value of $\mathrm{trace}(C^T X)$ represents the total cost of transporting the commodity from the suppliers to the recipients.

The Transportation Problem can then be presented as follows:

Let $s \in \mathbb{R}^m$, $d \in \mathbb{R}^n$, $C \in M_{m,n}(\mathbb{R})$, $e = (1)_{[n,1]}$,

$\overline{e} = (1)_{[m,1]}$, $s \geq (0)_{[m,1]}$, $d \geq (0)_{[n,1]}$, *and* $\overline{e}^T s \geq d^T e$.

Determine an $m \times n$ *matrix* X *with real coefficients*

so as minimize $\mathrm{trace}(C^T X)$ *subject to the constraints*

$X \geq (0)_{[m,n]}$,

$Xe \leq s$ *and*

$\overline{e}^T X \geq d^T$.

The Transportation Problem: Equality of Supply and Demand

We now restrict attention to the special case of the Transportation Problem in which total supply equals total demand. It then follows that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

and that

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n.$$

The Transportation Problem: Objective Function and Constraints

The Transportation Problem in the case where total supply equals total demand can be presented as follows:

determine $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$

so as minimize $\sum_{i,j} c_{i,j} x_{i,j}$

subject to the constraints

$x_{i,j} \geq 0$ *for all* i *and* j,

$\sum_{j=1}^{n} x_{i,j} = s_i$ *and* $\sum_{i=1}^{m} x_{i,j} = d_j$, *where*

$s_i \geq 0$ *and* $d_j \geq 0$ *for all* i *and* j, *and*

$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.

The Transportation Problem: The Problem in Matrix Notation

The Transportation Problem in the case where total supply equals total demand can also be presented in matrix notation as follows:

Let $s \in \mathbb{R}^m$, $d \in \mathbb{R}^n$, $C \in M_{m,n}(\mathbb{R})$, $e = (1)_{[n,1]}$,

$\overline{e} = (1)_{[m,1]}$, $s \geq (0)_{[m,1]}$, $d \geq (0)_{[n,1]}$ *and* $\overline{e}^T s = d^T e$.

Determine an $m \times n$ *matrix* X *with real coefficients*

so as minimize $\mathrm{trace}(C^T X)$ *subject to the constraints*

$X \geq (0)_{[m,n]}$,

$Xe = s$ *and*

$\overline{e}^T X = d^T$.