MA3484 Methods of Mathematical Economics
School of Mathematics, Trinity College
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Linear Programming

Module MA3484 should in particular discuss the following topics:—

- standard examples of linear programming problems, including the *transportation problem*;
- algorithms for solving linear programming problems based on the Simplex Method developed by George Dantzig;
- duality theorems in linear programming;
- relationships between linear programming and the theory of zero-sum two-person games;
- Leontief models in mathematical economics.

Fixed Point Theorems and General Equilibrium

Module MA3484 will also include the following topics:—

- the Brouwer Fixed Point Theorem;
- applications of the Brouwer Fixed Point Theorem to prove the existence of equilibrium prices matching supply to demand in simple economics models;
  pause standard examples of linear programming problems,
- the Katukani Fixed Point Theorem;
- application of the Katukani Fixed Point Theorem to prove the existence of Nash equilibria in games.

Mathematical Background: Linear Algebra

The following concepts and topics from Linear Algebra in particular are essential prerequisites for the discussion of linear programming:—

- definitions and basic properties of *real vector spaces*;
- *linear dependence* and *independence*;
- *bases* and *dimensions* of real vector spaces;
- the correspondence between linear transformations and matrices;
- the concepts of *rank* and *nullity* for linear transformations and matrices, and results that relate the dimensions of vector spaces to the rank and nullity of linear transformations between them.

Mathematical Background: Real Analysis

The exploration of duality in the theory of linear programming problems will require both results of linear algebra (concerning dual vector spaces) and also some theorems of real analysis.

The discussion of *duality* in the linear programming context will be based on results from *convexity theory*.

This will require understanding of basic definitions and results concerning convergence of sequences of points in n-dimensional Euclidean space $\mathbb{R}^n$.

In particular, proof of some results will require the *Bolzano-Weierstrass Theorem* in n dimensions: every bounded sequence of points in $\mathbb{R}^n$ has a convergent subsequence.

A proof of the Brouwer Fixed Point Theorem that may be discussed will require understanding of open and closed sets in Euclidean spaces, and of continuity of functions between subsets of Euclidean spaces, and will make use of the Lebesgue Lemma, as it applies to closed bounded subsets of Euclidean spaces.

Lebesgue Lemma (for closed bounded sets in $\mathbb{R}^n$)

Let X be a closed bounded set in $\mathbb{R}^n$. Then, given any open cover of X, there exists some positive real number δ with the following property: given any subset A of X whose diameter is less than δ, there exists some open set U belonging to the open cover such that $A \subset U$.

A Furniture Retailing Problem

A retail business is planning to devote a number of retail outlets to the sale of armchairs and sofas.
The retail prices of armchairs and sofas are determined by fierce competition in the furniture retailing business. Armchairs sell for €700 and sofas sell for €1000.
However

- the amount of floor space (and warehouse space) available for stocking the sofas and armchairs is limited;
- the amount of capital available for purchasing the initial stock of sofas and armchairs is limited;
- market research shows that the ratio of armchairs to sofas in stores should neither be too low nor too high.

Furniture Retailing: the Constraints

Specifically:

- there are 1000 square metres of floor space available for stocking the initial purchase of sofas and armchairs;
- each armchair takes up 1 square metre;
- each sofa takes up 2 square metres;
- the amount of capital available for purchasing the initial stock of armchairs and sofas is €351,000;
- the wholesale price of an armchair is €400;
- the wholesale price of a sofa is €600;
- market research shows that between 4 and 9 armchairs should be in stock for each 3 sofas in stock.

Furniture Retailing: modelling the Constraints

We suppose that the retail outlets are stocked with x armchairs and y sofas.
The armchairs (taking up 1 sq. metre each) and the sofas (taking up 2 sq. metres each) cannot altogether take up more than 1000 sq. metres of floor space. Therefore

$$x + 2y \leq 1000 \quad \text{(Floor space constraint).}$$

The cost of stocking the retail outlets with armchairs (costing €400 each) and sofas (costing €600 each) cannot exceed the available capital of €351000. Therefore

$$4x + 6y \leq 3510 \quad \text{(Capital constraint).}$$

Consumer research indicates that x and y should satisfy

$$4y \leq 3x \leq 9y \quad \text{(Armchair/Sofa ratio).}$$

Furniture Retailing: specifying the Feasible Region

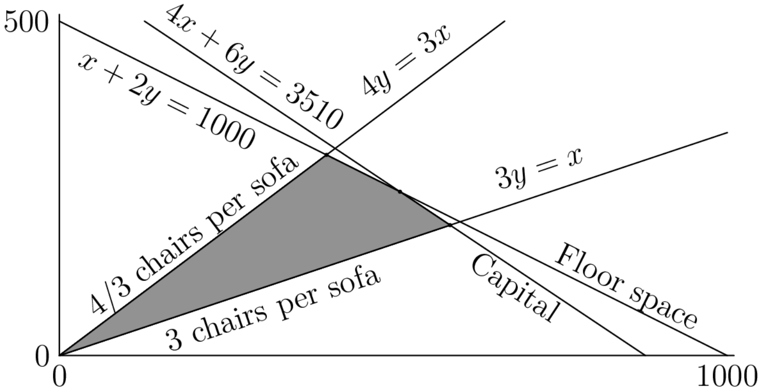
An ordered pair (x, y) of real numbers is said to specify a *feasible solution* to the linear programming problem if this pair of values meets all the relevant constraints.

An ordered pair (x, y) constitutes a feasible solution to the the Furniture Retailing problem if and only if all the following constraints are satisfied:

$$
\begin{aligned}
x - 3y &\leq 0;\\
4y - 3x &\leq 0;\\
x + 2y &\leq 1000;\\
4x + 6y &\leq 3510;\\
x &\geq 0;\\
y &\geq 0;
\end{aligned}
$$

Furniture Retailing: picturing the Feasible Region

The feasible region for the Furniture Retailing problem is depicted below:



Furniture Retailing: Gross Margin per Unit

We identify the *vertices* (or *corners*) of the feasible region for the Furniture Retailing problem. There are four of these:

- there is a vertex at $(0, 0)$;
- there is a vertex at $(400, 300)$ where the line $4y = 3x$ intersects the line $x + 2y = 1000$;
- there is a vertex at $(510, 245)$ where the line $x + 2y = 1000$ intersects the line $4x + 6y = 3510$;
- there is a vertex at $(585, 195)$ where the line $3y = x$ intersects the line $4x + 6y = 3510$.

These vertices are identified by inspection of the graph that depicts the constraints that determine the feasible region.

Furniture Retailing: Gross Margin per Unit

The furniture retail business obviously wants to confirm that the business will make a profit, and will wish to determine how many armchairs and sofas to purchase from the wholesaler to maximize expected profit.

There are fixed costs for wages, rental etc., and we assume that these are independent of the number of armchairs and sofas sold.

The *gross margin* on the sale of an armchair or sofa is the difference between the wholesale and retail prices of that item of furniture.

Armchairs cost €400 wholesale and sell for €700, and thus provide a gross margin of €300.

Sofas cost €600 wholesale and sell for €1000, and thus provide a gross margin of €400.

Furniture Retailing: Gross Profit

In a typical linear programming problem, one wishes to determine not merely *feasible* solutions to the problem. One wishes to determine an *optimal* solution that maximizes some *objective function* amongst all feasible solutions to the problem.
The objective function for the Furniture Retailing problem is the gross profit that would accrue from selling the furniture in stock.This gross profit is the difference between the cost of purchasing the furniture from the wholesaler and the return from selling that furniture.
This objective function is thus $f(x, y)$, where

$$f(x, y) = 300x + 400y.$$

We should determine the maximum value of this function on the feasible region.

Furniture Retailing: Determining the Optimal Solution

Because the objective function $f(x, y) = 300x + 400y$ is linear in x and y, its maximum value on the feasable region must be achieved at one of the vertices of the region.
Clearly this function is not maximized at the origin $(0, 0)$!
Now the remaining vertices of the feasible region are at $(400, 300)$, $(510, 245)$ and $(585, 195)$,and

$$\begin{aligned} f(400, 300) &= 240, 000, \\ f(510, 245) &= 251, 000, \\ f(585, 195) &= 253, 500. \end{aligned}$$

It follows that the objective function is maximized at $(585, 195)$.
The furniture retail business should therefore use up the available capital, stocking 3 armchairs for every sofa, despite the fact that this will not utilize the full amount of floor space available.

r Programming Problems in Von Neumann Maximizing

A linear programming problem may be presented in *Von Neumann maximizing form* as follows:

given real numbers c_i, $A_{i,j}$ *and* b_j *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$,

find real numbers $x_1, x_2, \ldots, x_n$ *so as to*

maximize $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

subject to constraints

$x_j \geq 0$ *for* $j = 1, 2, \ldots, n$, *and*

$A_{i,1} x_1 + A_{i,2} x_2 + \cdots + A_{i,n} x_n \leq b_i$ *for* $i = 1, 2, \ldots, m$.

Furniture Retailing Problem in Von Neumann Maximizing Form

The furniture retailing problem may be presented in Von Neumann maximizing form with $n = 2$, $m = 4$,

$$(c_1, c_2) = (300, 400),$$

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 4 \\ 1 & 2 \\ 4 & 6 \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1000 \\ 3510 \end{pmatrix}.$$

Here A represents the $m \times n$ whose coefficient in the ith row and jth column is $A_{i,j}$.

Matrix Notation Conventions

Linear programming problems may be presented in matrix form. We adopt the following notational conventions with regard to transposes, row and column vectors and vector inequalities:—

- vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ are represented as column vectors;
- we denote by M^{T} the $n \times m$ matrix that is the transpose of an $m \times n$ matrix M;
- in particular, given $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, where b and c are represented as column vectors, we denote by b^{T} and c^{T} the corresponding row vectors obtained on transposing the column vectors representing b and c;
- given vectors u and v in $\mathbb{R}^n$ for some positive integer n, we write $u \leq v$ (and $v \geq u$) if and only if $u_j \leq v_j$ for $j = 1, 2, \ldots, n$.

Matrix Notation for Linear Programming Problems

A linear programming problem in Von Neumann maximizing form may be presented in matrix notation as follows:—

Given an $m \times n$ *matrix* A *with real coefficients,*

and given column vectors $b \in \mathbb{R}^m$ *and* $c \in \mathbb{R}^n$*,*

find $x \in \mathbb{R}^n$ *so as to*

maximize $c^{\mathrm{T}} x$

subject to constraints $Ax \leq b$ *and* $x \geq 0$*.*