TRINITY COLLEGE DUBLIN
THE UNIVERSITY OF DUBLIN

School of Mathematics

JS and SS Mathematics **Trinity Term 2015**
JS and SS TSM Mathematics

MA3484 — Methods of Mathematical Economics

Sample Paper

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Instructions to Candidates:

Credit will be given for the best 3 questions answered.

All questions have equal weight.

Materials Permitted for this Examination:

Formulae and Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

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1. Let $c_{i,j}$ be the coefficient in the ith row and jth column of the cost matrix C, where

$$C = \begin{pmatrix} 6 & 8 & 9 & 6 \\ 5 & 10 & 3 & 7 \\ 3 & 9 & 2 & 4 \end{pmatrix}.$$

and let

$$s_1 = 7, \quad s_2 = 10, \quad s_3 = 13,$$

$$d_1 = 5, \quad d_2 = 10, \quad d_3 = 9, \quad d_4 = 6.$$

Determine non-negative real numbers $x_{i,j}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ that minimize $\sum_{i=1}^{3} \sum_{j=1}^{4} c_{i,j} x_{i,j}$ subject to the following constraints:

$$\sum_{j=1}^{4} x_{i,j} = s_i \quad \text{for} \quad i = 1, 2, 3,$$

$$\sum_{i=1}^{3} x_{i,j} = d_j \quad \text{for} \quad j = 1, 2, 3, 4,$$

and $x_{i,j} \geq 0$ for all i and j.

Also verify that the solution to this problem is indeed optimal.

(20 marks)

2. (a) Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where m and n are positive integers. Define what is meant by saying that a subset B of $I \times J$ is a *basis* for the Transportation Problem with m suppliers and n recipients.

(6 marks)

(b) Let m and n be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let K be a subset of $I \times J$. Suppose that there is no basis B of the Transportation Problem for which $K \subset B$. prove that there exists a non-zero $m \times n$ matrix Y with real coefficients which satisfies the following conditions:

- $\sum_{j=1}^{n} (Y)_{i,j} = 0$ for $i = 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} (Y)_{i,j} = 0$ for $j = 1, 2, \ldots, n$;
- $(Y)_{i,j} = 0$ when $(i, j) \notin K$.

(14 marks)

3. Consider the following linear programming problem:—

find real numbers x_1, x_2, x_3, x_4, x_5 *so as to minimize the objective function*

$$3x_1 + 2x_2 + 5x_3 + 9x_4 + 4x_5$$

subject to the following constraints:

$9x_1 + 3x_2 + 5x_3 + 2x_4 + x_5 = 14$;

$2x_1 + 7x_2 + 3x_3 + 4x_4 + 3x_5 = 26$;

$4x_1 + 2x_2 + 3x_3 + 6x_4 + 2x_5 = 13$;

$x_j \geq 0$ *for* $j = 1, 2, 3, 4, 5$.

A feasible solution to this problem is the following:

$$x_1 = 0, \quad x_2 = 2, \quad x_3 = 1, \quad x_4 = 0, \quad x_5 = 3,$$

(This feasible solution satisfies the constraints but does not necessarily minimize the objective function.) Find an optimal solution to the linear programming problem and verify that it is indeed optimal.

(20 marks)

4. Let $a_{i,j}$, b_i and c_j be real numbers for $i = 1, 2$ and $j = 1, 2, 3$, and let the **Primal Linear Programming Problem** and the **Dual Linear Programming Problem** be specified as set out below:—

   Primal Linear Programming Problem.

   Determine real numbers x_1, x_2, x_3, x_4 *so as to minimize*

   $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$

   subject to the following constraints:

   $a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 \geq b_1$;

   $a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 = b_2$;

   $a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 \geq b_3$;

   $x_2 \geq 0$ *and* $x_4 \geq 0$.

   Dual Linear Programming Problem.

   Determine real numbers p_1, p_2, p_3 *so as to maximize* $p_1b_1 + p_2b_2 + p_3b_3$

   subject to the following constraints:

   $p_1a_{1,1} + p_2a_{2,1} + p_3a_{3,1} = c_1$;

   $p_1a_{1,2} + p_2a_{2,2} + p_3a_{3,2} \leq c_2$;

   $p_1a_{1,3} + p_2a_{2,3} + p_3a_{3,3} = c_3$;

   $p_1a_{1,4} + p_2a_{2,4} + p_3a_{3,4} \leq c_4$;

   $p_1 \geq 0$ *and* $p_3 \geq 0$.

   (a) Prove that if (x_1, x_2, x_3, x_4) is a feasible solution of the **Primal Linear Programming Problem** and (p_1, p_2, p_3) is a feasible solution of the **Dual Linear Programming Problem** then

   $$p_1b_1 + p_2b_2 + p_3b_3 \leq c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4.$$

   Prove also that $p_1b_1 + p_2b_2 + p_3b_3 = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ if and only if the following *Complementary Slackness Conditions* are satisfied:—

Question continues on next page

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 &= b_1 \text{ if } p_1 > 0; \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 &= b_3 \text{ if } p_3 > 0; \\ p_1 a_{1,2} + p_2 a_{2,2} + p_3 a_{3,2} &= c_2 \text{ if } x_2 > 0; \\ p_1 a_{1,4} + p_2 a_{2,4} + p_3 a_{3,4} &= c_4 \text{ if } x_4 > 0. \end{aligned}$$

(8 marks)

(b) Let n be a positive integer, let I be a non-empty finite set, and, for each $i \in I$, let $\eta_i: \mathbb{R}^n \to \mathbb{R}$ be non-zero linear functional and let s_i be a real number. Let X be the convex polytope defined such that

$$X = \bigcap_{i \in I} \{\mathbf{x} \in \mathbb{R} : \eta_i(\mathbf{x}) \geq s_i\}.$$

(Thus a point $\mathbf{x}$ of $\mathbb{R}^n$ belongs to the convex polytope X if and only if $\eta_i(\mathbf{x}) \geq s_i$ for all $i \in I$.) Let $\varphi: \mathbb{R}^n \to \mathbb{R}$ be a non-zero linear functional on $\mathbb{R}^n$, and let $\mathbf{x}^* \in X$. Prove that $\varphi(\mathbf{x}^*) \leq \varphi(\mathbf{x})$ for all $\mathbf{x} \in X$ if and only if there exist non-negative real numbers g_i for all $i \in I$ such that $\varphi = \sum_{i \in I} g_i \eta_i$ and $g_i = 0$ whenever $\eta_i(\mathbf{x}^*) > s_i$.

(12 marks)

Note on Question 4

In answering 4(b), you may use, without proof, the following proposition that is a consequence of Farkas' Lemma.

Let n be a positive integer, let L be a non-empty finite set, let $\varphi: \mathbb{R}^n \to \mathbb{R}$ be a linear functional on $\mathbb{R}^n$, and, for each $i \in L$, let $\eta_i: \mathbb{R}^n \to \mathbb{R}$ be a linear functional on $\mathbb{R}^n$. Suppose that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ with the property that $\eta_i(\mathbf{v}) \geq 0$ for all $i \in L$. Then there exist non-negative real numbers g_i for all $i \in L$ such that $\varphi = \sum_{i \in L} g_i \eta_i$.