Module MA3484: Sample Paper
Worked solutions

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April 11, 2017

Module Website

The module website, with online lecture notes, problem sets. etc. are located at

`http://www.maths.tcd.ie/~dwilkins/Courses/MA3989/`

1. [Note: the following worked solution reproduces the detail of a large number of small steps to explain, demonstrate and validate the reasoning. A solution with this amount of detail would not be expected in a solution written out during the course of an examination.

   Moreover, in writing out a worked solution, it became apparent that, in this example, the process for passing from an initial basic feasible solution to an optimal solution required three changes of basis, which is probably a large number for a problem of this size, and as a result the time taken to complete the problem would exceed what would one would normally expect for problems of this size.]

   We now find a basic optimal solution to a transportation problem with 3 suppliers and 4 recipients. We find an initial basic feasible solution using the Minimum Cost Method, and then continue to find a basic optimal solution using a form of the Simplex Method adapted to the Transportation Problem.

   The supply vector is $(7, 10, 13)$ and the demand vector is $(5, 10, 9, 6)$. The components of both the supply vector and the demand vector add up to 30.

   The costs are as specified in the following cost matrix:—

$$\begin{pmatrix} 6 & 8 & 9 & 6 \\ 5 & 10 & 3 & 7 \\ 3 & 9 & 2 & 4 \end{pmatrix}.$$

   We fill in the row sums (or supplies), the column sums (or demands) and the costs $c_{i,j}$ for the given problem. The resultant tableau looks as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	s_i
1	6 ?	8 ?	9 ?	6 ?	7
2	5 ?	10 ?	3 ?	7 ?	10
3	3 ?	9 ?	2 ?	4 ?	13
d_j	5	10	9	6	30

We find an initial basic feasible solution using the Minimum Cost Method, in which we select a cell with minimum cost for which $x_{i,j}$ is as yet undetermined, then fill in the cell with the minimum of the supply s_i and demand d_j. Then either one column, or one row is completed with zeros. This reduces to a transportation-type problem of smaller size, and the Minimum Cost Method is applied recursively until the initial basic feasible solution has been found.

To start, the minimum cost cell is in position $(3,3)$, and the minimum of supply and demand is 9. We therefore fill in the third column follows:

$$x_{1,3} = 0, \quad x_{2,3} = 0, \quad x_{3,3} = 9.$$

(We also add a $\bullet$ symbol to cell $(3,3)$ to indicate that this cell represents an element of the initial basis.)

Then cell $(3,1)$ becomes the cell of lowest cost with undetermined $x_{i,j}$. Minimum of residual supply and demand for this cell is 4 (because we require that $x_{3,1} + x_{3,2} + 9 + x_{3,4} = 13$), and we complete the third row as follows:

$$x_{3,1} = 4, \quad x_{3,2} = 0, \quad x_{3,4} = 0.$$

(We also add a $\bullet$ symbol to cell $(3,1)$ to indicate that this cell represents an element of the initial basis.)

Then cell $(2,1)$ becomes the cell of lowest cost with undetermined $x_{i,j}$. Minimum of residual supply and demand for this cell is 1 (because we require that $x_{1,1} + x_{2,1} + 4 = 5$), and we complete the first column by setting $x_{1,1} = 0$ and $x_{2,1} = 1$.

The lowest cost cell with undetermined $x_{i,j}$ is then $(1,4)$. We set $x_{1,4} = 6$, and complete the fourth column by setting $x_{2,4} = 0$.

The lowest cost cell with undetermined $x_{i,j}$ is then $(1,2)$. We set $x_{1,2} = 1$.

We complete the process by setting $x_{2,2} = 9$.

The completed tableau after determining the initial basic solution by the Minimum Cost Method is as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	s_i
1	6	8 $\bullet$	9	6 $\bullet$	
	0	1	0	6	7
2	5 $\bullet$	10 $\bullet$	3	7	
	1	9	0	0	10
3	3 $\bullet$	9	2 $\bullet$	4	
	4	0	9	0	13
d_j	5	10	9	6	30

Our initial basic feasible solution is thus specified by the 4×3 matrix X, where

$$X = \begin{pmatrix} 0 & 1 & 0 & 6 \\ 1 & 9 & 0 & 0 \\ 4 & 0 & 9 & 0 \end{pmatrix}.$$

The initial basis B for the transportation problem is as follows:—

$$B = \{(1,2), (1,4), (2,1), (2,2), (3,1), (3,3)\}.$$

The basis has six elements as expected. (The number of basis elements should be $m + n - 1$, where m is the number of suppliers and n is the number of recipients.)

The coefficient of this matrix X in the ith row and jth column is the quantity $x_{i,j}$ that determines the quantity of the commodity to transport from the ith supplier to the jth recipient. Note that $x_{i,j} = 0$ when $(i,j) \notin B$. This corresponds to the requirement that $(x_{i,j})$ be a basic feasible solution determined by the basis B. Also $x_{i,j} > 0$ for all $(i,j) \in B$.

The cost of this initial feasible basic solution is

$$\begin{aligned} & 8 \times 1 + 6 \times 6 + 5 \times 1 + 10 \times 9 \\ & \quad + 3 \times 4 + 2 \times 9 \\ = \; & 8 + 36 + 5 + 90 + 12 + 18 \\ = \; & 169. \end{aligned}$$

We next determine whether the initial basic feasible solution found by the Minimum Cost Method is an optimal solution, and, if not, how to adjust the basis go obtain a solution of lower cost.

We determine u_1, u_2, u_3 and v_1, v_2, v_3, v_4 such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$, where B is the initial basis. We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	u_i
1	6 ?	8 • 0	9 ?	6 • 0	0
2	5 • 0	10 • 0	3 ?	7 ?	?
3	3 • 0	9 ?	2 • 0	4 ?	?
v_j	?	?	?	?	

We first calculate the quantities u_i and v_j.

Now $(1, 2) \in B$, $u_1 = 0$ and $c_{1,2} = 8$ force $v_2 = 8$.

Similarly $(1, 4) \in B$, $u_1 = 0$ and $c_{1,4} = 6$ force $v_4 = 6$.

Then $(2, 2) \in B$, $v_2 = 8$ and $c_{2,2} = 10$ force $u_2 = -2$.

Then $(2, 1) \in B$, $u_2 = -2$ and $c_{2,1} = 5$ force $v_1 = 3$.

Then $(3, 1) \in B$, $v_1 = 3$ and $c_{3,1} = 3$ force $u_3 = 0$.

Finally $(3, 3) \in B$, $u_3 = 0$ and $c_{3,3} = 2$ force $v_3 = 2$.

The tableau after the calculation of the u_i and v_j is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	u_i
1	6 ?	8 • 0	9 ?	6 • 0	0
2	5 • 0	10 • 0	3 ?	7 ?	−2
3	3 • 0	9 ?	2 • 0	4 ?	0
v_j	3	8	2	6	

We next determine $q_{i,q}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j. The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	u_i
1	6	8 •	9	6 •	0
	3	0	7	0	
2	5 •	10 •	3	7	-2
	0	0	-1	-1	
3	3 •	9	2 •	4	0
	0	1	0	-2	
v_j	3	8	2	6	

The initial basic feasible solution is not optimal because some of the quantities $q_{i,j}$ are negative. Indeed $q_{3,4} = -2$, We therefore seek to bring $(3,4)$ into the basis.

The procedure for achieving this requires us to determine a 3×4 matrix Y satisfying the following conditions:—

- $y_{3,4} = 1$;
- $y_{i,j} = 0$ when $(i,j) \notin B \cup \{(3,4)\}$;
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1	2	3	4	
1		? •		? •	0
2	? •	? •			0
3	? •		? •	1 ◦	0
	0	0	0	0	0

The fourth column sums to zero, and therefore $y_{1,4} = -1$.

Then the first row sums to zero, and therefore $y_{1,2} = 1$.

Then the second column sums to zero, and therefore $y_{2,2} = -1$.

Then the second row sums to zero, and therefore $y_{2,1} = 1$.

Then the first column sums to zero, and therefore $y_{3,1} = -1$.

Finally the third row sums to zero, and therefore $y_{3,3} = 0$.

The completed tableau is as follows:—

$y_{i,j}$	1	2	3	4	
1		$1 \quad \bullet$		$-1 \quad \bullet$	0
2	$1 \quad \bullet$	$-1 \quad \bullet$			0
3	$-1 \quad \bullet$		$0 \quad \bullet$	$1 \quad \circ$	0
	0	0	0	0	0

The following 3×4 matrix Y therefore satisfies our requirements:—

$$Y = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Now $X + \lambda Y$ is a feasible solution of the given transportation problem for all values of λ for which the coefficients are all non-negative. Now

$$X + \lambda Y = \begin{pmatrix} 0 & 1+\lambda & 0 & 6-\lambda \\ 1+\lambda & 9-\lambda & 0 & 0 \\ 4-\lambda & 0 & 9 & \lambda \end{pmatrix}.$$

We can increase λ, decreasing the cost by 2λ, up to $\lambda = 4$. This gives us a new basic feasible solution, which we take to be the current basic feasible solution.

Let X now denote the current basic feasible solution, and let B now denote the associated basis. Then

$$X = \begin{pmatrix} 0 & 5 & 0 & 2 \\ 5 & 5 & 0 & 0 \\ 0 & 0 & 9 & 4 \end{pmatrix}.$$

and

$$B = \{(1,2), (1,4), (2,1), (2,2), (3,3), (3,4)\}.$$

The new cost is 161, and one can verify that this is equal to $169 - 2 \times 4$, where 169 is the cost of the initial basic solution.

We next compute the numbers u_i and v_j and $q_{i,j}$ so that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$ and $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	u_i
1	6 ?	8 • 0	9 ?	6 • 0	0
2	5 • 0	10 • 0	3 ?	7 ?	?
3	3 ?	9 ?	2 • 0	4 • 0	?
v_j	?	?	?	?	

The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	u_i
1	6 3	8 • 0	9 5	6 • 0	0
2	5 • 0	10 • 0	3 −3	7 −1	−2
3	3 2	9 3	2 • 0	4 • 0	2
v_j	3	8	4	6	

The current basic feasible solution is not optimal because some of the quantities $q_{i,j}$ are negative. Indeed $q_{2,3} = -3$, We therefore seek to bring $(2,3)$ into the basis.

The procedure for achieving this requires us to determine a 3×4 matrix Y satisfying the following conditions:—

- $y_{2,3} = 1$;
- $y_{i,j} = 0$ when $(i,j) \notin B \cup \{(2,3)\}$;
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1	2	3	4	
1		? •		? •	0
2	? •	? •	1 ∘		0
3			? •	? •	0
	0	0	0	0	0

The third column sums to zero, and therefore $y_{3,3} = -1$.

Then the third row sums to zero, and therefore $y_{3,4} = 1$.

Then the fourth column sums to zero, and therefore $y_{1,4} = -1$.

Then the first row sums to zero, and therefore $y_{1,2} = 1$.

Then the second column sums to zero, and therefore $y_{2,2} = -1$.

Finally the second row sums to zero, and therefore $y_{2,1} = 0$.

The completed tableau is as follows:—

$y_{i,j}$	1	2	3	4	
1		1 •		−1 •	0
2	0 •	−1 •	1 ∘		0
3			−1 •	1 •	0
	0	0	0	0	0

The following 3×4 matrix Y therefore satisfies our requirements:—

$$Y = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Now $X + \lambda Y$ is a feasible solution of the given transportation problem for all values of λ for which the coefficients are all non-negative. Now

$$X + \lambda Y = \begin{pmatrix} 0 & 5+\lambda & 0 & 2-\lambda \\ 5 & 5-\lambda & \lambda & 0 \\ 0 & 0 & 9-\lambda & 4+\lambda \end{pmatrix}.$$

We can increase λ, decreasing the cost by 3λ, up to $\lambda = 2$. This gives us a new basic feasible solution, which we take to be the current basic feasible solution.

Let X now denote the current basic feasible solution, and let B now denote the associated basis. Then

$$X = \begin{pmatrix} 0 & 7 & 0 & 0 \\ 5 & 3 & 2 & 0 \\ 0 & 0 & 7 & 6 \end{pmatrix}.$$

and

$$B = \{(1,2), (2,1), (2,2), (2,3), (3,3), (3,4)\}.$$

The new cost is 155, and one can verify that this is equal to $161 - 3 \times 2$, where 161 is the cost of the previous basic solution.

We next compute the numbers u_i and v_j and $q_{i,j}$ so that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$ and $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1,2,3$ and $j = 1,2,3,4$.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	u_i
1	6 ?	8 • 0	9 ?	6 ?	0
2	5 • 0	10 • 0	3 • 0	7 ?	?
3	3 ?	9 ?	2 • 0	4 • 0	?
v_j	?	?	?	?	

The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	u_i
1	6 3	8 • 0	9 8	6 3	0
2	5 • 0	10 • 0	3 • 0	7 2	-2
3	3 -1	9 0	2 • 0	4 • 0	-1
v_j	3	8	1	3	

The current basic feasible solution is not optimal because one of the quantities $q_{i,j}$ is negative. Indeed $q_{3,1} = -1$, We therefore seek to bring $(3, 1)$ into the basis.

The procedure for achieving this requires us to determine a 3×4 matrix Y satisfying the following conditions:—

- $y_{3,1} = 1$;
- $y_{i,j} = 0$ when $(i, j) \notin B \cup \{(3, 1)\}$;
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1	2	3	4	
1		? •			0
2	? •	? •	? •		0
3	1 ○		? •	? •	0
	0	0	0	0	0

The first column sums to zero, and therefore $y_{2,1} = -1$.

The first row sums to zero and therefore $y_{1,2} = 0$.

Then the second column sums to zero, and therefore $y_{2,2} = 0$.

The fourth column sums to zero, and therefore $y_{3,4} = 0$.

Then the third row sums to zero, and therefore $y_{3,3} = -1$.

Finally the third column sums to zero, and therefore $y_{2,3} = 1$.

The completed tableau is as follows:—

$y_{i,j}$	1	2	3	4	
1		0 •			0
2	-1 •	0 •	1 •		0
3	1 ○		-1 •	0 •	0
	0	0	0	0	0

The following 3×4 matrix Y therefore satisfies our requirements:—

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Now $X + \lambda Y$ is a feasible solution of the given transportation problem for all values of λ for which the coefficients are all non-negative. Now

$$X + \lambda Y = \begin{pmatrix} 0 & 7 & 0 & 0 \\ 5 - \lambda & 3 & 2 + \lambda & 0 \\ \lambda & 0 & 7 - \lambda & 6 \end{pmatrix}.$$

We can increase λ, decreasing the cost by λ, up to $\lambda = 5$. This gives us a new basic feasible solution, which we take to be the current basic feasible solution.

Let X now denote the current basic feasible solution, and let B now denote the associated basis. Then

$$X = \begin{pmatrix} 0 & 7 & 0 & 0 \\ 0 & 3 & 7 & 0 \\ 5 & 0 & 2 & 6 \end{pmatrix}.$$

and

$$B = \{(1,2), (2,2), (2,3), (3,1), (3,3), (3,4)\}.$$

The new cost is 150, and one can verify that this is equal to $155 - 1 \times 5$, where 155 is the cost of the previous basic solution.

We next compute the numbers u_i and v_j and $q_{i,j}$ so that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$ and $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	u_i
1	6	8 $\bullet$	9	6	0
	?	0	?	?	
2	5	10 $\bullet$	3 $\bullet$	7	?
	?	0	0	?	
3	3 $\bullet$	9	2 $\bullet$	4 $\bullet$	?
	0	?	0	0	
v_j	?	?	?	?	

The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	u_i
1	6	8 $\bullet$	9	6	0
	4	0	8	3	
2	5	10 $\bullet$	3 $\bullet$	7	-2
	1	0	0	2	
3	3 $\bullet$	9	2 $\bullet$	4 $\bullet$	-1
	0	0	0	0	
v_j	2	8	1	3	

The fact that all $q_{i,j}$ are non-negative ensures that the current feasible solution is optimal.

Indeed let $\overline{x}_{i,j}$ be the components of a feasible solution to the problem. Then

$$\sum_{i=1}^{3}\sum_{j=1}^{4} c_{i,j}\overline{x}_{i,j} = \sum_{j=1}^{4} v_j d_j - \sum_{i=1}^{3} u_i s_i + \sum_{i=1}^{3}\sum_{j=1}^{4} q_{i,j}\overline{x}_{i,j}.$$

The last summand is always non-negative, and is equal to zero for the current feasible solution.

[Note that $q_{3,2} = 0$ despite the fact that $(3,2) \notin B$. A consequence of this is that the basic optimal solution to this particular problem is not unique. The following matrix

$$X = \begin{pmatrix} 0 & 7 & 0 & 0 \\ 0 & 1 & 9 & 0 \\ 5 & 2 & 0 & 6 \end{pmatrix}.$$

provides an alternative basic optimal solution to the problem.]

2. (a) [Definition.] Let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\},$$

and, for each $(i,j) \in I \times J$, let $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$, where $\overline{\mathbf{b}}^{(i)} \in \mathbb{R}^m$ and $\mathbf{b}^{(j)} \in \mathbb{R}^n$ are defined so that the ith component of $\overline{\mathbf{b}}^{(i)}$ and that jth component of $\mathbf{b}^{(j)}$ are equal to 1 and the other components of these vectors are zero. A subset B of $I \times J$ is said to be a *basis* for the Transportation Problem with m suppliers and n recipients if and only if the elements $\beta^{(i,j)}$ for which $(i,j) \in B$ constitute a basis of the real vector space W.

(b) [Bookwork.] Let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\},$$

let $\overline{\mathbf{b}}^{(1)}, \overline{\mathbf{b}}^{(2)}, \ldots, \overline{\mathbf{b}}^{(m)}$ be the standard basis of $\mathbb{R}^m$ and let $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(n)}$ be the standard basis of $\mathbb{R}^n$, where the ith component of $\overline{\mathbf{b}}^{(i)}$ and the jth component of $\mathbf{b}^{(j)}$ are equal to 1 and the other components of these vectors are zero, and let $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$ for all $(i,j) \in I \times J$.

Now the elements $\beta^{(i,j)}$ for $(i,j) \in I \times J$ span the vector space W. It therefore follows from a standard result of linear algebra that if the elements $\beta^{(i,j)}$ for which $(i,j) \in K$ were linearly independent then there would exist a subset B of $I \times J$ satisfying $K \subset B$ such that the elements $\beta^{(i,j)}$ for which $(i,j) \in B$ would constitute a basis of W. This subset B of $I \times J$ would then be a basis for the Transportation Problem. But the subset K is not contained in any basis for the Transportation Problem. It follows that the elements $\beta^{(i,j)}$ for which $(i,j) \in K$ must be linearly dependent. Therefore there exists a non-zero $m \times n$ matrix Y with real coefficients such that $(Y)_{i,j} = 0$ when $(i,j) \notin K$ and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \beta^{(i,j)} = \mathbf{0}_W.$$

Now $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$ for all $i \in I$ and $j \in J$. It follows that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \overline{\mathbf{b}}^{(i)} = \mathbf{0}$$

and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \mathbf{b}^{(j)} = \mathbf{0},$$

and therefore

$$\sum_{j=1}^{n} (Y)_{i,j} = 0 \quad (i = 1, 2, \ldots, m)$$

and

$$\sum_{i=1}^{m} (Y)_{i,j} = 0 \quad (j = 1, 2, \ldots, n),$$

as required.

3. [Seen similar.]

Note: *There are several ways of organizing the calculation using tableaux. Any method that arrives at and verifies the optimal solution is acceptable.*

The problem is to minimize $\mathbf{c}^T\mathbf{x}$ subject to constraints $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{pmatrix} 9 & 3 & 5 & 2 & 1 \\ 2 & 7 & 3 & 4 & 3 \\ 4 & 2 & 3 & 6 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 14 \\ 26 \\ 13 \end{pmatrix}$$

and

$$\mathbf{c}^T = \begin{pmatrix} 3 & 2 & 5 & 9 & 4 \end{pmatrix}, \quad \mathbf{x}^T = \begin{pmatrix} x_1 & x_2 & x_3 & x_9 & x_5 \end{pmatrix}.$$

We denote by $\mathbf{a}^{(j)}$ the 3-dimensional vector specified by the jth column of the matrix A.

We have an initial solution $\mathbf{x} = (0, 2, 1, 0, 3)$ with initial basis $B = \{2, 3, 5\}$ and initial cost 21. We find $\mathbf{p} \in \mathbb{R}^3$ to satisfy the matrix equation

$$\begin{pmatrix} 2 & 5 & 4 \end{pmatrix} = (c_2, c_3, c_5) = \mathbf{c}_B^T = \mathbf{p}^T M_B,$$

where

$$M_B = \begin{pmatrix} 3 & 5 & 1 \\ 7 & 3 & 3 \\ 2 & 3 & 2 \end{pmatrix}.$$

Now

$$\begin{aligned} \det M_B &= 3 \times (3 \times 2 - 3 \times 3) + 5 \times (2 \times 3 - 7 \times 2) + 1 \times (7 \times 3 - 2 \times 3) \\ &= 3 \times (-3) + 5 \times (-8) + 15 = -34. \end{aligned}$$

and

$$M_B^{-1} = \frac{-1}{34} \begin{pmatrix} -3 & -7 & 12 \\ -8 & 4 & -2 \\ 15 & 1 & -26 \end{pmatrix},$$

and thus

$$\begin{aligned}\mathbf{p}^T &= \begin{pmatrix} c_2 & c_3 & c_5 \end{pmatrix} M_B^{-1} = \frac{-1}{34} \begin{pmatrix} 2 & 5 & 4 \end{pmatrix} \begin{pmatrix} -3 & -7 & 12 \\ -8 & 4 & -2 \\ 15 & 1 & -26 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{14}{34} & -\frac{10}{34} & \frac{90}{34} \end{pmatrix}.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{c}^T - \mathbf{p}^T A &= \begin{pmatrix} 3 & 2 & 5 & 9 & 4 \end{pmatrix} - \begin{pmatrix} -\frac{14}{34} & -\frac{10}{34} & \frac{90}{34} \end{pmatrix} \begin{pmatrix} 9 & 3 & 5 & 2 & 1 \\ 2 & 7 & 3 & 4 & 3 \\ 4 & 2 & 3 & 6 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 & 5 & 9 & 4 \end{pmatrix} - \begin{pmatrix} \frac{214}{34} & 2 & 5 & \frac{472}{34} & 4 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{112}{34} & 0 & 0 & -\frac{166}{34} & 0 \end{pmatrix}\end{aligned}$$

Let $\overline{\mathbf{x}}$ be a feasible solution, where Let $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5)$, Then $A\overline{\overline{x}} = \mathbf{b}$ and $\overline{x}_j \geq 0$ for $j = 1, 2, 3, 4, 5$. Then

$$\begin{aligned}\mathbf{c}^T\overline{\mathbf{x}} &= \mathbf{p}^T A\overline{\mathbf{x}} + \mathbf{q}^T\overline{\mathbf{x}} = \mathbf{p}^T\mathbf{b} + \mathbf{q}^T\overline{\mathbf{x}} = 21 + \mathbf{q}^T\overline{\mathbf{x}} \\ &= 21 - \frac{112}{34}\overline{x}_1 - \frac{166}{34}\overline{x}_4,\end{aligned}$$

where $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$. We look for a basis that includes 4. Now

$$\mathbf{a}^{(4)} = t_{2,4}\mathbf{a}^{(2)} + t_{3,4}\mathbf{a}^{(3)} + t_{5,4}\mathbf{a}^{(5)} = M_B \begin{pmatrix} t_{2,4} \\ t_{3,4} \\ t_{5,4} \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} t_{2,4} \\ t_{3,4} \\ t_{5,4} \end{pmatrix} = M_B^{-1}\mathbf{a}^{(4)} = \frac{-1}{34} \begin{pmatrix} -3 & -7 & 12 \\ -8 & 4 & -2 \\ 15 & 1 & -26 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -\frac{38}{34} \\ \frac{12}{34} \\ \frac{122}{34} \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 0 & 2 + \frac{38}{34}\lambda & 1 - \frac{12}{34}\lambda & \lambda & 3 - \frac{122}{34}\lambda \end{pmatrix}$$

is a feasible solution of the problem whenever all components are non-negative. We obtain another basic feasible solution with lower cost on determining λ to be the largest non-negative real number satisfying

$$1 - \frac{12}{34}\lambda \geq 0 \quad \text{and} \quad 3 - \frac{122}{34}\lambda \geq 0.$$

Now $\frac{122}{34}$ lies between 3 and 4. We should therefore take $\lambda = 3 \times \frac{34}{122} = \frac{102}{122}$, and the new basic feasible solution is

$$\left(\begin{array}{ccccc} 0 & \frac{358}{122} & \frac{86}{122} & \frac{102}{122} & 0 \end{array} \right)$$

We now let this row vector represent the current basic solution. The current cost is then $\frac{2064}{122}$ and the current basis B is given by $B = \{2, 3, 4\}$. Now let M_B now consist of the 2nd and 3rd and 4th columns of the matrix A. We find that

$$M_B = \begin{pmatrix} 3 & 5 & 2 \\ 7 & 3 & 4 \\ 2 & 3 & 6 \end{pmatrix}, \quad M_B^{-1} = -\frac{1}{122} \begin{pmatrix} 6 & -24 & 14 \\ -34 & 14 & 2 \\ 15 & 1 & -26 \end{pmatrix}.$$

We then let

$$\begin{aligned} \mathbf{p}^T &= \left(\begin{array}{ccc} c_2 & c_3 & c_4 \end{array} \right) M_B^{-1} = -\frac{1}{122} \left(\begin{array}{ccc} 2 & 5 & 9 \end{array} \right) \begin{pmatrix} 6 & -24 & 14 \\ -34 & 14 & 2 \\ 15 & 1 & -26 \end{pmatrix} \\ &= \left(\begin{array}{ccc} \frac{23}{122} & -\frac{31}{122} & \frac{196}{122} \end{array} \right). \end{aligned}$$

Then

$$\begin{aligned} \mathbf{c}^T - \mathbf{p}^T A &= \left(\begin{array}{ccccc} 3 & 2 & 5 & 9 & 4 \end{array} \right) - \left(\begin{array}{ccc} \frac{23}{122} & -\frac{31}{122} & \frac{196}{122} \end{array} \right) \begin{pmatrix} 9 & 3 & 5 & 2 & 1 \\ 2 & 7 & 3 & 4 & 3 \\ 4 & 2 & 3 & 6 & 2 \end{pmatrix} \\ &= \left(\begin{array}{ccccc} 3 & 2 & 5 & 9 & 4 \end{array} \right) - \left(\begin{array}{ccccc} \frac{929}{122} & 2 & 5 & 9 & \frac{322}{122} \end{array} \right) \\ &= \left(\begin{array}{ccccc} -\frac{563}{122} & 0 & 0 & 0 & \frac{166}{122} \end{array} \right) \end{aligned}$$

The components of this vector are not all non-negative, and therefore the current basic solution is not optimal. Because the first component is negative we seek to bring 1 into the basis. Now

$$\mathbf{a}^{(1)} = t_{2,1}\mathbf{a}^{(2)} + t_{3,1}\mathbf{a}^{(3)} + t_{4,1}\mathbf{a}^{(4)} = M_B \begin{pmatrix} t_{2,1} \\ t_{3,1} \\ t_{4,1} \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} t_{2,1} \\ t_{3,1} \\ t_{4,1} \end{pmatrix} = M_B^{-1}\mathbf{a}^{(1)} = \frac{-1}{122}\begin{pmatrix} 6 & -24 & 14 \\ -34 & 14 & 2 \\ 15 & 1 & -26 \end{pmatrix}\begin{pmatrix} 9 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{62}{122} \\ \frac{270}{122} \\ -\frac{33}{122} \end{pmatrix}.$$

It follows that

$$\left(\lambda \quad \tfrac{358}{122} + \tfrac{62}{122}\lambda \quad \tfrac{86}{122} - \tfrac{270}{122}\lambda \quad \tfrac{102}{122} + \tfrac{33}{122}\lambda \quad 0 \right)$$

is a feasible solution of the problem whenever all components are non-negative. We obtain another basic feasible solution with lower cost than the current feasible solution on setting $\lambda = \frac{86}{270}$, and the new current basic feasible solution is

$$\left(\tfrac{86}{270} \quad \tfrac{836}{270} \quad 0 \quad \tfrac{249}{270} \quad 0 \right)$$

We now let this vector represent the current basic feasible solution $\mathbf{x}$. The new basis B is given by $B = \{1, 2, 4\}$. We must test the current basic feasible solution for optimality.

Accordingly we calculate $\mathbf{p}$ such that $\mathbf{p}^T = \mathbf{c}_B^T M_B^{-1}$, where

$$\mathbf{c}_B^T = \left(c_1 \quad c_2 \quad c_4 \right) = \left(3 \quad 2 \quad 9 \right).$$

and

$$M_B = \begin{pmatrix} 9 & 3 & 2 \\ 2 & 7 & 4 \\ 4 & 2 & 6 \end{pmatrix}, \quad M_B^{-1} = \frac{1}{270}\begin{pmatrix} 34 & -14 & -2 \\ 4 & 46 & -32 \\ -24 & -6 & 57 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{p}^T &= \frac{1}{270}\left(3 \quad 2 \quad 9 \right)\begin{pmatrix} 34 & -14 & -2 \\ 4 & 46 & -32 \\ -24 & -6 & 57 \end{pmatrix} \\ &= \left(-\tfrac{106}{270} \quad -\tfrac{4}{270} \quad \tfrac{443}{270} \right). \end{aligned}$$

Then

$$\begin{aligned} \mathbf{c}^T - \mathbf{p}^T A &= \left(3 \quad 2 \quad 5 \quad 9 \quad 4 \right) - \left(-\tfrac{106}{270} \quad -\tfrac{4}{270} \quad \tfrac{443}{270} \right)\begin{pmatrix} 9 & 3 & 5 & 2 & 1 \\ 2 & 7 & 3 & 4 & 3 \\ 4 & 2 & 3 & 6 & 2 \end{pmatrix} \\ &= \left(3 \quad 2 \quad 5 \quad 9 \quad 4 \right) - \left(3 \quad 2 \quad \tfrac{787}{270} \quad 9 \quad \tfrac{768}{270} \right) \\ &= \left(0 \quad 0 \quad \tfrac{563}{270} \quad 0 \quad \tfrac{312}{270} \right) \end{aligned}$$

Because all components of this last row vector are non-negative, the current basic feasible solution is optimal. The cost of this solution is $\frac{4171}{270}$. The cost of any feasible solution $(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5)$ is

$$\frac{4171}{270} + \frac{563}{270}\overline{x}_3 + \frac{312}{270}\overline{x}_5,$$

where $\overline{x}_3 \geq 0$ and $\overline{x}_5 \geq 0$. Thus the current basic feasible solution $(\frac{86}{270}, \frac{836}{270}, 0, \frac{249}{270}, 0)$ is indeed optimal.
Cancelling common factors from numerator and denominator, we find that this basic optimal solution is $(\frac{43}{135}, \frac{418}{135}, 0, \frac{83}{90}, 0)$.

4. (a) [Seen Similar.] The inequalities satisfied by the feasible solutions (x_1, x_2, x_3, x_4) and (p_1, p_2, p_3) of the corresponding linear programming problems ensure that

$$\begin{aligned} c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 - p_1b_1 - p_2b_2 + p_3b_3 \\ &= (c_1 - p_1a_{1,1} - p_2a_{2,1} - p_3a_{3,1})x_1 \\ &\quad + (c_2 - p_1a_{1,2} - p_2a_{2,2} - p_3a_{3,2})x_2 \\ &\quad + (c_3 - p_1a_{1,3} - p_2a_{2,3} - p_3a_{3,3})x_3 \\ &\quad + (c_4 - p_1a_{1,4} - p_2a_{2,4} - p_3a_{3,4})x_4 \\ &\quad + p_1(a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 - b_1) \\ &\quad + p_2(a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 - b_2) \\ &\quad + p_3(a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 - b_3) \\ &= (c_2 - p_1a_{1,2} - p_2a_{2,2} - p_3a_{3,2})x_2 \\ &\quad + (c_4 - p_1a_{1,4} - p_2a_{2,4} - p_3a_{3,4})x_4 \\ &\quad + p_1(a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 - b_1) \\ &\quad + p_3(a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 - b_3) \end{aligned}$$

because

$$\begin{aligned} a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 - b_2 &= 0, \\ a_{4,1}x_1 + a_{4,2}x_2 + a_{4,3}x_3 + a_{4,4}x_4 - b_4 &= 0, \\ c_1 - p_1a_{1,1} - p_2a_{2,1} - p_3a_{3,1} &= 0, \\ c_3 - p_1a_{1,3} - p_2a_{2,3} - p_3a_{3,3} &= 0. \end{aligned}$$

Now the constraints in the relevant programming problems ensure that the summands in the above expression for

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 - p_1b_1 - p_2b_2 - p_3b_3$$

are all non-negative. It follows that

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 - p_1b_1 - p_2b_2 - p_3b_3 \geq 0.$$

Moreover equality holds if and only if all summands are zero, in which case

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3} + a_{1,4} &= b_1 \text{ if } p_1 > 0, \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3} + a_{3,4} &= b_3 \text{ if } p_3 > 0, \\ p_1a_{1,2} + p_2a_{2,2} + p_3a_{3,2} &= c_2 \text{ if } x_2 > 0, \\ p_1a_{1,4} + p_2a_{2,4} + p_3a_{3,4} &= c_4 \text{ if } x_4 > 0, \end{aligned}$$

as required.

(b) [Bookwork.] Let $K = \{i \in I : \eta_i(\mathbf{x}^*) > s_i\}$. Suppose that there do not exist non-negative real numbers g_i for all $i \in I$ such that $\varphi = \sum_{i \in I} g_i \eta_i$ and $g_i = 0$ when $i \in K$. On applying the proposition stated on in the Note on the examination paper (with $L = I \setminus K$), we deduce that there must exist some $\mathbf{v} \in \mathbb{R}^n$ such that $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I \setminus K$ and $\varphi(\mathbf{v}) < 0$. Then

$$\eta_i(\mathbf{x}^* + \lambda \mathbf{v}) = \eta_i(\mathbf{x}^*) + \lambda \eta_i(\mathbf{v}) \geq s_i$$

for all $i \in I \setminus K$ and for all $\lambda \geq 0$. If $i \in K$ then $\eta_i(\mathbf{x}^*) > s_i$. The set K is finite. It follows that there must exist some real number λ_0 satisfying $\lambda_0 > 0$ such that $\eta_i(\mathbf{x}^* + \lambda \mathbf{v}) \geq s_i$ for all $i \in K$ and for all real numbers λ satisfying $0 \leq \lambda \leq \lambda_0$.

Combining the results in the cases when $i \in I \setminus K$ and when $i \in K$, we find that $\eta_i(\mathbf{x}^* + \lambda \mathbf{v}) \geq s_i$ for all $i \in I$ and $\lambda \in [0, \lambda_0]$, and therefore $\mathbf{x}^* + \lambda \mathbf{v} \in X$ for all real numbers λ satisfying $0 \leq \lambda \leq \lambda_0$. But

$$\varphi(\mathbf{x}^* + \lambda \mathbf{v}) = \varphi(\mathbf{x}^*) + \lambda \varphi(\mathbf{v}) < \varphi(\mathbf{x}^*)$$

whenever $\lambda > 0$. It follows that the linear functional φ cannot attain a minimum value in X at any point $\mathbf{x}^*$ for which either $K = I$ or for which K is a proper subset of I but there exist non-negative real numbers g_i for all $i \in I \setminus K$ such that $\varphi = \sum_{i \in I \setminus K} g_i \eta_i$.

The result follows.