Module MA3484: Transportation Problem
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Transportation Problem Examples

D. R. Wilkins

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Contents

1 A Transportation Problem Example with Four Suppliers and Five Recipients

We discuss in detail how to solve a particular example of the Transportation Problem with 4 suppliers and 5 recipients, where the supply and demand vectors and the cost matrix are as follows:—

$$s^T = (9, 11, 4, 5), \quad d^T = (6, 7, 5, 3, 8).$$

$$C = \begin{pmatrix} 2 & 4 & 3 & 7 & 5 \\ 4 & 8 & 5 & 1 & 8 \\ 5 & 9 & 4 & 4 & 2 \\ 7 & 2 & 5 & 5 & 3 \end{pmatrix}.$$

We find a basic feasible solution to the Transportation Problem example with 4 suppliers and 5 recipients, where the supply and demand vectors are as follows:—

$$s^T = (9, 11, 4, 5), \quad d^T = (6, 7, 5, 3, 8).$$

We find an initial basic feasible solution using a method known as the Northwest Corner Method. We then apply an iterative procedure to find a basic optimal solution, using a version of the Simplex Method adapted to the Transportation Problem.

In order to find an initial basic feasible solution, we need to fill in the entries in a tableau of the following form:—

$x_{i,j}$	1	2	3	4	5	s_i
1	·	·	·	·	·	9
2	·	·	·	·	·	11
3	·	·	·	·	·	4
4	·	·	·	·	·	5
d_j	6	7	5	3	8	29

In the tableau just presented the labels on the left hand side identify the suppliers, the labels at the top identify the recipients, the numbers on the right hand side list the number of units that the relevant supplier must provide, and the numbers at the bottom identify the number of units that the relevant recipient must obtain. Number in the bottom right hand corner gives the common value of the total supply and the total demand.

The values in the individual cells must be non-zero, the rows must sum to the value on the right, and the columns must sum to the value on the bottom.

The Northwest Corner Method is applied recursively. At each stage the undetermined cell in at the top left (the northwest corner) is given the maximum possible value allowable with the constraints. The remainder of either the first row or the first column must then be completed with zeros. This leads to a reduced tableau to be determined with either one fewer row or else one fewer column. One continues in this fashion, as exemplified in the solution of this particular problem, until the entire tableau has been completed.

The method also determines a basis associated with the basic feasible solution determined by the Northwest Corner Method. This basis lists the cells that play the role of northwest corner at each stage of the method.

At the first stage, the northwest corner cell is associated with supplier 1 and recipient 1. This cell is assigned a value equal to the mimimum of the corresponding column and row sums. Thus, this example, the northwest corner cell, is given the value 6, which is the desired column sum. The remaining cells in that row are given the value 0.

The tableau then takes the following form:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	·	·	·	·	9
2	0	·	·	·	·	11
3	0	·	·	·	·	4
4	0	·	·	·	·	5
d_j	6	7	5	3	8	29

The ordered pair $(1, 1)$ commences the list of elements making up the associated basis.

At the second stage, one applies the Northwest Corner Method to the following reduced tableau:—

$x_{i,j}$	2	3	4	5	s_i
1	·	·	·	·	3
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	7	5	3	8	23

The required value for the first row sum of the reduced tableau has been reduced to reflect the fact that the values in the remaining undetermined cells of the first row must sum to the value 3.

The value 3 is then assigned to the northwest corner cell of the reduced tableau (as 3 is the maximum possible value for this cell subject to the constraints on row and column sums). The reduced tableau therefore takes the following form after the second stage:—

$x_{i,j}$	2	3	4	5	s_i
1	3	0	0	0	3
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	7	5	3	8	23

The main tableau at the completion of the second stage then stands as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	·	·	·	·	11
3	0	·	·	·	·	4
4	0	·	·	·	·	5
d_j	6	7	5	3	8	29

The list of ordered pairs representing the basis elements determined at the second stage then stands as follows:—

Basis: $(1,1),(1,2),\ldots$.

The reduced tableau for the third stage then stands as follows:—

$x_{i,j}$	2	3	4	5	s_i
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	4	5	3	8	20

Accordingly the northwest corner of the reduced tableau should be assigned the value 4, and the remaining elements of the first column should be assigned the value 0.

The reduced tableau at the completion of the third stage stands as follows:—

$x_{i,j}$	2	3	4	5	s_i
2	4	·	·	·	11
3	0	·	·	·	4
4	0	·	·	·	5
d_j	4	5	3	8	20

The main tableau and list of basis elements at the completion of the third stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	·	·	·	11
3	0	0	·	·	·	4
4	0	0	·	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(1,2),(2,2),\ldots$.

The reduced tableau at the completion of the fourth stage is as follows:—

$x_{i,j}$	3	4	5	s_i
2	5	·	·	7
3	0	·	·	4
4	0	·	·	5
d_j	5	3	8	16

The main tableau and list of basis elements at the completion of the fourth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	·	·	11
3	0	0	0	·	·	4
4	0	0	0	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(1,2),(2,2),(2,3),\ldots$.

At the fifth stage the sum of the undetermined cells for the 2nd supplier must sum to 2. Therefore the main tableau and list of basis elements at the completion of the fifth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	·	·	4
4	0	0	0	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(1,2),(2,2),(2,3),(2,4),\ldots$.

At the sixth stage the sum of the undetermined cells for the 4th recipient must sum to 1. Therefore the main tableau and list of basis elements at the completion of the sixth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	·	4
4	0	0	0	0	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),\ldots$.

Two further stages suffice to complete the tableau. Moreover, at the completion of the eighth and final stage the main tableau and list of basis elements stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	3	4
4	0	0	0	0	5	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)$.

The following table establishes that the cost of the initial basic feasible solution is 108:—

(i,j)	$x_{i,j}$	$c_{i,j}$	$c_{i,j}x_{i,j}$
$(1,1)$	6	2	12
$(1,2)$	3	4	12
$(2,2)$	4	8	32
$(2,3)$	5	5	25
$(2,4)$	2	1	2
$(3,4)$	1	4	4
$(3,5)$	3	2	6
$(4,5)$	5	3	15
Total			108

Now the basic feasible solution produced by applying the Northwest Corner Method is just one amongst many basic feasible solutions. There are many others. Some of these may be obtained on applying the Northwest Corner Method after reordering the rows and columns (thus renumbering the suppliers and recipients).

It would take significant work to calculate all basic feasible solutions and then calculate the cost associated with each one.

However there is a method for passing from one feasible solution to another so as to progressively lower the cost until the feasible solution have been found and verified to be the optimal solution to the problem.

Let B be the basis consisting of the ordered pairs

$$(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5).$$

The following tableau records the costs $c_{i,j}$ associated with those ordered pairs (i,j) that belong to the basis B:

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				u_1
2		8	5	1		u_2
3				4	2	u_3
4					3	u_4
v_j	v_1	v_2	v_3	v_4	v_5	

We determine real numbers u_i for $i = 1,2,3,4$ and v_j for $j = 1,2,3,4,5$ such that $v_j - u_i = c_{i,j}$ for all $(i,j) \in B$. These real numbers u_i and v_j are not required to be non-negative: they may be positive, negative or zero.

(We postpone till later an explanation as to why finding values u_i and v_j satisfying the above equation actually helps us in solving the problem.)

Now if the real numbers u_i and v_j provide a solution to these equations, then another solution is obtained on replacing u_i and v_j by $u_i + k$ and $v_j + k$, where k is some fixed constant. It follows that one of the required values can be set to an arbitrary value. Accordingly we seek a solution with $u_1 = 0$. Then we must have $v_1 = 2$ and $v_2 = 4$ in order to satisfy the equations determined by the costs associated with the basis elements with $i = 1$.

After setting $u_1 = 0$, and then determining the values of v_1 and v_2, the tableau for finding the numbers u_i and v_j takes the following form:—

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				0
2		8	5	1		u_2
3				4	2	u_3
4					3	u_4
v_j	2	4	v_3	v_4	v_5	

The equations $v_2 = 4$ and $c_{2,2} = 8$ then force $u_2 = -4$, which in turn forces $v_3 = 1$ and $v_4 = -3$.

After setting $u_1 = 0$, and then successively determining the values of v_1 v_2, u_2, v_3 and v_4, the tableau takes the following form:—

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				0
2		8	5	1		-4
3				4	2	u_3
4					3	u_4
v_j	2	4	1	-3	v_5	

The equations $v_4 - u_3 = c_{3,4}$, $v_5 - u_3 = c_{3,5}$ and $v_5 - u_4 = c_{4,5}$ then successively force $u_3 = -7$, $v_5 = -5$ and $u_4 = -8$.

The completed tableau for determining the values of u_i and v_j thus takes the following form:—

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				0
2		8	5	1		-4
3				4	2	-7
4					3	-8
v_j	2	4	1	-3	-5	

We now return to the discussion of the solution of the particular example of the Transportation Problem with 4 suppliers and 5 recipients, where the supply and demand vectors and the cost matrix are as follows:—

$$s^T = (9, 11, 4, 5), \quad d^T = (6, 7, 5, 3, 8).$$

$$C = \begin{pmatrix} 2 & 4 & 3 & 7 & 5 \\ 4 & 8 & 5 & 1 & 8 \\ 5 & 9 & 4 & 4 & 2 \\ 7 & 2 & 5 & 5 & 3 \end{pmatrix}.$$

We have already determined a basic feasible solution associated with the basis B, where

$$B = \{(1,1), (1,2), (2,2), (2,3), (2,4), (3,4), (3,5), (4,5)\}.$$

The basic feasible solution and the basis B were determined using the North-west Corner Method. The non-zero coefficients of the basic feasible solution are as follows:—

$$x_{1,1} = 6, \quad x_{1,2} = 3, \quad x_{2,2} = 4, \quad x_{2,3} = 5,$$

$$x_{2,4} = 2, \quad x_{3,4} = 1, \quad x_{3,5} = 3, \quad x_{4,5} = 5,$$

The components of the basic feasible solution corresponding to the basis B are this recorded in the following tableau:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	3	4
4	0	0	0	0	5	5
d_j	6	7	5	3	8	29

We have also determined real numbers u_i and v_j such that $c_{i,j} = v_j - u_i$ for $(i,j) \in B$. These values are recorded in the following tableau, which records the costs in the body of the tableau, the values u_i to the right and the values v_j along the bottom:—

$c_{i,j}$	1	2	3	4	5	u_i
1	2	4				0
2		8	5	1		-4
3				4	2	-7
4					3	-8
v_j	2	4	1	-3	-5	

Thus if

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = -7, \quad , u_4 = -8,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = -5,$$

then $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$.

However the values of $v_j - u_i$ can differ from $c_{i,j}$ when $(i,j) \notin B$. We can construct a tableau which records, in the top left of every cell of the body, the cost $c_{i,j}$ in the top left and the slackness $q_{i,j}$ in the bottom right, where $q_{i,j} = c_{i,j} + u_i - v_j$. The symbol • in the top right of a cell indicates that the cell represents an element of the basis. This tableau is then as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 2	7 10	5 10	0
2	4 −2	8 • 0	5 • 0	1 • 0	8 9	−4
3	5 −4	9 −2	4 −4	4 • 0	2 • 0	−7
4	7 −3	2 −10	5 −4	5 0	3 • 0	−8
v_j	2	4	1	−3	−5	

Let

$$Q = \begin{pmatrix} 0 & 0 & 2 & 10 & 10 \\ -2 & 0 & 0 & 0 & 9 \\ -4 & -2 & -4 & 0 & 0 \\ -3 & -10 & -4 & 0 & 0 \end{pmatrix}.$$

Then the costs $c_{i,j}$ satisfy $c_{i,j} = v_j - u_i + q_{i,j}$, where

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = -7, \quad , u_4 = -8,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = -5,$$

and $q_{i,j} = (Q)_{i,j}$ for $i = 1,2,3,4$ and $j = 1,2,3,4,5$.

Note that $q_{i,j} = 0$ for all $(i,j) \in B$.

Now let $\overline{X}$ be any feasible solution of the Transportation Problem under consideration, and let $\overline{x}_{i,j} = (\overline{X})_{i,j}$ for $i = 1,2,3,4$ and $j = 1,2,3,4,5$. Then $\overline{x}_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{5} \overline{x}_{i,j} = s_i$ for $i = 1,2,3,4$ and $\sum_{i=1}^{4} \overline{x}_{i,j} = d_j$ for $j = 1,2,3,4,5$. Then

$$\begin{aligned} \operatorname{trace}(C^T \overline{X}) &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} \overline{x}_{i,j} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} (v_j - u_i + q_{i,j}) \overline{x}_{i,j} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{n} s_i u_i + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j} \\
&= \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{n} s_i u_i + \sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}.
\end{aligned}
$$

Note that the summand $\sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}$. is determined by the values of $q_{i,j}$ and $\overline{x}_{i,j}$ for those ordered pairs (i,j) that do not belong to the basis B. Thus if we apply this formula to the basic feasible solution $(x_{i,j})$ determined by the basis B, we find that

$$
\mathrm{trace}(C^T X) = \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{n} s_i u_i,
$$

because $x_{i,j} = 0$ when $(i,j) \notin B$. It follows that

$$
\mathrm{trace}(C^T \overline{X}) = \mathrm{trace}(C^T X) + \sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}
$$

for any feasible solution $\overline{X}$ of the specified transportation problem.

If it were the case that all the coefficients $q_{i,j}$ were non-negative then it would follow that $\sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j} \geq 0$ for all feasible solutions $\overline{X}$, and therefore X would be a basic optimal solution.

However the matrix Q has negative entries, and indeed $q_{4,2} = -10$. It turns out that we can construct a new basis, including the ordered pair $(4,2)$ so as to ensure that this new basis determines a basic feasible solution whose cost is lower than that of the original basic feasible solution.

We complete the following tableau so that coefficients are determined only for those cells determined by the $\bullet$ symbol so as to ensure that all rows and columns of the body of the tableau add up to zero:—

$y_{i,j}$	1	2	3	4	5	
1	? $\bullet$	? $\bullet$				0
2		? $\bullet$	? $\bullet$	? $\bullet$		0
3				? $\bullet$	? $\bullet$	0
4		1 $\circ$			? $\bullet$	0
	0	0	0	0	0	0

Examining the bottom row, we find that $y_{4,5} = -1$. Then, from the rightmost column we find that $y_{3,5} = 1$. Proceeding in this fashion, we successively determine $y_{3,4}$ and $y_{2,4}$. At this stage we see that

$$y_{4,5} = -1, \quad y_{3,5} = 1, \quad , y_{3,4} = -1 \quad \text{and} \quad y_{2,4} = 1.$$

To proceed further we note that $y_{1,1} = 0$, because the first column must sum to zero. This gives $y_{1,2} = 0$ and $y_{2,2} = -1$. This then forces $y_{2,3} = 0$. Thus the components $(Y)_{i,j}$ for which $(i, j) \in B$ are determined as in the following tableau:—

$y_{i,j}$	1	2	3	4	5	
1	0 $\bullet$	0 $\bullet$				0
2		-1 $\bullet$	0 $\bullet$	1 $\bullet$		0
3				-1 $\bullet$	1 $\bullet$	0
4		1 $\circ$			-1 $\bullet$	0
	0	0	0	0	0	0

The remaining coefficients of the matrix Y must equal zero, and thus

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

Then

$$X + \lambda Y = \begin{pmatrix} 6 & 3 & 0 & 0 & 0 \\ 0 & 4-\lambda & 5 & 2+\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda & 3+\lambda \\ 0 & \lambda & 0 & 0 & 5-\lambda \end{pmatrix}.$$

An earlier calculation established that $\text{trace}(C^T X) = 108$. Therefore

$$\begin{aligned} \text{trace}(C^T(X + \lambda Y)) &= \text{trace}(C^T X) + \lambda q_{4,2}(Y)_{4,2} \\ &= 108 - 10\lambda. \end{aligned}$$

The matrix $X + \lambda Y$ will represent a feasible solution when $0 \leq \lambda \leq 1$. But examination of the coefficient in the 3rd row and 4th column of $X + \lambda Y$ shows that $X + \lambda Y$ will not be a feasible solution for $\lambda > 1$. Thus taking $\lambda = 1$ determines a new basic feasible solution $\overline{X}$ determined by the basis obtained on adding $(4, 2)$ and removing $(3, 4)$ from the basis B.

Now let X and B denote the new feasible solution and the new basis associated with that feasible solution. Then

$$X = \begin{pmatrix} 6 & 3 & 0 & 0 & 0 \\ 0 & 3 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 4 \end{pmatrix}$$

and

$$B = \{(1,1), (1,2), (2,2), (2,3), (2,4), (3,5), (4,2), (4,5)\}.$$

The cost associated with this new basic solution is 98.

(i,j)	$x_{i,j}$	$c_{i,j}$	$c_{i,j}x_{i,j}$
$(1,1)$	6	2	12
$(1,2)$	3	4	12
$(2,2)$	3	8	24
$(2,3)$	5	5	25
$(2,4)$	3	1	3
$(3,5)$	4	2	8
$(4,2)$	1	2	2
$(4,5)$	4	3	12
Total			98

We now let X denote the new feasible solution and let B denote the new basis. We determine new values of the associated quantities u_i for $i = 1,2,3,4$ and $j = 1,2,3,4,5$, and calculate the new matrix Q so that $c_{i,j} = v_j - u_i + q_{i,j}$, where $q_{i,j} = (Q)_{i,j}$ for all i and j.

To determine u_i, v_j and $q_{i,j}$ we complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 ?	7 ?	5 ?	0
2	4 ?	8 • 0	5 • 0	1 • 0	8 ?	?
3	5 ?	9 ?	4 ?	4 ?	2 • 0	?
4	7 ?	2 • 0	5 ?	5 ?	3 • 0	?
v_j	?	?	?	?	?	

Now we choose $u_1 = 0$. Then, in order to ensure that $c_{i,j} = v_j - u_i$ for the two basis elements $(1,1)$ and $(1,2)$ we must take $v_1 = 2$ and $v_2 = 4$. Then

$v_2 = 4$ and $c_{2,2} = 8$ force $u_2 = -4$. Then $u_2 = -4$, $c_{2,3} = 5$ and $c_{2,4} = 1$ force $v_3 = 1$ and $v_4 = -3$. Also $v_2 = 4$ and $c_{2,4} = 2$ force $u_4 = 2$. Then $u_4 = 2$ and $c_{4,5} = 3$ force $v_5 = 5$. Then $v_5 = 5$ and $c_{3,5} = 2$ force $u_3 = 3$.

Thus

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = 3, \quad u_4 = 2,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = 5.$$

After determining u_i for $i = 1, 2, 3, 4$ and v_j for $j = 1, 2, 3, 4, 5$, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 ?	7 ?	5 ?	0
2	4 ?	8 • 0	5 • 0	1 • 0	8 ?	-4
3	5 ?	9 ?	4 ?	4 ?	2 • 0	3
4	7 ?	2 • 0	5 ?	5 ?	3 • 0	2
v_j	2	4	1	-3	5	

Let $q_{i,j} = c_{i,j} + u_i - v_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. Then $q_{i,j}$ when $(i, j) \in B$. We calculate and record in the tableau the values of $q_{i,j}$ when $(i, j) \notin B$. We find that

$$q_{1,3} = 2, \quad q_{1,4} = 10, \quad q_{1,5} = 0, \quad q_{2,1} = -2,$$

$$q_{2,5} = -1, \quad q_{3,1} = 6, \quad q_{3,2} = 8, \quad q_{3,3} = 6,$$

$$q_{3,4} = 10, \quad q_{4,1} = 7, \quad q_{4,3} = 6, \quad q_{4,4} = 10.$$

After calculating $q_{i,j}$ so that such that $q_{i,j} = c_{i,j} + u_i - v_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 • 0	4 • 0	3 2	7 10	5 0	0
2	4 -2	8 • 0	5 • 0	1 • 0	8 -1	-4
3	5 6	9 8	4 6	4 10	2 • 0	3
4	7 7	2 • 0	5 6	5 10	3 • 0	2
v_j	2	4	1	-3	5	

Let

$$Q = \begin{pmatrix} 0 & 0 & 2 & 10 & 0 \\ -2 & 0 & 0 & 0 & -1 \\ 6 & 8 & 6 & 10 & 0 \\ 7 & 0 & 6 & 10 & 0 \end{pmatrix}.$$

Then the costs $c_{i,j}$ satisfy $c_{i,j} = v_j - u_i + q_{i,j}$, where

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = 3, \quad u_4 = 2,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = 5.$$

and $q_{i,j} = (Q)_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$.

The matrix Q has some negative coefficients, and therefore the current basic feasible solution is not optimal. Because $q_{2,1}$ is the most negative coefficient of the matrix Q, we determine a new basis, consisting of the ordered pair $(2, 1)$ together with all but one ordered pair in the existing basis, so that the new basis corresponds to a new basic feasible solution with lower cost.

In order to determine the new basis, we complete the tableau below, (in which elements of the current basis are marked with the $\bullet$ symbol) in order that the rows and columns sum to zero. The tableau to be completed is as follows:—

$y_{i,j}$	1	2	3	4	5	
1	? $\bullet$	? $\bullet$				0
2	1 $\circ$	? $\bullet$	? $\bullet$	? $\bullet$		0
3					? $\bullet$	0
4		? $\bullet$			? $\bullet$	0
	0	0	0	0	0	0

The completed tableau is as follows:—

$y_{i,j}$	1	2	3	4	5	
1	-1 $\bullet$	1 $\bullet$				0
2	1 $\circ$	-1 $\bullet$	0 $\bullet$	0 $\bullet$		0
3					0 $\bullet$	0
4		0 $\bullet$			0 $\bullet$	0
	0	0	0	0	0	0

The remaining coefficients of the matrix Y must equal zero, and thus

$$Y = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we add λX to the current feasible solution X. We find that the coefficients of the resulting matrix $X + \lambda Y$ are as follows:—

$$X + \lambda Y = \begin{pmatrix} 6-\lambda & 3+\lambda & 0 & 0 & 0 \\ \lambda & 3-\lambda & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 4 \end{pmatrix}.$$

Then $X + \lambda Y$ determines a feasible solution with cost $98 - 2\lambda$ provided that $0 \leq \lambda < 3$. But the matrix does not represent a feasible solution for $\lambda > 3$. Accordingly we take $\lambda = 3$ to determine a new basic feasible solution. The ordered pair $(2, 1)$ enters the basis, and the ordered pair $(2, 2)$ leaves the basis.

We now let X denote the new feasible solution and let B denote the new basis associated with that feasible solution. We find that

$$X = \begin{pmatrix} 3 & 6 & 0 & 0 & 0 \\ 3 & 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 4 \end{pmatrix}.$$

and

$$B = \{(1,1), (1,2), (2,1), (2,3), (2,4), (3,5), (4,2), (4,5)\}.$$

The cost of this new feasible solution should be 92.

The following tableau verifies that the cost of the new feasible solution is indeed 92:—

(i, j)	$x_{i,j}$	$c_{i,j}$	$c_{i,j}x_{i,j}$
$(1, 1)$	3	2	6
$(1, 2)$	6	4	24
$(2, 1)$	3	4	12
$(2, 3)$	5	5	25
$(2, 4)$	3	1	3
$(3, 5)$	4	2	8
$(4, 2)$	1	2	2
$(4, 5)$	4	3	12
Total			92

To determine u_i, v_j and $q_{i,j}$ we complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 $\bullet$ 0	4 $\bullet$ 0	3 ?	7 ?	5 ?	0
2	4 $\bullet$ 0	8 ?	5 $\bullet$ 0	1 $\bullet$ 0	8 ?	?
3	5 ?	9 ?	4 ?	4 ?	2 $\bullet$ 0	?
4	7 ?	2 $\bullet$ 0	5 ?	5 ?	3 $\bullet$ 0	?
v_j	?	?	?	?	?	

To complete this tableau we first need to find values of u_i for $i = 1, 2, 3, 4$ and v_j for $j = 1, 2, 3, 4, 5$ such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. (The elements of the basis B have been marked with the $\bullet$ symbol.)

We set $u_1 = 0$. Then the $u_1 = 0$, $c_{1,1} = 2$ and $c_{1,2} = 4$ force $v_1 = 2$ and $v_2 = 4$. Then $v_1 = 2$ and $c_{1,2} = 4$ force $u_2 = -2$. Also $v_2 = 4$ and $c_{4,2} = 2$ force $u_4 = 2$. Then $u_2 = -3$, $c_{2,3} = 5$ and $c_{2,4} = 1$ force $v_3 = 3$ and $v_4 = -1$. Also $u_4 = 2$ and $c_{4,5} = 3$ forces $v_5 = 5$, which in turn forces $u_3 = 3$.

After determining u_i for $i = 1, 2, 3, 4$ and v_j for $j = 1, 2, 3, 4, 5$, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 $\bullet$ 0	4 $\bullet$ 0	3 ?	7 ?	5 ?	0
2	4 $\bullet$ 0	8 ?	5 $\bullet$ 0	1 $\bullet$ 0	8 ?	-2
3	5 ?	9 ?	4 ?	4 ?	2 $\bullet$ 0	3
4	7 ?	2 $\bullet$ 0	5 ?	5 ?	3 $\bullet$ 0	2
v_j	2	4	3	-1	5	

After calculating $q_{i,j}$ so that such that $q_{i,j} = c_{i,j} + u_i - v_j$ for $i = 1, 2, 3, 4$

and $j = 1, 2, 3, 4, 5$, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	2 $\bullet$	4 $\bullet$	3	7	5	0
	0	0	0	8	0	
2	4 $\bullet$	8	5 $\bullet$	1 $\bullet$	8	-2
	0	2	0	0	1	
3	5	9	4	4	2 $\bullet$	3
	6	8	4	8	0	
4	7	2 $\bullet$	5	5	3 $\bullet$	2
	7	0	4	8	0	
v_j	2	4	3	-1	5	

Let

$$Q = \begin{pmatrix} 0 & 0 & 0 & 8 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 6 & 8 & 4 & 8 & 0 \\ 7 & 0 & 4 & 8 & 0 \end{pmatrix}.$$

Then the costs $c_{i,j}$ satisfy $c_{i,j} = v_j - u_i + q_{i,j}$, where

$$u_1 = 0, \quad u_2 = -4, \quad u_3 = 3, \quad u_4 = 2,$$

$$v_1 = 2, \quad v_2 = 4, \quad v_3 = 1, \quad v_4 = -3, \quad v_5 = 5.$$

and $q_{i,j} = (Q)_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$.

Note that $q_{i,j} \geq 0$ for all integers i and j satisfying $1 \leq i \leq 4$ and $1 \leq j \leq 5$, and that $q_{i,j} = 0$ for all $(i, j) \in B$.

The current basic feasible solution is an optimal solution to the Transportation problem with the given supply and demand vectors and the given cost matrix.

Indeed let $\overline{X}$ be any feasible solution of the Transportation Problem under consideration, and let $\overline{x}_{i,j} = (\overline{X})_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. Then $\overline{x}_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{5} \overline{x}_{i,j} = s_i$ for $i = 1, 2, 3, 4$ and $\sum_{i=1}^{4} \overline{x}_{i,j} = d_j$ for $j = 1, 2, 3, 4, 5$. Then, as we showed earlier, the cost $\operatorname{trace}(C^T\overline{X})$ of the feasible solution $\overline{X}$ satisfies

$$\begin{aligned} \operatorname{trace}(C^T\overline{X}) &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} (v_j - u_i + q_{i,j})\overline{x}_{i,j} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{n} s_i u_i + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} \overline{x}_{i,j} \\
&= \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{n} s_i u_i + \sum_{(i,j) \notin B} q_{i,j} \overline{x}_{i,j}.
\end{aligned}
$$

Note that the summand $\sum_{(i,j)\notin B} q_{i,j}\overline{x}_{i,j}$. is determined by the values of $q_{i,j}$ and $\overline{x}_{i,j}$ for those ordered pairs (i, j) that do not belong to the basis B. Thus if we apply this formula to the basic feasible solution $(x_{i,j})$ determined by the basis B, we find that

$$
\operatorname{trace}(C^T X) = \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{n} s_i u_i,
$$

because $x_{i,j} = 0$ when $(i, j) \notin B$. It follows that

$$
\operatorname{trace}(C^T \overline{X}) = \operatorname{trace}(C^T X) + \sum_{(i,j) \notin B} q_{i,j} \overline{x}_{i,j}
$$

for any feasible solution $\overline{X}$ of the specified transportation problem.

But we have calculed a basic feasible solution X of the transportation problem with the property that $q_{i,j} \geq 0$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. It follows that every feasible solution $\overline{X}$ of this transportation problem satisfies $\operatorname{trace}(C^T \overline{X}) \geq \operatorname{trace}(C^T X)$. Therefore the current basic feasible solution is an optimal feasible solution minimizing transportation costs.

We now summarize the conclusions of our numerical example. The problem was to determine a basic optimal solution of the transportation problem with given supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C, where

$$
s^T = (9, 11, 4, 5), \quad d^T = (6, 7, 5, 3, 8)
$$

and

$$
C = \begin{pmatrix} 2 & 4 & 3 & 7 & 5 \\ 4 & 8 & 5 & 1 & 8 \\ 5 & 9 & 4 & 4 & 2 \\ 7 & 2 & 5 & 5 & 3 \end{pmatrix}.
$$

We have determined a basic optimal solution

$$
X = \begin{pmatrix} 3 & 6 & 0 & 0 & 0 \\ 3 & 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 4 \end{pmatrix}.
$$

associated to a basis B, where

$$B = \{(1,1),(1,2),(2,1),(2,3),(2,4),(3,5),(4,2),(4,5)\}.$$

The cost of this new feasible solution is 92.

Moreover we have also determined quantities u_i for $i = 1,2,3,4$ and v_j for $j = 1,2,3,4,5$ satisfying the following two conditions:—

- $c_{i,j} \geq v_j - u_i$ for $i = 1,2,3,4$ and $j = 1,2,3,4,5$;
- $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$.

2 A Transportation Problem Example with Six Suppliers and Five Recipients

We now find a basic optimal solution to a transportation problem with 6 suppliers and 5 recipients. We find an initial basic feasible solution using the Minimum Cost Method, and then continue to find a basic optimal solution using a form of the Simplex Method adapted to the Transportation Problem.

The supply vector is $(9, 14, 5, 16, 7, 9)$ and the demand vector is $(8, 17, 6, 14, 15)$. The components of both the supply vector and the demand vector add up to 60.

The costs are as specified in the following cost matrix:—

$$\begin{pmatrix} 12 & 8 & 9 & 4 & 6 \\ 5 & 10 & 8 & 9 & 5 \\ 6 & 4 & 12 & 12 & 4 \\ 5 & 7 & 12 & 10 & 8 \\ 4 & 6 & 8 & 10 & 12 \\ 7 & 3 & 7 & 12 & 8 \end{pmatrix}.$$

We fill in the row sums (or supplies), the column sums (or demands) and the costs $c_{i,j}$ for the given problem. The resultant tableau looks as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12 ?	8 ?	9 ?	4 ?	6 ?	9
2	5 ?	10 ?	8 ?	9 ?	5 ?	14
3	6 ?	4 ?	12 ?	12 ?	4 ?	5
4	5 ?	7 ?	12 ?	10 ?	8 ?	16
5	4 ?	6 ?	8 ?	10 ?	12 ?	7
6	7 ?	3 ?	7 ?	12 ?	8 ?	9
d_j	8	17	6	14	15	60

In order to apply the Minimum Cost Method to find an initial basic feasible solution, we identify the cell with the minimum cost associated to it. The minimum cost is 3, and this cost is the value of $c_{i,j}$ only when $i = 6$ and $j = 2$. We therefore set $x_{6,2}$ to be the minimum of the supply s_6 and the demand d_2. Now $s_6 = 9$ and $d_2 = 17$. Therefore we set $x_{6,2} = 9$. Now the values of $x_{i,j}$ in the row $i = 6$ must be non-negative and must sum to 9. It follows that $x_{6,j} = 0$ when $j \neq 2$. We therefore fill in the values of $x_{i,j}$ for $i = 6$, and enter a $\bullet$ symbol in the tableau cell for $(i, j) = (6, 2)$:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12	8	9	4	6	
	?	?	?	?	?	9
2	5	10	8	9	5	
	?	?	?	?	?	14
3	6	4	12	12	4	
	?	?	?	?	?	5
4	5	7	12	10	8	
	?	?	?	?	?	16
5	4	6	8	10	12	
	?	?	?	?	?	7
6	7	3 $\bullet$	7	12	8	
	0	9	0	0	0	9
d_j	8	17	6	14	15	60

We next look for the cell or cells of minimum cost amongst those where the value of $x_{i,j}$ is still to be determined. The minimum cost amongst such cells is 4, and $c_{i,j} = 4$ when (i, j) is equal to one of the ordered pairs $(1, 4)$, $(3, 2)$, $(3, 5)$ and $(5, 1)$. Now it makes sense to choose the cell with $c_{i,j} = 4$ for which the value of $x_{i,j}$ will be the maximum possible, since that ensures, in the context of the transformation problem, that the largest amount of the commodity is transported at this cheap price. Thus we chose $(1, 4)$ as our next basis element, set $x_{1,4} = 9$, and set $x_{1,j} = 0$ when $j \neq 0$. The tableau is then as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12 / 0	8 / 0	9 / 0	4 • / 9	6 / 0	9
2	5 / ?	10 / ?	8 / ?	9 / ?	5 / ?	14
3	6 / ?	4 / ?	12 / ?	12 / ?	4 / ?	5
4	5 / ?	7 / ?	12 / ?	10 / ?	8 / ?	16
5	4 / ?	6 / ?	8 / ?	10 / ?	12 / ?	7
6	7 / 0	3 • / 9	7 / 0	12 / 0	8 / 0	9
d_j	8	17	6	14	15	60

The minimum cost amongst those cells for which $x_{i,j}$ is still to be determined is still 4. Taking $(i, j) = (5, 1)$ permits the largest possible value of $x_{i,j}$ amongst those cells for which $c_{i,j} = 4$ and $x_{i,j}$ is still to be determined. Therefore we set $x_{5,1}$ to the minimum value of s_5 and d_1, and thus set $x_{5,1} = 7$. In order to achieve a feasible solution we must then take $x_{5,j} = 0$ when $j \neq 1$. Accordingly the first three basis elements determined are

$$(6, 2), \quad (1, 4), \quad (5, 1),$$

and the tableau at the completion of this stage is as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12 / 0	8 / 0	9 / 0	4 • / 9	6 / 0	9
2	5 / ?	10 / ?	8 / ?	9 / ?	5 / ?	14
3	6 / ?	4 / ?	12 / ?	12 / ?	4 / ?	5
4	5 / ?	7 / ?	12 / ?	10 / ?	8 / ?	16
5	4 • / 7	6 / 0	8 / 0	10 / 0	12 / 0	7
6	7 / 0	3 • / 9	7 / 0	12 / 0	8 / 0	9
d_j	8	17	6	14	15	60

There are now two remaining cells with cost equal to 4. These are the cells where (i, j) is one of the ordered pairs $(3, 2)$ and $(3, 5)$. At each of these cells, the minimum of s_i and d_j has the value 5. We arbitrarily choose $(3, 2)$ as the next element for the basis, set $x_{3,2} = 5$, and set $x_{3,j} = 0$ for $j \neq 2$. Accordingly the first four basis elements determined are

$$(6, 2), \quad (1, 4), \quad (5, 1), \quad (3, 2)$$

and the tableau at the completion of this stage is as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12	8	9	4 •	6	
	0	0	0	9	0	9
2	5	10	8	9	5	
	?	?	?	?	?	14
3	6	4 •	12	12	4	
	0	5	0	0	0	5
4	5	7	12	10	8	
	?	?	?	?	?	16
5	4 •	6	8	10	12	
	7	0	0	0	0	7
6	7	3 •	7	12	8	
	0	9	0	0	0	9
d_j	8	17	6	14	15	60

We next look for the cell or cells of minimum cost amongst those where the value of $x_{i,j}$ is still to be determined. The minimum cost amongst such cells is 5, and $c_{i,j} = 5$ when (i, j) is equal to one of the ordered pairs $(2, 1)$, $(2, 5)$, and $(4, 1)$. Now taking $(i, j) = (2, 1)$ or $(i, j) = (4, 1)$ would result in $x_{2,1}$ or $x_{4,1}$ having the value 1, because the numbers in the first column of the body of the tableau must sum to 8. We obtain a larger possible value of $x_{i,j}$ with $(i, j) = (2, 5)$, and accordingly we add $(2, 5)$ to our basis, set $x_{2,5} = 14$, and set $x_{2,j} = 0$ when $j \neq 5$. Accordingly the first five basis elements determined are

$$(6, 2), \quad (1, 4), \quad (5, 1), \quad (3, 2), \quad (2, 5)$$

and the tableau at the completion of this stage is as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12 0	8 0	9 0	4 • 9	6 0	9
2	5 0	10 0	8 0	9 0	5 • 14	14
3	6 0	4 • 5	12 0	12 0	4 0	5
4	5 ?	7 ?	12 ?	10 ?	8 ?	16
5	4 • 7	6 0	8 0	10 0	12 0	7
6	7 0	3 • 9	7 0	12 0	8 0	9
d_j	8	17	6	14	15	60

The ordered pair $(4, 1)$ is now the only remaining ordered pair (i, j) for which $c_{i,j} = 5$ and $x_{i,j}$ is still to be determined. We add $(4, 1)$ to our basis. The first column must add up to 8, and accordingly $x_{4,1} = 1$. There are no further cells in the first column for which $x_{i,0}$ is undetermined. Accordingly the first six basis elements determined are

$$(6, 2), \ (1, 4), \ (5, 1), \ (3, 2), \ (2, 5), \ (4, 1)$$

and the tableau at the completion of this stage is as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12 0	8 0	9 0	4 • 9	6 0	9
2	5 0	10 0	8 0	9 0	5 • 14	14
3	6 0	4 • 5	12 0	12 0	4 0	5
4	5 • 1	7 ?	12 ?	10 ?	8 ?	16
5	4 • 7	6 0	8 0	10 0	12 0	7
6	7 0	3 • 9	7 0	12 0	8 0	9
d_j	8	17	6	14	15	60

The only ordered pairs (i,j) for which $x_{i,j}$ is still to be determined at those with $i = 4$ and $j = 2,3,4,5$. The minimum cost associated with these cells is 7, and corresponds to $j = 2$. Now the second column must sum to 17, and the values of $x_{i,2}$ for $i \neq 4$ sum to 14. Accordingly we must take $x_{4,2} = 3$. (This is compatible with the sum of the row $i = 4$ being equal to 16.) Accordingly the first seven basis elements determined are

$$(6,2),\ (1,4),\ (5,1),\ (3,2),\ (2,5),\ (4,1),\ (4,2)$$

and the tableau at the completion of this stage is as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12	8	9	4 •	6	
	0	0	0	9	0	9
2	5	10	8	9	5 •	
	0	0	0	0	14	14
3	6	4 •	12	12	4	
	0	5	0	0	0	5
4	5 •	7 •	12	10	8	
	1	3	?	?	?	16
5	4 •	6	8	10	12	
	7	0	0	0	0	7
6	7	3 •	7	12	8	
	0	9	0	0	0	9
d_j	8	17	6	14	15	60

The next lowest cost with undetermined $x_{i,j}$ is 8, and occurs for $(i,j) = (4,5)$. We add $(4,5)$ to the basis. We must take $x_{4,5} = 1$ in order to ensure that $\sum_{i=1}^{6} x_{i,5} = d_5 = 15$. Accordingly the first eight basis elements determined are

$$(6,2),\ (1,4),\ (5,1),\ (3,2),\ (2,5),\ (4,1),\ (4,2)$$
$$(4,5)$$

and the tableau at the completion of this stage is as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12	8	9	4 •	6	
	0	0	0	9	0	9
2	5	10	8	9	5 •	
	0	0	0	0	14	14
3	6	4 •	12	12	4	
	0	5	0	0	0	5
4	5 •	7 •	12	10	8 •	
	1	3	?	?	1	16
5	4 •	6	8	10	12	
	7	0	0	0	0	7
6	7	3 •	7	12	8	
	0	9	0	0	0	9
d_j	8	17	6	14	15	60

The next lowest cost with undetermined $x_{i,j}$ is 10, and occurs for $(i,j) = (4,4)$. We add $(4,4)$ to the basis. We must take $x_{4,4} = 5$ in order to ensure that $\sum_{i=1}^{6} x_{i,4} = d_4 = 14$. Accordingly the first nine basis elements determined are

$$(6,2),\ (1,4),\ (5,1),\ (3,2),\ (2,5),\ (4,1),\ (4,2)$$
$$(4,5),\ (4,4)$$

and the tableau at the completion of this stage is as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12	8	9	4 •	6	
	0	0	0	9	0	9
2	5	10	8	9	5 •	
	0	0	0	0	14	14
3	6	4 •	12	12	4	
	0	5	0	0	0	5
4	5 •	7 •	12	10 •	8 •	
	1	3	?	5	1	16
5	4 •	6	8	10	12	
	7	0	0	0	0	7
6	7	3 •	7	12	8	
	0	9	0	0	0	9
d_j	8	17	6	14	15	60

We complete the table by adding $(4,3)$ to the basis and setting $x_{4,3} = 6$, so as to ensure that $\sum_{j=1}^{5} x_{4,j} = s_4 = 16$. and $\sum_{i=1}^{6} x_{i,3} = d_3 = 6$. Accordingly the complete basis consists of the ordered pairs

$$\begin{array}{l} (6,2),\ \ (1,4),\ \ (5,1),\ \ (3,2),\ \ (2,5),\ \ (4,1),\ \ (4,2) \\ (4,5),\ \ (4,4)\ \ (4,3) \end{array}$$

and the tableau at the completion of this stage is as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12 0	8 0	9 0	4 • 9	6 0	9
2	5 0	10 0	8 0	9 0	5 • 14	14
3	6 0	4 • 5	12 0	12 0	4 0	5
4	5 • 1	7 • 3	12 • 6	10 • 5	8 • 1	16
5	4 • 7	6 0	8 0	10 0	12 0	7
6	7 0	3 • 9	7 0	12 0	8 0	9
d_j	8	17	6	14	15	60

We have now found an initial basic feasible solution to this transportation problem. This initial basic feasible solution is determined by basis B, where

$$\begin{aligned} B &= \{(6,2),\ \ (1,4),\ \ (5,1),\ \ (3,2),\ \ (2,5),\ \ (4,1), \\ &\qquad (4,2),\ \ (4,5),\ \ (4,4)\ \ (4,3)\}. \end{aligned}$$

The basic feasible solution is represented by the 6×5 matrix X, where

$$X = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 5 & 0 & 0 & 0 \\ 1 & 3 & 6 & 5 & 1 \\ 7 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

The coefficient of this matrix X in the ith row and jth column is the quantity $x_{i,j}$ that determines the quantity of the commodity to transport from the ith supplier to the jth recipient. Note that $x_{i,j} = 0$ when $(i,j) \notin B$. This corresponds to the requirement that $(x_{i,j})$ be a basic feasible solution determined by the basis B. Also $x_{i,j} > 0$ for all $(i,j) \in B$.

The cost of this initial feasible basic solution is

$$\begin{aligned} & 4 \times 9 + 5 \times 14 + 4 \times 5 + 5 \times 1 + 7 \times 3 \\ & \qquad + 12 \times 6 + 10 \times 5 + 8 \times 1 + 4 \times 7 + 3 \times 9 \\ = {} & 36 + 70 + 20 + 5 + 21 + 72 + 50 + 8 + 28 + 27 \\ = {} & 337. \end{aligned}$$

The average transportation cost per unit of the commodity is then 5.617.

We next determine whether the initial basic feasible solution found by the Minimum Cost Method is an optimal solution, and, if not, how to adjust the basis go obtain a solution of lower cost.

We determine $u_1, u_2, u_3, u_4, u_5, u_6$ and v_1, v_2, v_3, v_4, v_5 such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$, where B is the initial basis, specified as follows:

$$\begin{aligned} B = {} & \{(6,2),\ (1,4),\ (5,1),\ (3,2),\ (2,5),\ (4,1), \\ & (4,2),\ (4,5),\ (4,4)\ (4,3)\}. \end{aligned}$$

We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 ?	8 ?	9 ?	4 • 0	6 ?	0
2	5 ?	10 ?	8 ?	9 ?	5 • 0	?
3	6 ?	4 • 0	12 ?	12 ?	4 ?	?
4	5 • 0	7 • 0	12 • 0	10 • 0	8 • 0	?
5	4 • 0	6 ?	8 ?	10 ?	12 ?	?
6	7 ?	3 • 0	7 ?	12 ?	8 ?	?
v_j	?	?	?	?	?	

Now $(1,4) \in B$, $u_1 = 0$ and $c_{1,4} = 4$ force $v_4 = 4$. Then $(4,4) \in B$, $v_4 = 4$ and $c_{4,4} = 10$ force $u_4 = -6$. After entering these values, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 / ?	8 / ?	9 / ?	4 • / 0	6 / ?	0
2	5 / ?	10 / ?	8 / ?	9 / ?	5 • / 0	?
3	6 / ?	4 • / 0	12 / ?	12 / ?	4 / ?	?
4	5 • / 0	7 • / 0	12 • / 0	10 • / 0	8 • / 0	-6
5	4 • / 0	6 / ?	8 / ?	10 / ?	12 / ?	?
6	7 / ?	3 • / 0	7 / ?	12 / ?	8 / ?	?
v_j	?	?	?	4	?	

Next we note that $(4, j) \in B$ for all j. Therefore $u_4 = -6$ and $c_{4,j} = v_j - u_4$ force $v_j = c_{4,j} - 6$ for all j. Therefore $v_1 = -1$, $v_2 = 1$, $v_3 = 6$ and $v_5 = 2$. After entering these values, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 / ?	8 / ?	9 / ?	4 • / 0	6 / ?	0
2	5 / ?	10 / ?	8 / ?	9 / ?	5 • / 0	?
3	6 / ?	4 • / 0	12 / ?	12 / ?	4 / ?	?
4	5 • / 0	7 • / 0	12 • / 0	10 • / 0	8 • / 0	-6
5	4 • / 0	6 / ?	8 / ?	10 / ?	12 / ?	?
6	7 / ?	3 • / 0	7 / ?	12 / ?	8 / ?	?
v_j	-1	1	6	4	2	

Next $(2, 5) \in B$, $c_{2,5} = 5$ and $v_5 = 2$ forces $u_2 = -3$. Also $(3, 2) \in B$, $c_{3,2} = 4$ and $v_2 = 1$ forces $u_3 = -3$. Also $(5, 1) \in B$, $c_{5,1} = 4$ and $v_1 = -1$ forces $u_5 = -5$. Also $(6, 2) \in B$, $c_{6,2} = 3$ and $v_2 = 1$ forces $u_6 = -2$.

After entering these values, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 / ?	8 / ?	9 / ?	4 • / 0	6 / ?	0
2	5 / ?	10 / ?	8 / ?	9 / ?	5 • / 0	−3
3	6 / ?	4 • / 0	12 / ?	12 / ?	4 / ?	−3
4	5 • / 0	7 • / 0	12 • / 0	10 • / 0	8 • / 0	−6
5	4 • / 0	6 / ?	8 / ?	10 / ?	12 / ?	−5
6	7 / ?	3 • / 0	7 / ?	12 / ?	8 / ?	−2
v_j	−1	1	6	4	2	

We have thus found the following values for the u_i and v_j:

$$u_1 = 0, \;\; u_2 = -3, \;\; u_3 = -3, \;\; u_4 = -6, \;\; u_5 = -5, \;\; u_6 = -2.$$

$$v_1 = -1, \;\; , v_1 = 1, \;\; v_2 = 6, \;\; v_4 = 4, \;\; v_5 = 2.$$

We next calculate $q_{i,j}$ for all i and j so that $c_{i,j} = v_j - u_i + q_{i,j}$. Note that the determination of the numbers u_i and v_j ensures that $q_{i,j} = 0$ for all $(i, j) \in B$.

After entering the value of $q_{i,j}$ for all i and j, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 / 13	8 / 7	9 / 3	4 • / 0	6 / 4	0
2	5 / 3	10 / 6	8 / −1	9 / 2	5 • / 0	−3
3	6 / 4	4 • / 0	12 / 3	12 / 5	4 / −1	−3
4	5 • / 0	7 • / 0	12 • / 0	10 • / 0	8 • / 0	−6
5	4 • / 0	6 / 0	8 / −3	10 / 1	12 / 5	−5
6	7 / 6	3 • / 0	7 / −1	12 / 6	8 / 4	−2
v_j	−1	1	6	4	2	

The initial basic feasible solution is not optimal because some of the quantities $q_{i,j}$ are negative. Indeed $q_{2,3} = -1$, $q_{3,5} = -1$, $q_{5,3} = -3$ and $q_{6,3} = -1$. The most negative of these is $q_{5,3}$. We therefore seek to bring $(5,3)$ into the basis.

The procedure for achieving this requires us to determine a 6×5 matrix Y satisfying the following conditions:—

- $y_{5,3} = 1$;
- $y_{i,j} = 0$ when $(i,j) \notin B \cup \{(5,3)\}$;
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1	2	3	4	5	
1				? •		0
2					? •	0
3		? •				0
4	? •	? •	? •	? •	? •	0
5	? •		1 ◦			0
6		? •				0
	0	0	0	0	0	0

The constraint that the rows and columns of the table all sum to zero determines $y_{4,3}$, and also determines $y_{i,j}$ for all $(i,j) \in B$ satisfying $i \neq 4$. These values of $y_{i,j}$ are recorded in the following tableau:—

$y_{i,j}$	1	2	3	4	5	
1				0 •		0
2					0 •	0
3		0 •				0
4	? •	? •	−1 •	? •	? •	0
5	−1 •		1 ◦			0
6		0 •				0
	0	0	0	0	0	0

The remaining values of $y_{i,j}$ for $(i,j) \in B$ are then readily determined, and the tableau is completed as follows:—

$y_{i,j}$	1	2	3	4	5	
1				$0 \bullet$		0
2					$0 \bullet$	0
3		$0 \bullet$				0
4	$1 \bullet$	$0 \bullet$	$-1 \bullet$	$0 \bullet$	$0 \bullet$	0
5	$-1 \bullet$		$1 \circ$			0
6		$0 \bullet$				0
	0	0	0	0	0	0

We now determine those values of λ for which $X+\lambda Y$ is a feasible solution, where

$$X + \lambda Y = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 5 & 0 & 0 & 0 \\ 1+\lambda & 3 & 6-\lambda & 5 & 1 \\ 7-\lambda & 0 & \lambda & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

From this matrix, it is clear that $X + \lambda Y$ is a feasible solution for $0 \leq \lambda \leq 6$. Moreover the next basis is obtained by adding $(5,3)$ to the existing basis and removing $(4,3)$. The new basic feasible solution corresponding to the new basis is obtained from $X + \lambda Y$ by setting $\lambda = 6$.

We now let B denote the new basis and let X denote the new basic feasible solution corresponding to the new basis. Accordingly

$$\begin{aligned} B &= \{(6,2),\ (1,4),\ (5,1),\ (3,2),\ (2,5),\ (4,1), \\ &\quad (4,2),\ (4,5),\ (4,4)\ (5,3)\}. \end{aligned}$$

and

$$X = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 5 & 0 & 0 & 0 \\ 7 & 3 & 0 & 5 & 1 \\ 1 & 0 & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover

$$\text{Cost} = \text{Old Cost} + 6 * (-3) = 337 - 18 = 319.$$

The cost of the current feasible solution can also be obtained from the data recorded in the following tableau that represents the current feasible solution:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12 / 0	8 / 0	9 / 0	4 • / 9	6 / 0	9
2	5 / 0	10 / 0	8 / 0	9 / 0	5 • / 14	14
3	6 / 0	4 • / 5	12 / 0	12 / 0	4 / 0	5
4	5 • / 7	7 • / 3	12 / 0	10 • / 5	8 • / 1	16
5	4 • / 1	6 / 0	8 • / 6	10 / 0	12 / 0	7
6	7 / 0	3 • / 9	7 / 0	12 / 0	8 / 0	9
d_j	8	17	6	14	15	60

In order to determine whether or not the new basic feasible solution is optimal, and, if not, how to improve it, we determine u_i for $1 \leq i \leq 5$ and v_j for $1 \leq j \leq 6$ such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$, where B is now the current basis. We then calculate $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1,2,3,4,5,6$ and $j = 1,2,3,4,5$.

Accordingly we determine the numbers u_i and v_j, setting $u_1 = 0$ and using the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 / ?	8 / ?	9 / ?	4 • / 0	6 / ?	0
2	5 / ?	10 / ?	8 / ?	9 / ?	5 • / 0	?
3	6 / ?	4 • / 0	12 / ?	12 / ?	4 / ?	?
4	5 • / 0	7 • / 0	12 / ?	10 • / 0	8 • / 0	?
5	4 • / 0	6 / ?	8 • / 0	10 / ?	12 / ?	?
6	7 / ?	3 • / 0	7 / ?	12 / ?	8 / ?	?
v_j	?	?	?	?	?	

Now $(1,4) \in B$, $u_1 = 0$ and $c_{1,4} = 4$ force $v_4 = 4$. Then $(4,4) \in B$, $v_4 = 4$ and $c_{4,4} = 10$ force $u_4 = -6$. Then $u_4 = -6$ and $(4,1),(4,2),(4,5) \in B$ force

$v_1 = -1$, $v_2 = 1$ and $v_5 = 2$. After recording these values the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12	8	9	4 •	6	0
	?	?	?	0	?	
2	5	10	8	9	5 •	?
	?	?	?	?	0	
3	6	4 •	12	12	4	?
	?	0	?	?	?	
4	5 •	7 •	12	10 •	8 •	-6
	0	0	?	0	0	
5	4 •	6	8 •	10	12	?
	0	?	0	?	?	
6	7	3 •	7	12	8	?
	?	0	?	?	?	
v_j	-1	1	?	4	2	

Then $(2,5) \in B$, $v_5 = 2$ and $c_{2,5} = 5$ force $u_2 = -3$. Also $(3,2) \in B$, $v_2 = 1$ and $c_{3,2} = 4$ force $u_3 = -3$. Also $(5,1) \in B$, $v_1 = -1$ and $c_{6,2} = 4$ force $u_5 = -5$. Also $(6,2) \in B$, $v_2 = 1$ and $c_{6,2} = 3$ force $u_6 = -2$. Then $(5,3) \in B$, $u_5 = -5$ and $c_{5,3} = 8$ force $v_3 = 3$.

Thus after recording the values of u_i and v_j for all i and j the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12	8	9	4 •	6	0
	?	?	?	0	?	
2	5	10	8	9	5 •	-3
	?	?	?	?	0	
3	6	4 •	12	12	4	-3
	?	0	?	?	?	
4	5 •	7 •	12	10 •	8 •	-6
	0	0	?	0	0	
5	4 •	6	8 •	10	12	-5
	0	?	0	?	?	
6	7	3 •	7	12	8	-2
	?	0	?	?	?	
v_j	-1	1	3	4	2	

The next stage is to compute the values of $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1,2,3,4,5,6$ and $j = 1,2,3,4,5$. The values of $q_{i,j}$ are accordingly recorded in the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 13	8 7	9 6	4 • 0	6 4	0
2	5 3	10 6	8 2	9 2	5 • 0	-3
3	6 4	4 • 0	12 6	12 5	4 -1	-3
4	5 • 0	7 • 0	12 3	10 • 0	8 • 0	-6
5	4 • 0	6 0	8 • 0	10 1	12 5	-5
6	7 6	3 • 0	7 2	12 6	8 4	-2
v_j	-1	1	3	4	2	

The fact that $q_{3,5} = -1$ shows that the current basic feasible solution is not optimal. We therefore seek to bring $(3,5)$ into the basis, and, to achieve this, we calculate the coefficients $y_{i,j}$ of a 6×5 matrix Y satisfying the following conditions:—

- $y_{3,5} = 1$;
- $y_{i,j} = 0$ when $(i,j) \notin B \cup \{(3,5)\}$;
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1	2	3	4	5	
1				? •		0
2					? •	0
3		? •			1 ○	0
4	? •	? •		? •	? •	0
5	? •		? •			0
6		? •				0
	0	0	0	0	0	0

The constraints that the 1st, 2nd, 3rd and 6th rows of the body of the table sum to zero force $y_{1,4} = 0$, $y_{2,5} = 0$, $y_{3,2} = -1$ and $y_{6,2} = 0$. The constraint that the 3rd column of the body of the table sum to zero forces $y_{5,3} = 0$. After entering these values, the tableau is as follows:—

$y_{i,j}$	1	2	3	4	5	
1				0 •		0
2					0 •	0
3		−1 •			1 ∘	0
4	? •	? •		? •	? •	0
5	? •		0 •			0
6		0 •				0
	0	0	0	0	0	0

The constraints that the 5th row and the 2nd, 4th and 5th column of the body of the table sum to zero force $y_{5,1} = 0$, $y_{4,2} = 1$, $y_{4,4} = 0$ and $y_{4,5} = -1$.

The then requirement that the first column of the body of the table, and the identity $y_{5,1} = 0$ together force $y_{4,1} = 0$. The completed tableau is thus as follows:—

$y_{i,j}$	1	2	3	4	5	
1				0 •		0
2					0 •	0
3		−1 •			1 ∘	0
4	0 •	1 •		0 •	−1 •	0
5	0 •		0 •			0
6		0 •				0
	0	0	0	0	0	0

We now determine those values of λ for which $X + \lambda Y$ is a feasible solution, where

$$X + \lambda Y = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 5-\lambda & 0 & 0 & \lambda \\ 7 & 3+\lambda & 0 & 5 & 1-\lambda \\ 1 & 0 & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

From this matrix, it is clear that $X + \lambda Y$ is a feasible solution for $0 \leq \lambda \leq 1$. Moreover the next basis is obtained by adding $(3, 5)$ to the existing basis and removing $(4, 5)$. The new basic feasible solution corresponding to the new basis is obtained from $X + \lambda Y$ by setting $\lambda = 1$.

We now let B denote the new basis and let X denote the new basic feasible solution corresponding to the new basis. Accordingly

$$\begin{aligned} B \;=\; \{&(6,2),\;(1,4),\;(5,1),\;(3,2),\;(2,5),\;(4,1),\\ &(4,2),\;(3,5),\;(4,4)\;(5,3)\}, \end{aligned}$$

and

$$X = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 4 & 0 & 0 & 1 \\ 7 & 4 & 0 & 5 & 0 \\ 1 & 0 & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover

$$\text{Cost} = \text{Old Cost} + 1 * (-1) = 319 - 1 = 318.$$

The cost of the current feasible solution can also be obtained from the data recorded in the following tableau that represents the current feasible solution:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	4	5	s_i
1	12 / 0	8 / 0	9 / 0	4 • / 9	6 / 0	9
2	5 / 0	10 / 0	8 / 0	9 / 0	5 • / 14	14
3	6 / 0	4 • / 4	12 / 0	12 / 0	4 • / 1	5
4	5 • / 7	7 • / 4	12 / 0	10 • / 5	8 / 0	16
5	4 • / 1	6 / 0	8 • / 6	10 / 0	12 / 0	7
6	7 / 0	3 • / 9	7 / 0	12 / 0	8 / 0	9
d_j	8	17	6	14	15	60

In order to determine whether or not the new basic feasible solution is optimal, and, if not, how to improve it, we determine u_i for $1 \leq i \leq 5$ and v_j for $1 \leq j \leq 6$ such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$, where B is now the current basis. We then calculate $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3, 4, 5, 6$ and $j = 1, 2, 3, 4, 5$.

Accordingly we determine the numbers u_i and v_j, setting $u_1 = 0$ and using the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 / ?	8 / ?	9 / ?	4 • / 0	6 / ?	0
2	5 / ?	10 / ?	8 / ?	9 / ?	5 • / 0	?
3	6 / ?	4 • / 0	12 / ?	12 / ?	4 • / 0	?
4	5 • / 0	7 • / 0	12 / ?	10 • / 0	8 / ?	?
5	4 • / 0	6 / ?	8 • / 0	10 / ?	12 / ?	?
6	7 / ?	3 • / 0	7 / ?	12 / ?	8 / ?	?
v_j	?	?	?	?	?	

Solving the equations determining u_i and v_j, we find, successively, $u_1 = 0$, $v_4 = 4$, $u_4 = -6$, $v_1 = -1$, $v_2 = 1$, $u_5 = -5$, $v_3 = 3$, $u_3 = -3$, $u_6 = -2$, $v_5 = 1$ and $u_2 = -4$.

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 / ?	8 / ?	9 / ?	4 • / 0	6 / ?	0
2	5 / ?	10 / ?	8 / ?	9 / ?	5 • / 0	−4
3	6 / ?	4 • / 0	12 / ?	12 / ?	4 • / 0	−3
4	5 • / 0	7 • / 0	12 / ?	10 • / 0	8 / ?	−6
5	4 • / 0	6 / ?	8 • / 0	10 / ?	12 / ?	−5
6	7 / ?	3 • / 0	7 / ?	12 / ?	8 / ?	−2
v_j	−1	1	3	4	1	

The next stage is to compute the values of $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3, 4, 5, 6$ and $j = 1, 2, 3, 4, 5$. The values of $q_{i,j}$ are accordingly recorded in the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	4	5	u_i
1	12 13	8 7	9 6	4 • 0	6 5	0
2	5 2	10 5	8 1	9 1	5 • 0	−4
3	6 4	4 • 0	12 6	12 5	4 • 0	−3
4	5 • 0	7 • 0	12 3	10 • 0	8 1	−6
5	4 • 0	6 0	8 • 0	10 1	12 6	−5
6	7 6	3 • 0	7 2	12 6	8 5	−2
v_j	−1	1	3	4	1	

We now summarize what has been achieved. The problem was to find a basic optimal solution to a transportation problem with 6 suppliers and 5 recipients.

The supply vector is $(9, 14, 5, 16, 7, 9)$ and the demand vector is $(8, 17, 6, 14, 15)$. The components of both the supply vector and the demand vector add up to 60.

The costs are as specified in the following cost matrix:—

$$\begin{pmatrix} 12 & 8 & 9 & 4 & 6 \\ 5 & 10 & 8 & 9 & 5 \\ 6 & 4 & 12 & 12 & 4 \\ 5 & 7 & 12 & 10 & 8 \\ 4 & 6 & 8 & 10 & 12 \\ 7 & 3 & 7 & 12 & 8 \end{pmatrix}.$$

The solution is provided by the following matrix, whose coefficient in the ith row and jth column represents the quantity of the commodity to be transported from the ith supplier to the jth recipient:

$$X = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 4 & 0 & 0 & 1 \\ 7 & 4 & 0 & 5 & 0 \\ 1 & 0 & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

This solution is a basic solution associated with the following basis:

$$\begin{aligned} B &= \{(1,4),\ (2,5),\ (3,2),\ (3,5),\ (4,1),\ (4,2),\\ &\qquad (4,4),\ (5,1),\ (5,3),\ (6,2)\}. \end{aligned}$$

The cost of this basic optimal solution is 318. We have determined values of $u_1, u_2, u_3, u_4, u_5, u_6$ and v_1, v_2, v_3, v_4, v_5 such that the cost $c_{i,j}$ of transporting the commoditity from the ith supplier to the jth recipient satisfies $c_{i,j} = v_j - u_i$ whenever $(i,j) \in B$. These numbers have the following values:—

$$u_1 = 0,\ \ u_2 = -4,\ \ u_3 = -3,\ \ u_4 = -6,\ \ u_5 = -5,\ \ u_6 = -2.$$

$$v_1 = -1,\ \ ,v_1 = 1,\ \ v_2 = 3,\ \ v_4 = 4,\ \ v_5 = 1.$$

Moreover we have determined numbers $q_{i,j}$ such that $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1,2,3,4,5$ and $j = 1,2,3,4$. Let Q be the matrix with $(Q)_{i,j} = q_{i,j}$ for all i and j. Then

$$Q = \begin{pmatrix} 13 & 7 & 6 & 0 & 5 \\ 2 & 5 & 1 & 1 & 0 \\ 4 & 0 & 6 & 5 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \\ 6 & 0 & 2 & 6 & 5 \end{pmatrix}.$$

The manner in which the matrix Q has been constructed ensures that its component $q_{i,j}$ in the ith row and jth column satisfies $q_{i,j} = 0$ whenever $(i,j) \in B$. Thus the matrix Q must have at least ten coefficients equal to zero. In fact there is an eleventh coefficient equal to zero, since $q_{5,2} = 0$, though $(5,2) \notin B$. The significance of this is that this particular transportation problem has a second basic optimal solution.

To find this second optimal solution, we determine a 6×5 matrix Y, with coefficient $y_{i,j}$ in the ith row and jth column, where this matrix Y satisfies the following conditions:—

- $y_{5,2} = 1$;
- $y_{i,j} = 0$ when $(i,j) \notin B \cup \{(5,2)\}$;
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1		2		3		4		5		
1							?	$\bullet$			0
2									?	$\bullet$	0
3			?	$\bullet$					?	$\bullet$	0
4	?	$\bullet$	?	$\bullet$			?	$\bullet$			0
5	?	$\bullet$	1	$\circ$	?	$\bullet$					0
6			?	$\bullet$							0
	0		0		0		0		0		0

The completed tableau is as follows:—

$y_{i,j}$	1		2		3		4		5		
1							0	$\bullet$			0
2									0	$\bullet$	0
3			0	$\bullet$					0	$\bullet$	0
4	1	$\bullet$	-1	$\bullet$			0	$\bullet$			0
5	-1	$\bullet$	1	$\circ$	0	$\bullet$					0
6			0	$\bullet$							0
	0		0		0		0		0		0

Then $X + \lambda Y$ is an optimal solution for $0 \leq \lambda \leq 1$, where

$$X + \lambda Y = \begin{pmatrix} 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 14 \\ 0 & 4 & 0 & 0 & 1 \\ 7+\lambda & 4-\lambda & 0 & 5 & 0 \\ 1-\lambda & \lambda & 6 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

The fact that all these solutions are optimal stems from the fact that the costs satisfy $c_{4,1} + c_{5,2} = c_{5,1} + c_{4,2}$. Indeed $c_{4,1} + c_{5,2} = 5 + 6 = 11$ and $c_{5,1} + c_{4,2} = 4 + 7 = 11$.