Module MA3484: Methods of Mathematical Economics
Hilary Term 2015

D. R. Wilkins

Contents

3 The Transportation Problem

3.1 Transportation in the Dairy Industry

We discuss an example of the Transportation Problem of Linear Programming, as it might be applied to optimize transportation costs in the dairy industry.

A food business has milk-processing plants located in various towns in a small country. We shall refer to these plants as *dairies*. Raw milk is supplied by numerous farmers with farms located throughout that country, and is transported by milk tanker from the farms to the dairies. The problem is to determine the catchment areas of the dairies so as to minimize transport costs.

We suppose that there are m farms, labelled by integers from 1 to m that supply milk to n dairies, labelled by integers from 1 to n. Suppose that, in a given year, the ith farm has the capacity to produce and supply a s_i litres of milk for $i = 1, 2, \ldots, m$, and that the jth dairy needs to receive at least d_j litres of milk for $j = 1, 2, \ldots, n$ to satisfy the business obligations.

The quantity $\sum_{i=1}^{m} s_i$ then represents that *total supply* of milk, and the quantity $\sum_{j=1}^{n} d_j$ represents the *total demand* for milk.

We suppose that $x_{i,j}$ litres of milk are to be transported from the ith farm to the jth dairy, and that $c_{i,j}$ represents the cost per litre of transporting this milk.

Then the total cost of transporting milk from the farms to the dairies is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

The quantities $x_{i,j}$ of milk to be transported from the farms to the dairies should then be determined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ so as to minimize the total cost of transporting milk.

However the ith farm can supply no more than s_i litres of milk in a given year, and that jth dairy requires at least d_j litres of milk in that year. It follows that the quantities $x_{i,j}$ of milk to be transported between farms and dairy are constrained by the requirements that

$$\sum_{j=1}^{n} x_{i,j} \le s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} \geq d_j \quad \text{for } j = 1, 2, \ldots, m.$$

3.2 The General Transportation Problem

The Transportation Problem can be expressed generally in the following form. Some commodity is supplied by m suppliers and is transported from those suppliers to n recipients. The ith supplier can supply at most to s_i units of the commodity, and the jth recipient requires at least d_j units of the commodity. The cost of transporting a unit of the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

The total transport cost is then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

where $x_{i,j}$ denote the number of units of the commodity transported from the ith supplier to the jth recipient.

The Transportation Problem can then be presented as follows:

determine $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$ *so as minimize* $\sum_{i,j} c_{i,j} x_{i,j}$ *subject to the constraints* $x_{i,j} \geq 0$ *for all* i *and* j, $\sum_{j=1}^{n} x_{i,j} \leq s_i$ *and* $\sum_{i=1}^{m} x_{i,j} \geq d_j$, *where* $s_i \geq 0$ *for all* i, $d_j \geq 0$ *for all* i, *and* $\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$.

3.3 Transportation Problems in which Total Supply equals Total Demand

Consider an instance of the Transportation Problem with m suppliers and n recipients. The following proposition shows that a solution to the Transportation Problem can only exist if total supply of the relevant commodity exceeds total demand for that commodity.

Proposition 3.1 *Let* $s_1, s_2, \ldots, s_m$ *and* $d_1, d_2, \ldots, d_n$ *be non-negative real numbers. Suppose that there exist non-negative real numbers* $x_{i,j}$ *be for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$ *that satisfy the inequalities*

$$\sum_{j=1}^{n} x_{i,j} \leq s_i \quad \textit{and} \quad \sum_{i=1}^{m} x_{i,j} \geq d_j.$$

Then

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} s_i.$$

Moreover if it is the case that

$$\sum_{j=1}^{n} d_j = \sum_{i=1}^{m} s_i.$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \textit{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \textit{for } j = 1, 2, \ldots, n.$$

Proof The inequalities satisfied by the non-negative real numbers $x_{i,j}$ ensure that

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \leq \sum_{i=1}^{m} s_i.$$

Thus the total supply must equal or exceed the total demand.

If it is the case that $\sum\limits_{j=1}^{n} x_{i,j} < s_i$ for at least one value of i then $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} x_{i,j} < \sum_{i=1}^{m} s_i$. Similarly if it is the case that $\sum\limits_{i=1}^{m} x_{i,j} > d_j$ for at least one value of j then $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} x_{i,j} > \sum_{j=1}^{n} d_j$.

It follows that if total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n,$$

as required. ∎

We analyse the Transportation Problem in the case where total supply equals total demand. The optimization problem in this case can then be stated as follows:—

> *determine* $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$ *so as minimize* $\sum_{i,j} c_{i,j}x_{i,j}$ *subject to the constraints* $x_{i,j} \geq 0$ *for all* i *and* j*,* $\sum_{j=1}^{n} x_{i,j} = s_i$ *and* $\sum_{i=1}^{m} x_{i,j} = d_j$*, where* $s_i \geq 0$ *and* $d_j \geq 0$ *for all* i *and* j*, and* $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.

Definition A *feasible* solution to the Transportation Problem (with equality of total supply and total demand) takes the form of real numbers $x_{i,j}$, where

- $x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;
- $\sum_{j=1}^{n} x_{i,j} = s_i$;
- $\sum_{i=1}^{m} x_{i,j} = d_j$.

Definition A feasible solution $(x_{i,j})$ of the Transportation Problem is said to be *optimal* if it minimizes cost amongst all feasible solutions of the Transportation Problem.

3.4 Row Sums and Column Sums of Matrices

We commence the analysis of the Transportation Problem by studying the interrelationships between the various real vector spaces and linear transformations that arise naturally from the statement of the Transportation Problem.

The quantities $x_{i,j}$ to be determined are coefficients of an $m \times n$ matrix X. This matrix X is represented as an element of the real vector space $M_{m,n}(\mathbb{R})$ that consists of all $m \times n$ matrices with real coefficients.

The non-negative quantities $s_1, s_2, \ldots, s_m$ that specify the sums of the coefficients in the rows of the unknown matrix X are the components of a *supply vector* $\mathbf{s}$ belonging to the m-dimensional real vector space $\mathbb{R}^m$.

Similarly the non-negative quantities $d_1, d_2, \ldots, d_n$ that specify the sums of the coefficients in the columns of the unknown matrix X are the components of a *demand vector* $\mathbf{d}$ belonging to the n-dimensional space $\mathbb{R}^n$.

The requirement that total supply equals total demand translates into a requirement that the sum $\sum_{i=1}^{m} (\mathbf{s})_i$ of the components of the supply vector $\mathbf{s}$ must equal the sum $\sum_{j=1}^{n} (\mathbf{d})_j$ of the components of the demand vector $\mathbf{d}$.

Accordingly we introduce a real vector space W consisting of all ordered pairs $(\mathbf{y}, \mathbf{z})$ for which $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{z} \in \mathbb{R}^n$ and $\sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j$.

Lemma 3.2 *Let m and n be positive integers, and let*

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

Then the dimension of the real vector space W is $m + n - 1$. ∎

Proof It is easy to see that the vector space W is isomorphic to $\mathbb{R}^m$ when $n = 1$. The result then follows directly in the case when $n = 1$. Thus suppose that $n > 1$.

Given real numbers $y_1, y_2, \ldots, y_n$ and $z_1, z_2, \ldots, z_{n-1}$, there exists exactly one element $(\mathbf{y}, \mathbf{z})$ of W that satisfies $(\mathbf{y})_i = y_i$ for $i = 1, 2, \ldots, m$ and $(\mathbf{z})_j = z_j$ for $j = 1, 2, \ldots, n - 1$. The remaining component $(\mathbf{z})_n$ of the n-dimensional vector $\mathbf{z}$ is then determined by the equation

$$(\mathbf{z})_n = \sum_{i=1}^{m} y_i - \sum_{j=1}^{m-1} z_j.$$

It follows from this that $\dim W = m + n - 1$, as required. ∎

The supply and demand constraints on the sums of the rows and columns of the unknown matrix X can then be specified by means of linear transformations

$$\rho: M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$$

and

$$\sigma: M_{m,n}(\mathbb{R}) \to \mathbb{R}^n,$$

where, for each $X \in M_{m,n}(\mathbb{R})$, the components of the m-dimensional vector $\rho(X)$ are the sums of the coefficients along each row of X, and the components of the n-dimensional vector $\sigma(X)$ are the sums of the coefficients along each column of X.

Let $X \in M_{m,n}(\mathbb{R})$. Then the ith component $\rho(X)_i$ of the vector $\rho(X)$ is determined by the equation

$$\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j} \quad \text{for } i = 1, 2, \ldots, m,$$

for $i = 1, 2, \ldots, m$, and the jth component $\sigma(X)_j$ of $\sigma(X)$ is determined by the equation

$$\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j} \quad \text{for } j = 1, 2, \ldots, n.$$

for $j = 1, 2, \ldots, n$.

The costs $c_{i,j}$ are the components of an $m \times n$ matrix C, the *cost matrix*, that in turn determines a linear functional

$$f\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}$$

on the vector space $M_{m,n}(\mathbb{R})$ defined such that

$$f(X) = \operatorname{trace}(C^T X) = \sum_{i=1}^{m} \sum_{j=1}^{n} (C)_{i,j} X_{i,j}$$

for all $X \in M_{m,n}(\mathbb{R})$.

An instance of the problem is specified by specifying a supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C. The components of $\mathbf{s}$ and $\mathbf{d}$ are required to be non-negative real numbers. Moreover $(\mathbf{s}, \mathbf{d}) \in W$, where W is the real vector space consisting of all ordered pairs $(\mathbf{s}, \mathbf{d})$ with $\mathbf{s} \in \mathbb{R}^m$ and $\mathbf{d} \in \mathbb{R}^n$ for which the sum of the components of the vector $\mathbf{s}$ equals the sum of the components of the vector $\mathbf{d}$.

A feasible solution of the Transportation Problem with given supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C is represented by an $m \times n$ matrix X satisfying the following three conditions:—

- The coefficients of X are all non-negative;
- $\rho(X) = \mathbf{s}$;
- $\sigma(X) = \mathbf{d}$.

The cost functional $f\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}$ is defined so that

$$f(X) = \operatorname{trace}(C^T X)$$

for all $X \in M_{m,n}(\mathbb{R})$.

A feasible solution X of the Transportation problem is optimal if and only if $f(X) \le f(\overline{X})$ for all feasible solutions $\overline{X}$ of that problem.

Lemma 3.3 *Let $M_{m,n}(\mathbb{R})$ be the real vector space consisting of all $m \times n$ matrices with real coefficients, let $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ be the linear transformations defined so that the ith component $\rho(X)_i$ of $\rho(X)$ satisfies*

$$\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j} \quad \textit{for } i = 1, 2, \ldots, m,$$

for $i = 1, 2, \ldots, m$, and the jth component $\sigma(X)_j$ of $\sigma(X)$ satisfies

$$\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j} \quad \textit{for } j = 1, 2, \ldots, n.$$

for $j = 1, 2, \ldots, n$. Then $(\rho(X), \sigma(X)) \in W$ for all $X \in M_{m,n}(\mathbb{R})$, where

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

Proof Let $X \in M_{m,n}(\mathbb{R})$. Then

$$\sum_{i=1}^{m} \rho(X)_i = \sum_{i=1}^{m} \sum_{j=1}^{n} (X)_{i,j} = \sum_{j=1}^{n} \sigma(X)_j.$$

It follows that $(\rho(X), \sigma(X)) \in W$ for all $X \in M_{m,n}(\mathbb{R})$, as required. ∎

3.5 Bases for the Transportation Problem

The real vector space $M_{m,n}(\mathbb{R})$ consisting of all $m \times n$ matrices with real coefficients has a natural basis consisting of the matrices $E^{(i,j)}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where, for each i and j, the coefficient of the matrix $E^{(i,j)}$ in the ith row and jth column has the value 1, and all other coefficients are zero. Indeed

$$X = \sum_{i=1}^{m} \sum_{j=1}^{n} (X)_{i,j} E^{(i,j)}$$

for all $X \in M_{m,n}(\mathbb{R})$.

Let $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ be the linear transformations defined such that $(\rho(X))_i = \sum\limits_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum\limits_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$. Then $\rho(E^{(i,j)}) = \overline{\mathbf{b}}^{(i)}$ for $i = 1, 2, \ldots, m$, where $\overline{\mathbf{b}}^{(i)}$ denotes the ith vector in the standard basis of $\mathbb{R}^m$, defined such that

$$(\overline{\mathbf{b}}^{(i)})_k = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

Similarly $\sigma(E^{(i,j)}) = \mathbf{b}^{(j)}$ for $j = 1, 2, \ldots, n$, where $\mathbf{b}^{(j)}$ denotes the jth vector in the standard basis of $\mathbb{R}^n$, defined such that

$$(\mathbf{b}^{(j)})_l = \begin{cases} 1 & \text{if } j = l; \\ 0 & \text{if } j \neq l. \end{cases}$$

Now $(\rho(X), \sigma(X)) \in W$ for all $X \in M_{m,n}(\mathbb{R})$, where

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let

$$\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$$

for all $(i, j) \in I \times J$. Then the elements $\beta^{(i,j)}$ span the vector space W. It follows from basic linear algebra that there exist subsets B of $I \times J$ such that the elements $\beta^{(i,j)}$ of W for which $(i, j) \in B$ constitute a basis of the real vector space W (see Corollary 2.3).

Definition Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where m and n are positive integers, let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\},$$

and, for each $(i, j) \in I \times J$, let $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$, where $\overline{\mathbf{b}}^{(i)} \in \mathbb{R}^m$ and $\mathbf{b}^{(j)} \in \mathbb{R}^n$ are defined so that the ith component of $\overline{\mathbf{b}}^{(i)}$ and that jth component of $\mathbf{b}^{(j)}$ are equal to 1 and the other components of these vectors are zero. A subset B of $I \times J$ is said to be a *basis* for the Transportation Problem with m suppliers and n recipients if and only if the elements $\beta^{(i,j)}$ for which $(i, j) \in B$ constitute a basis of the real vector space W.

The real vector space W is of dimension $m+n-1$, where m is the number of suppliers and n is the number of recipients. It follows that any basis for the Transportation Problem with m suppliers and n recipients has $m+n-1$ members.

Proposition 3.4 *Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where m and n are positive integers. Then a subset B of $I \times J$ is a basis for the transportation problem if and only if, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^{m} (\mathbf{y})_m = \sum_{j=1}^{n} (\mathbf{z})_n$, there exists a unique $m \times n$ matrix X with real coefficients satisfying the following properties:—*

(i) $\sum_{j=1}^{n} (X)_{i,j} = (\mathbf{y})_i$ *for* $i = 1, 2, \ldots, m$;

(ii) $\sum_{i=1}^{m} (X)_{i,j} = (\mathbf{z})_j$ *for* $j = 1, 2, \ldots, n$;

(ii) $(X)_{i,j} = 0$ *unless* $(i,j) \in B$.

Proof For each $(i,j) \in I \times J$, let $E^{(i,j)}$ denote the matrix whose coefficient in the ith row and jth column are equal to 1 and whose other coefficients are zero, and let $\rho(X) \in \mathbb{R}^m$ and $\sigma(X) \in \mathbb{R}^n$ be defined for all $m \times n$ matrices X with real coefficients so that $(\rho(X))_i = \sum_{j=1}^{n} (X)_{i,j}$ and $(\sigma(X))_j = \sum_{i=1}^{m} (X)_{i,j}$. Then $\rho(E^{(i,j)}) = \overline{\mathbf{b}}^{(i)}$ for $i = 1, 2, \ldots, m$, where $\overline{\mathbf{b}}^{(i)}$ denotes the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero. Similarly $\sigma(E^{(i,j)}) = \mathbf{b}^{(j)}$ for $j = 1, 2, \ldots, n$, where $\mathbf{b}^{(j)}$ denotes the vector in $\mathbb{R}^m$ whose jth component is equal to 1 and whose other components are zero.

Let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

Then $(\rho(X), \sigma(X)) \in W$ for all $X \in M_{m,n}(\mathbb{R})$, and

$$(\rho(E^{(i,j)}), \sigma(E^{(i,j)}) = \beta^{(i,j)}$$

for all $(i,j) \in I \times J$ where

$$\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)}).$$

Let B be a subset of $I \times J$, let $\mathbf{y}$ and $\mathbf{z}$ be elements of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively that satisfy $(\mathbf{y}, \mathbf{z}) \in W$, and let X be an $m \times n$ matrix with real coefficients with the property that $(X)_{i,j} = 0$ unless $(i,j) \in B$. Then

$$\rho(X) = \sum_{(i,j) \in B} (X)_{(i,j)} \rho(E^{(i,j)}) = \sum_{(i,j) \in B} (X)_{(i,j)} \overline{\mathbf{b}}^{(i)}$$

and

$$\sigma(X) = \sum_{(i,j) \in B} (X)_{(i,j)} \sigma(E^{(i,j)}) = \sum_{(i,j) \in B} (X)_{(i,j)} \mathbf{b}^{(j)},$$

and therefore

$$(\rho(X), \sigma(X)) = \sum_{(i,j) \in B} (X)_{(i,j)} \beta^{(i,j)}.$$

Suppose that, given any $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ for which $(\mathbf{y}, \mathbf{z}) \in W$, there exists a unique $m \times n$ matrix X such that $\mathbf{y} = \rho(X)$, $\mathbf{z} = \sigma(X)$ and $(X)_{i,j} = 0$ for all $(i, j) \in B$. Then the elements $\beta^{(i,j)}$ of W for which $(i, j) \in B$ must span W and must also be linearly independent. These elements must therefore constitute a basis for the vector space B. It then follows that the subset B of $I \times J$ must be a basis for the Transportation Problem.

Conversely if B is a basis for the Transportation Problem then, given any $(\mathbf{y}, \mathbf{z}) \in W$, there must exist a unique $m \times n$ matrix X with real coefficients such that $(\mathbf{y}, \mathbf{z}) = \sum_{(i,j)\in B} (X)_{i,j}\beta^{(i,j)}$ and $(X)_{i,j} = 0$ unless $(i, j) \in B$. The result follows. ∎

Lemma 3.5 *Let m and n be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let K be a subset of $I \times J$. Suppose that there is no basis B of the Transportation Problem for which $K \subset B$. Then there exists a non-zero $m \times n$ matrix Y with real coefficients which satisfies the following conditions:*

- $\sum_{j=1}^{n} (Y)_{i,j} = 0$ *for* $i = 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} (Y)_{i,j} = 0$ *for* $j = 1, 2, \ldots, n$;
- $(Y)_{i,j} = 0$ *when* $(i, j) \notin K$.

Proof Let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\},$$

let $\overline{\mathbf{b}}^{(1)}, \overline{\mathbf{b}}^{(2)}, \ldots, \overline{\mathbf{b}}^{(m)}$ be the standard basis of $\mathbb{R}^m$ and let $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(n)}$ be the standard basis of $\mathbb{R}^n$, where the ith component of $\overline{\mathbf{b}}^{(i)}$ and the jth component of $\mathbf{b}^{(j)}$ are equal to 1 and the other components of these vectors are zero, and let $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$ for all $(i, j) \in I \times J$.

Now follows from Proposition 2.2 that if the elements $\beta^{(i,j)}$ for which $(i, j) \in K$ were linearly independent then there would exist a subset B of $I \times J$ satisfying $K \subset B$ such that the elements $\beta^{(i,j)}$ for which $(i, j) \in B$ would constitute a basis of W. This subset B of $I \times J$ would then be a basis for the Transportation Problem. But the subset K is not contained in any basis for the Transportation Problem. It follows that the elements $\beta^{(i,j)}$ for which $(i, j) \in K$ must be linearly dependent. Therefore there exists a non-zero $m \times n$ matrix Y with real coefficients such that $(Y)_{i,j} = 0$ when $(i,j) \notin K$ and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \beta^{(i,j)} = \mathbf{0}_W.$$

Now $\beta^{(i,j)} = (\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)})$ for all $i \in I$ and $j \in J$. It follows that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \overline{\mathbf{b}}^{(i)} = \mathbf{0}$$

and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \mathbf{b}^{(j)} = \mathbf{0},$$

and therefore

$$\sum_{j=1}^{n} (Y)_{i,j} = 0 \quad (i = 1, 2, \ldots, m)$$

and

$$\sum_{i=1}^{m} (Y)_{i,j} = 0 \quad (j = 1, 2, \ldots, n),$$

as required. ∎

3.6 Basic Feasible Solutions of Transportation Problems

Consider the Transportation Problem with m suppliers and n recipients, where the ith supplier can provide at most s_i units of some given commodity, where $s_i \geq 0$, and the jth recipient requires at least d_j units of that commodity, where $d_j \geq 0$. We suppose also that total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

The cost of transporting the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

The concept of a *basis* for the Transportation Problem was introduced in Subsection 3.5. We recall some results established in that subsection.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. A subset B of $I \times J$ is a basis for the Transportation Problem if and only if, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and

$\mathbf{z} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j$, there exists a unique matrix X with real coefficients such that $\sum_{j=1}^{n} (X)_{i,j} = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} (X)_{i,j} = (\mathbf{z})_j$ for $j = 1, 2, \ldots, n$ and $(X)_{i,j} = 0$ unless $(i, j) \in B$ (see Proposition 3.4). A basis for the transportation problem has $m + n - 1$ elements.

Also if K is a subset of $I \times J$ that is not contained in any basis for the Transportation Problem then there exists a non-zero $m \times n$ matrix Y such that $\sum_{j=1}^{n} (Y)_{i,j} = 0$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} (X)_{i,j} = 0$ for $j = 1, 2, \ldots, n$ and $(Y)_{i,j} = 0$ unless $(i, j) \in K$ (see Lemma 3.5).

Definition A feasible solution $(x_{i,j})$ of a Transportation Problem is said to be *basic* if there exists a basis B for that Transportation Problem such that $x_{i,j} = 0$ whenever $(i, j) \notin B$.

Example Consider the instance of the Transportation Problem where $m = n = 2$, $s_1 = 8$, $s_2 = 3$, $d_1 = 2$, $d_2 = 9$, $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$.

A feasible solution takes the form of a 2×2 matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

with non-negative components which satisfies the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}.$$

A basic feasible solution will have at least one component equal to zero. There are four matrices with at least one zero component which satisfy the required equations. They are the following:—

$$\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 8 & 0 \\ -6 & 9 \end{pmatrix}, \quad \begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 9 \\ 3 & 0 \end{pmatrix}.$$

The first and third of these matrices have non-negative components. These two matrices represent basic feasible solutions to the problem, and moreover they are the only basic feasible solutions.

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$.

The cost of the basic feasible solution $\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}$ is

$$8c_{1,2} + 2c_{2,1} + c_{2,2} = 24 + 8 + 1 = 33.$$

The cost of the basic feasible solution $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is

$$2c_{1,1} + 6c_{1,2} + 3c_{2,2} = 4 + 18 + 3 = 25.$$

Now any 2×2 matrix $\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ satisfying the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}$$

must be of the form

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} \lambda & 8 - \lambda \\ 2 - \lambda & 1 + \lambda \end{pmatrix}$$

for some real number λ.

But the matrix $\begin{pmatrix} \lambda & 8 - \lambda \\ 2 - \lambda & 1 + \lambda \end{pmatrix}$ has non-negative components if and only if $0 \leq \lambda \leq 2$. It follows that the set of feasible solutions of this instance of the transportation problem is

$$\left\{ \begin{pmatrix} \lambda & 8 - \lambda \\ 2 - \lambda & 1 + \lambda \end{pmatrix} : \lambda \in \mathbb{R} \text{ and } 0 \leq \lambda \leq 2 \right\}.$$

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$. Therefore, for each real number λ satisfying $0 \leq \lambda \leq 2$, the cost $f(\lambda)$ of the feasible solution $\begin{pmatrix} \lambda & 8 - \lambda \\ 2 - \lambda & 1 + \lambda \end{pmatrix}$ is given by

$$f(\lambda) = 2\lambda + 3(8 - \lambda) + 4(2 - \lambda) + (1 + \lambda) = 33 - 4\lambda.$$

Cost is minimized when $\lambda = 2$, and thus $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is the optimal solution of this instance of the Transportation Problem. The cost of this optimal solution is 25.

Proposition 3.6 *Given any feasible solution of the Transportation Problem, there exists a basic feasible solution with whose cost does not exceed that of the given solution.*

Proof Let m and n be positive integers, and let $\mathbf{s}$ and $\mathbf{d}$ be elements of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively that satisfy $(\mathbf{s})_i \geq 0$ for $i = 1, 2, \ldots, m$, $(\mathbf{d})_i \geq 0$ for $j = 1, 2, \ldots, n$ and $\sum_{i=1}^{m} (\mathbf{s})_i = \sum_{j=1}^{n} (\mathbf{d})_j$, let C be an $m \times n$ matrix whose components are non-negative real numbers, and let X be a feasible solution of the resulting instance of the Transportation Problem with cost matrix C.

Let $s_i = (\mathbf{s})_i$, $d_j = (\mathbf{d})_j$, $x_{i,j} = (X)_{i,j}$ and $c_{i,j} = (C)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then $x_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$. The cost of the feasible solution X is then $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$.

If the feasible solution X is itself basic then there is nothing to prove. Suppose therefore that X is not a basic solution. We show that there then exists a feasible solution $\overline{X}$ with fewer non-zero components than the given feasible solution.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let

$$K = \{(i, j) \in I \times J : x_{i,j} > 0\}.$$

Because X is not a basic solution to the Transportation Problem, there does not exist any basis B for the Transportation Problem satisfying $K \subset B$. It therefore follows from Lemma 3.5 that there exists a non-zero $m \times n$ matrix Y which satisfies the following conditions:—

- $\sum_{j=1}^{n} (Y)_{i,j} = 0$ for $i = 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} (Y)_{i,j} = 0$ for $j = 1, 2, \ldots, n$;
- $(Y)_{i,j} = 0$ when $(i, j) \notin K$.

We can assume without loss of generality that $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} (Y)_{i,j} \geq 0$, because otherwise we can replace Y with $-Y$.

Let $Z_\lambda = X - \lambda Y$ for all real numbers λ. Then $(Z_\lambda)_{i,j} = x_{i,j} - \lambda y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where $x_{i,j} = (X)_{i,j}$ and $y_{i,j} = (Y)_{i,j}$.

Moreover the matrix Z_λ has the following properties:—

- $\sum_{j=1}^{n} (Z_\lambda)_{i,j} = s_i$;
- $\sum_{i=1}^{m} (Z_\lambda)_{i,j} = d_j$;
- $(Z_\lambda)_{i,j} = 0$ whenever $(i,j) \notin K$;
- $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j}(Z_\lambda)_{i,j} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j}(X)_{i,j}$ whenever $\lambda \geq 0$.

Now the matrix Y is a non-zero matrix whose rows and columns all sum to zero. It follows that at least one of its coefficients must be strictly positive. Thus there exists at least one ordered pair (i,j) belonging to the set K for which $y_{i,j} > 0$. Let

$$\lambda_0 = \text{minimum} \left\{ \frac{x_{i,j}}{y_{i,j}} : (i,j) \in K \text{ and } y_{i,j} > 0 \right\}.$$

Then $\lambda_0 > 0$. Moreover if $0 \leq \lambda < \lambda_0$ then $x_{i,j} - \lambda y_{i,j} > 0$ for all $(i,j) \in K$, and if $\lambda > \lambda_0$ then there exists at least one element (i_0, j_0) of K for which $x_{i_0,j_0} - \lambda y_{i_0,j_0} < 0$. It follows that $x_{i,j} - \lambda_0 y_{i,j} \geq 0$ for all $(i,j) \in K$, and $x_{i_0,j_0} - \lambda_0 y_{i_0,j_0} = 0$.

Thus Z_{λ_0} is a feasible solution of the given Transportation Problem whose cost does not exceed that of the given feasible solution X. Moreover Z_{λ_0} has fewer non-zero components than the given feasible solution X.

If Z_{λ_0} is itself a basic feasible solution, then we have found the required basic feasible solution whose cost does not exceed that of the given feasible solution. Otherwise we can iterate the process until we arrive at the required basic feasible solution whose cost does not exceed that of the given feasible solution. ∎

A given instance of the Transportation Problem has only finitely many basic feasible solutions. Indeed there are only finitely many bases for the problem, and any basis is associated with at most one basic feasible solution. Therefore there exists a basic feasible solution whose cost does not exceed the cost of any other basic feasible solution. It then follows from Proposition 3.6 that the cost of this basic feasible solution cannot exceed the cost of any other feasible solution of the given instance of the Transportation Problem. This basic feasible solution is thus a basic optimal solution of the Transportation Problem.

The Transportation Problem determined by the supply vector, demand vector and cost matrix has only finitely many basic feasible solutions, because there are only finitely many bases for the problem, and each basis can determine at most one basic feasible solution. Nevertheless the number of basic feasible solutions may be quite large.

But it can be shown that the Transportation Problem always has a basic optimal solution. It can be found using an algorithm that implements the Simplex Method devised by George B. Dantzig in the 1940s. This algorithm involves passing from one basis to another, lowering the cost at each stage, until one eventually finds a basis that can be shown to determine a basic optimal solution of the Transportation Problem.

3.7 An Example illustrating the Procedure for finding an Initial Basic Feasible Solution to a Transportation Problem using the Minimum Cost Method

We discuss the method for finding a basic optimal solution of the Transportation Problem by working through a particular example. First we find an initial basic feasible solution using a method known as the *Minimum Cost Method*. Then we test whether or not this initial basic feasible solution is optimal. It turns out that, in this example, the initial basic solutions is not optimal. We then commence an iterative process for finding a basic optimal solution.

Let $c_{i,j}$ be the coefficient in the ith row and jth column of the cost matrix C, where

$$C = \begin{pmatrix} 8 & 4 & 16 \\ 3 & 7 & 2 \\ 13 & 8 & 6 \\ 5 & 7 & 8 \end{pmatrix}.$$

and let

$$s_1 = 13, \quad s_2 = 8, \quad s_3 = 11, \quad s_4 = 13,$$

$$d_1 = 19, \quad d_2 = 12, \quad d_3 = 14.$$

We seek to non-negative real numbers $x_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ that minimize $\sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} x_{i,j}$ subject to the following constraints:

$$\sum_{j=1}^{3} x_{i,j} = s_i \quad \text{for} \quad i = 1, 2, 3, 4,$$

$$\sum_{i=1}^{4} x_{i,j} = d_j \quad \text{for} \quad j = 1, 2, 3,$$

and $x_{i,j} \geq 0$ for all i and j.

For this problem the supply vector is $(13, 8, 11, 13)$ and the demand vector is $(19, 12, 14)$. The components of both the supply vector and the demand vector add up to 45.

In order to start the process of finding an initial basic solution for this problems, we set up a tableau that records the row sums (or supplies), the column sums (or demands) and the costs $c_{i,j}$ for the given problem, whilst leaving cells to be filled in with the values of the non-negative real numbers $x_{i,j}$ that will specify the initial basic feasible solution. The resultant tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 ?	4 ?	16 ?	13
2	3 ?	7 ?	2 ?	8
3	13 ?	8 ?	6 ?	11
4	5 ?	7 ?	8 ?	13
d_j	19	12	14	45

We apply the minimum cost method to find an initial basic solution.

The cell with lowest cost is the cell $(2, 3)$. We assign to this cell the maximum value possible, which is the minimum of s_2, which is 8, and d_3, which is 14. Thus we set $x_{2,3} = 8$. This forces $x_{2,1} = 0$ and $x_{2,2} = 0$. The pair $(2, 3)$ is added to the current basis.

The next undetermined cell of lowest cost is $(1, 2)$. We assign to this cell the minimum of s_1, which is 13, and $d_2 - x_{2,2}$, which is 12. Thus we set $x_{1,2} = 12$. This forces $x_{3,2} = 0$ and $x_{4,2} = 0$. The pair $(1, 2)$ is added to the current basis.

The next undetermined cell of lowest cost is $(4, 1)$. We assign to this cell the minimum of $s_4 - x_{4,2}$, which is 13, and $d_1 - x_{2,1}$, which is 19. Thus we set $x_{4,1} = 13$. This forces $x_{4,3} = 0$. The pair $(4, 1)$ is added to the current basis.

The next undetermined cell of lowest cost is $(3, 3)$. We assign to this cell the minimum of $s_3 - x_{3,2}$, which is 11, and $d_3 - x_{2,3} - x_{4,3}$, which is 6 $(= 14 - 8 - 0)$. Thus we set $x_{3,3} = 6$. This forces $x_{1,3} = 0$. The pair $(3, 3)$ is added to the current basis.

The next undetermined cell of lowest cost is $(1, 1)$. We assign to this cell the minimum of $s_1 - x_{1,2} - x_{1,3}$, which is 1, and $d_1 - x_{2,1} - x_{4,1}$, which is 6. Thus we set $x_{1,1} = 1$. The pair $(1, 1)$ is added to the current basis.

The final undetermined cell is $(3, 1)$. We assign to this cell the common value of $s_3 - x_{3,2} - x_{3,3}$ and $d_1 - x_{1,1} - x_{2,1} - x_{4,1}$, which is 5. Thus we set $x_{3,1} = 5$. The pair $(3, 1)$ is added to the current basis.

The values of the elements $x_{i,j}$ of the initial basic feasible solution are tabulated (with basis elements marked by the • symbol) as follows:—

$c_{i,j} \searrow x_{i,j}$	1		2		3		s_i
1	8	•	4	•	16		
		1		12		0	13
2	3		7		2	•	
		0		0		8	8
3	13	•	8		6	•	
		5		0		6	11
4	5	•	7		8		
		13		0		0	13
d_j		19		12		14	45

Thus the initial basis is B where

$$B = \{(1, 1),\ (1, 2),\ (2, 3),\ (3, 1),\ (3, 3),\ (4, 1)\}.$$

The basic feasible solution is represented by the 6×5 matrix X, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 0 & 0 & 8 \\ 5 & 0 & 6 \\ 13 & 0 & 0 \end{pmatrix}.$$

The cost of this initial feasible basic solution is

$$\begin{aligned} & 8 \times 1 + 4 \times 12 + 2 \times 8 + 13 \times 5 + 6 \times 6 \\ & \qquad + 5 \times 13 \\ = \; & 8 + 48 + 16 + 65 + 36 + 65 \\ = \; & 238. \end{aligned}$$

3.8 An Example illustrating the Procedure for finding a Basic Optimal Solution to a Transportation Problem

We continue with the study of the optimization problem discussed in the previous section.

We seek to determine non-negative real numbers $x_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ that minimize $\sum_{i=1}^{4}\sum_{j=1}^{3} c_{i,j}x_{i,j}$, where $c_{i,j}$ is the coefficient in the ith row and jth column of the cost matrix C, where

$$C = \begin{pmatrix} 8 & 4 & 16 \\ 3 & 7 & 2 \\ 13 & 8 & 6 \\ 5 & 7 & 8 \end{pmatrix}.$$

subject to the constraints

$$\sum_{j=1}^{3} x_{i,j} = s_i \quad (i = 1, 2, 3, 4)$$

and

$$\sum_{i=1}^{4} x_{i,j} = d_j \quad (j = 1, 2, 3),$$

where

$$s_1 = 13, \quad s_2 = 8, \quad s_3 = 11, \quad s_4 = 13,$$

$$d_1 = 19, \quad d_2 = 12, \quad d_3 = 14.$$

We have found an initial basic feasible solution by the Minimum Cost Method. This solution satisfies $x_{i,j} = (X)_{i,j}$ for all i and j, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 0 & 0 & 8 \\ 5 & 0 & 6 \\ 13 & 0 & 0 \end{pmatrix}.$$

We next determine whether this initial basic feasible solution is an optimal solution, and, if not, how to adjust the basis to obtain a solution of lower cost.

We determine u_1, u_2, u_3, u_4 and v_1, v_2, v_3 such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$, where B is the initial basis.

We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1		2		3		u_i
1	8	•	4	•	16		0
		0		0		?	
2	3		7		2	•	?
		?		?		0	
3	13	•	8		6	•	?
		0		?		0	
4	5	•	7		8		?
		0		?		?	
v_j	?		?		?		

Now $u_1 = 0$, $(1,1) \in B$ and $(1,2) \in B$ force $v_1 = 8$ and $v_2 = 4$.
Then $v_1 = 8$, $(3,1) \in B$ and $(4,1) \in B$ force $u_3 = -5$ and $u_4 = 3$.
Then $u_3 = -5$ and $(3,3) \in B$ force $v_3 = 1$.
Then $v_3 = 1$ and $(2,3) \in B$ force $u_2 = -1$.
After entering the numbers u_i and v_j, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1		2		3		u_i
1	8	•	4	•	16		0
		0		0		?	
2	3		7		2	•	-1
		?		?		0	
3	13	•	8		6	•	-5
		0		?		0	
4	5	•	7		8		3
		0		?		?	
v_j	8		4		1		

Computing the numbers $q_{i,j}$ such that $c_{i,j} + u_i = v_j + q_{i,j}$, we find that $q_{1,3} = 15$, $q_{2,1} = -6$, $q_{2,2} = 2$, $q_{3,2} = -1$, $q_{4,2} = 6$ and $q_{4,3} = 10$.

The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 $\bullet$	4 $\bullet$	16	0
	0	0	15	
2	3	7	2 $\bullet$	-1
	-6	2	0	
3	13 $\bullet$	8	6 $\bullet$	-5
	0	-1	0	
4	5 $\bullet$	7	8	3
	0	6	10	
v_j	8	4	1	

The initial basic feasible solution is not optimal because some of the quantities $q_{i,j}$ are negative. To see this, suppose that the numbers $\overline{x}_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ constitute a feasible solution to the given problem. Then $\sum_{j=1}^{3} \overline{x}_{i,j} = s_i$ for $i = 1, 2, 3$ and $\sum_{i=1}^{4} \overline{x}_{i,j} = d_j$ for $j = 1, 2, 3, 4$. It follows that

$$\begin{aligned} \sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} \overline{x}_{i,j} &= \sum_{i=1}^{4} \sum_{j=1}^{3} (v_j - u_i + q_{i,j}) \overline{x}_{i,j} \\ &= \sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i + \sum_{i=1}^{4} \sum_{j=1}^{3} q_{i,j} \overline{x}_{i,j}. \end{aligned}$$

Applying this identity to the initial basic feasible solution, we find that $\sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i = 238$, given that 238 is the cost of the initial basic feasible solution. Thus the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ satisfies

$$\overline{C} = 238 + 15\overline{x}_{1,3} - 6\overline{x}_{2,1} + 2\overline{x}_{2,2} - \overline{x}_{3,2} + 6\overline{x}_{4,2} + 10\overline{x}_{4,3}.$$

One could construct feasible solutions with $\overline{x}_{2,1} < 0$ and $\overline{x}_{i,j} = 0$ for $(i, j) \notin B \cup \{(2, 1)\}$, and the cost of such feasible solutions would be lower than that of the initial basic solution. We therefore seek to bring $(2, 1)$ into the basis, removing some other element of the basis to ensure that the new basis corresponds to a feasible basic solution.

The procedure for achieving this requires us to determine a 4×3 matrix Y satisfying the following conditions:—

- $y_{2,1} = 1$;
- $y_{i,j} = 0$ when $(i, j) \notin B \cup \{(2, 1)\}$;

- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1	2	3	
1	? •	? •		0
2	1 ◦		? •	0
3	? •		? •	0
4	? •			0
	0	0	0	0

The constraints that $y_{2,1} = 1$, $y_{i,j} = 0$ when $(i,j) \notin B$ and the constraints requiring the rows and columns to sum to zero determine the values of $y_{i,j}$ for all $y_{i,j} \in B$. These values are recorded in the following tableau:—

$y_{i,j}$	1	2	3	
1	0 •	0 •		0
2	1 ◦		−1 •	0
3	−1 •		1 •	0
4	0 •			0
	0	0	0	0

We now determine those values of λ for which $X+\lambda Y$ is a feasible solution, where

$$X + \lambda Y = \begin{pmatrix} 1 & 12 & 0 \\ \lambda & 0 & 8-\lambda \\ 5-\lambda & 0 & 6+\lambda \\ 13 & 0 & 0 \end{pmatrix}.$$

In order to drive down the cost as far as possible, we should make λ as large as possible, subject to the requirement that all the coefficients of the above matrix should be non-negative numbers. Accordingly we take $\lambda = 5$. Our new basic feasible solution X is then as follows:—

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{pmatrix}.$$

We regard X of as the current feasible basic solution.

The cost of the current feasible basic solution X is

$$\begin{aligned} & 8 \times 1 + 4 \times 12 + 3 \times 5 + 2 \times 3 + 6 \times 11 \\ & \qquad + 5 \times 13 \\ = \; & 8 + 48 + 15 + 6 + 66 + 65 \\ = \; & 208. \end{aligned}$$

The cost has gone down by 30, as one would expect (the reduction in the cost being $-\lambda q_{2,1}$ where $\lambda = 5$ and $q_{2,1} = -6$).

The current basic feasible solution X is associated with the basis B where

$$B = \{(1,1), \ (1,2), \ (2,1), \ (2,3), \ (3,3), \ (4,1)\}.$$

We now compute, for the current feasible basic solution We determine, for the current basis B values u_1, u_2, u_3, u_4 and v_1, v_2, v_3 such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$. the initial basis.

We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 • 0	4 • 0	16 ?	0
2	3 • 0	7 ?	2 • 0	?
3	13 ?	8 ?	6 • 0	?
4	5 • 0	7 ?	8 ?	?
v_j	?	?	?	

Now $u_1 = 0$, $(1,1) \in B$ and $(1,2) \in B$ force $v_1 = 8$ and $v_2 = 4$.

Then $v_1 = 8$, $(2,1) \in B$ and $(4,1) \in B$ force $u_2 = 5$ and $u_4 = 3$.

Then $u_2 = 5$ and $(3,3) \in B$ force $v_3 = 7$.

Then $v_3 = 7$ and $(3,3) \in B$ force $u_3 = 1$.

Computing the numbers $q_{i,j}$ such that $c_{i,j} + u_i = v_j + q_{i,j}$, we find that $q_{1,3} = 9$, $q_{2,2} = 8$, $q_{3,1} = 6$, $q_{3,2} = 5$, $q_{4,2} = 6$ and $q_{4,3} = 4$.

The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 • 0	4 • 0	16 9	0
2	3 • 0	7 8	2 • 0	5
3	13 6	8 5	6 • 0	1
4	5 • 0	7 6	8 4	3
v_j	8	4	7	

All numbers $q_{i,j}$ are non-negative for the current feasible basic solution. This solution is therefore optimal. Indeed, arguing as before we find that the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ satisfies

$$\overline{C} = 208 + 9\overline{x}_{1,3} + 8\overline{x}_{2,2} + 6\overline{x}_{3,1} + 5\overline{x}_{3,2} + 6\overline{x}_{4,2} + 4\overline{x}_{4,3}.$$

We conclude that X is an basic optimal solution, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{pmatrix}.$$

3.9 A Result concerning the Construction of Bases for the Transportation Problem

The following general proposition ensures that certain standard methods for determining an initial basic solution of the Transportation Problem, including the *Northwest Corner Method* and the *Minimum Cost Method* will succeed in determining a basic feasible solution to the Transportation Problem.

Proposition 3.7 *Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where m and n are positive integers, let $i_1, i_2, \ldots, i_{m+n-1}$ be elements of I and let $j_1, j_2, \ldots, j_{m+n-1}$ be elements of J, and let*

$$B = \{(i_k, j_k) : k = 1, 2, \ldots, m + n - 1\}.$$

Suppose that there exist subsets $I_0, I_1, \ldots, I_{m+n-1}$ of I and $J_0, J_1, \ldots, J_{m+n+1}$ of J such that $I_0 = I$, $J_0 = J$, and such that, for each integer k between 1 and $m + n - 1$, exactly one of the following two conditions is satisfied:—

(i) $i_k \notin I_k$, $j_k \in J_k$, $I_{k-1} = I_k \cup \{i_k\}$ *and* $J_{k-1} = J_k$;

(ii) $i_k \in I_k$, $j_k \notin J_k$, $I_{k-1} = I_k$ *and* $J_{k-1} = J_k \cup \{j_k\}$;

Then, given any real numbers $a_1, a_1, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ satisfying

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j,$$

there exist uniquely-determined real numbers $x_{i,j}$ for all $i \in I$ and $j \in J$ such that $\sum_{j \in J} x_{i,j} = a_i$ for all $i \in I$, $\sum_{i \in I} x_{i,j} = b_j$ for all $j \in J$, and $x_{i,j} = 0$ whenever $(i, j) \notin B$.

Proof We prove the result by induction on $m + n$. The result is easily seen to be true when $m = n = 1$. Thus suppose as our inductive hypothesis that the corresponding results are true when I and J are replaced by I_1 and J_1, so that, given any real numbers a'_i for $i \in I_1$ and b'_j for $j \in J_1$ satisfying $\sum_{i \in I_1} a'_i = \sum_{j \in J_1} b'_j$, there exist uniquely-determined real numbers $x_{i,j}$ for $i \in I_1$ and $j \in J_1$ such that $\sum_{j \in J_1} x_{i,j} = a_i$ for all $i \in I_1$ and $\sum_{i \in I_1} x_{i,j} = b_j$ for all $j \in I_1$. We prove that the corresponding results are true for the given sets I and J.

Now the conditions in the statement of the Proposition ensure that either $i_1 \notin I_1$ or else $j_1 \notin J_1$.

Suppose that $i_1 \notin I_1$. Then $I = I_1 \cup \{i_1\}$ and $J_1 = J$. Now I_k and J_k are subsets of I_1 and J_1 for $k = 1, 2, \ldots, m+n-1$. Moreover $(i_k, j_k) \in I_{k-1} \times J_{k-1}$

for all integers k satisfying $1 \leq k \leq m + n + 1$. It follows that $i_k \in I_1$ and therefore $i_k \neq i_1$ whenever $2 \leq k \leq m$. It follows that the conclusions of the proposition are true if and only if there exist uniquely-determined real numbers $x_{i,j}$ for $i \in I$ and $j \in I$ such that

$$\begin{aligned}
x_{i_1,j_1} &= a_{i_1}, \\
x_{i_1,j} &= 0 \text{ whenever } j \neq j_1, \\
\sum_{j \in J} x_{i,j} &= a_i \text{ whenever } i \neq i_1, \\
\sum_{i \in I_1} x_{i,j_1} &= b_{j_1} - a_{i_1}, \\
\sum_{i \in I_1} x_{i,j} &= b_j \text{ whenever } j \neq j_1, \\
x_{i,j} &= 0 \text{ whenever } (i,j) \notin B
\end{aligned}$$

The induction hypothesis ensures the existence and uniqueness of the real numbers $x_{i,j}$ for $i \in I_1$ and $j \in J$ determined so as to satisfy the above conditions. Thus the induction hypothesis ensures that the required result is true in the case where $i_1 \notin I_1$.

An analogous argument shows that the required result is true in the case where $j_1 \notin J_1$. The result follows. ∎

Proposition 3.7 ensures that if $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ and if a subset B of $I \times J$ is determined so as to satisfy the requirements of Proposition 3.7, then that subset B of $I \times J$ is a basis for the Transportation Problem with m suppliers and n recipients.

The algorithms underlying the *Minimal Cost Method* and the *Northwest Corner Method* give rise to subsets I_k and J_k of I and J respectively for $k = 0, 1, 2, \ldots, m + n - 1$ that satisfy the conditions of Proposition 3.7. This proposition therefore ensures that *Minimal Cost Method* and the *Northwest Corner Method* do indeed determine basic feasible solutions to the Transportation Problem.

Remark One can prove a converse result to Proposition 3.7 which establishes that, given any basis B for an instance of the Transportation Problem with m suppliers and n recipients, there exist subsets I_k of I and J_k of J, for $i = 1, 2, \ldots, m + n - 1$, where $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, so that these subsets I_k and J_k of I and J are related to one another and to the basis B in the manner described in the statement of Proposition 3.7.

3.10 The Minimum Cost Method

We describe the *Minimum Cost Method* for finding an initial basic feasible solution to the Transportation Problem.

Consider an instance of the Transportation Problem specified by positive integers m and n and non-negative real numbers $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$, where $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. Let $I = \{1, 2, \ldots, m\}$ and let $J = \{1, 2, \ldots, n\}$. A feasible solution consists of an array of non-negative real numbers $x_{i,j}$ for $i \in I$ and $j \in J$ with the property that $\sum_{j \in J} x_{i,j} = s_i$ for all $i \in I$ and $\sum_{i \in I} x_{i,j} = d_j$ for all $j \in J$. The objective of the problem is to find a feasible solution that minimizes cost, where the cost of a feasible solution $(x_{i,j} : i \in I \text{ and} j \in J)$ is $\sum_{i \in I} \sum_{j \in J} c_{i,j} x_{i,j}$.

In applying the Minimal Cost Method to find an initial basic solution to the Transportation we apply an algorithm that corresponds to the determination of elements $(i_1, j_1), (i_2, j_2), \ldots, (i_{m+n-1}, j_{m+n-1})$ of $I \times J$ and of subsets $I_0, I_1, \ldots, I_{m+n-1}$ of I and $J_0, J_1, \ldots, J_{m+n-1}$ of J such that the conditions of Proposition 3.7 are satisfied.

Indeed let $I_0 = I$, $J_0 = J$ and $B_0 = \{0\}$. The Minimal Cost Method algorithm is accomplished in $m + n - 1$ stages.

Let k be an integer satisfying $1 \leq k \leq m + n - 1$ and that subsets I_{k-1} of I, J_{k-1} of J and B_{k-1} of $I \times J$ have been determined in accordance with the rules that apply at previous stages of the Minimal Cost algorithm. Suppose also that non-negative real numbers $x_{i,j}$ have been determined for all ordered pairs (i, j) in $I \times J$ that satisfy either $i \notin I_{k-1}$ or $j \notin J_{k-1}$ so as to satisfy the following conditions:—

- $\sum_{j \in J \setminus J_{k-1}} x_{i,j} \leq s_i$ whenever $i \in I_{k-1}$;
- $\sum_{j \in J} x_{i,j} = s_i$ whenever $i \notin I_{k-1}$;
- $\sum_{i \in I \setminus I_{k-1}} x_{i,j} \leq d_j$ whenever $j \in J_{k-1}$;
- $\sum_{i \in I} x_{i,j} = d_j$ whenever $j \notin J_{k-1}$.

The Minimal Cost Method specifies that one should choose $(i_k, j_k) \in I_{k-1} \times J_{k-1}$ so that

$$c_{i_k, j_k} \leq c_{i,j} \quad \text{for all } (i, j) \in I_{k-1} \times J_{k-1},$$

and set $B_k = B_{k-1} \cup \{(i_k, j_k)\}$. Having chosen (i_k, j_k), the non-negative real number x_{i_k,j_k} is then determined so that

$$x_{i_k,j_k} = \min\left(s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j},\; d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k} \right).$$

The subsets I_k and J_k of I and J respectively are then determined, along with appropriate values of $x_{i,j}$, according to the following rules:—

(i) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} < d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we set $I_k = I_{k-1} \setminus \{i_k\}$ and $J_k = J_{k-1}$, and we also let $x_{i_k,j} = 0$ for all $j \in J_{k-1} \setminus \{j_k\}$;

(ii) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} > d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we set $J_k = J_{k-1} \setminus \{j_k\}$ and $I_k = I_{k-1}$, and we also let $x_{i,j_k} = 0$ for all $i \in I_{k-1} \setminus \{i_k\}$;

(iii) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} = d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we determine I_k and J_k and the corresponding values of $x_{i,j}$ either in accordance with the specification in rule (i) above or else in accordance with the specification in rule (ii) above.

These rules ensure that the real numbers $x_{i,j}$ determined at this stage are all non-negative, and that the following conditions are satisfied at the conclusion of the kth stage of the Minimal Cost Method algorithm:—

- $\sum_{j \in J \setminus J_k} x_{i,j} \leq s_i$ whenever $i \in I_k$;
- $\sum_{j \in J} x_{i,j} = s_i$ whenever $i \notin I_k$;
- $\sum_{i \in I \setminus I_k} x_{i,j} \leq d_j$ whenever $j \in J_k$;
- $\sum_{i \in I} x_{i,j} = d_j$ whenever $j \notin J_k$.

At the completion of the final stage (for which $k = m + n - 1$) we have determined a subset B of $I \times J$, where $B = B_{m+n-1}$, together with non-negative real numbers $x_{i,j}$ for $i \in I$ and $j \in I$ that constitute a feasible solution to the given instance of the Transportation Problem. Moreover Proposition 3.7 ensures that this feasible solution is a basic feasible solution of the problem with associated basis B.

3.11 The Northwest Corner Method

The *Northwest Corner Method* for finding a basic feasible solution proceeds according to the stages of the *Minimum Cost Method* above, differing only from that method in the choice of the ordered pair (i_k, j_k) at the kth stage of the method. In the Minimum Cost Method, the ordered pair (i_k, j_k) is chosen such that $(i_k, j_k) \in I_{k-1} \times J_{k-1}$ and

$$c_{i_k,j_k} \leq c_{i,j} \quad \text{for all } (i,j) \in I_{k-1} \times J_{k-1}$$

(where the sets I_{k-1}, J_{k-1} are determined as in the specification of the Minimum Cost Method). In applying the Northwest Corner Method, costs associated with ordered pairs (i,j) in $I \times J$ are not taken into account. Instead (i_k, j_k) is chosen so that i_k is the minimum of the integers in I_{k-1} and j_k is the minimum of the integers in J_{k-1}. Otherwise the specification of the Northwest Corner Method corresponds to that of the Minimum Cost Method, and results in a basic feasible solution of the given instance of the Transportation Problem.

3.12 The Iterative Procedure for Solving the Transportation Problem, given an Initial Basic Feasible Solution

We now describe in general terms the method for solving the Transportation Problem, in the case where total supply equals total demand.

We suppose that an initial basic feasible solution has been obtained. We apply an iterative method (based on the general Simplex Method for the solution of linear programming problems) that will test a basic feasible solution for optimality and, in the event that the feasible solution is shown not to be optimal, establishes information that (with the exception of certain 'degenerate' cases of the Transportation Problem) enables one to find a basic feasible solution with lower cost. Iterating this procedure a finite number of times, one should arrive at a basic feasible solution that is optimal for the given instance of the the Transportation Problem.

We suppose that the given instance of the Transportation Problem involves m suppliers and n recipients. The required supplies are specified by non-negative real numbers $s_1, s_2, \ldots, s_m$, and the required demands are specified by non-negative real numbers $d_1, d_2, \ldots, d_n$. We further suppose that $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. A *feasible solution* is represented by non-negative real numbers $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. A subset B of $I \times J$ is a *basis* for the Transportation Problem if and only if, given any real numbers $y_1, y_2, \ldots, y_m$ and $z_1, z_2, \ldots, z_n$, there exist uniquely determined real numbers $\overline{x}_{i,j}$ for $i \in I$ and $j \in J$ such that $\sum_{j=1}^{n} \overline{x}_{i,j} = y_i$ for $i \in I$, $\sum_{i=1}^{m} \overline{x}_{i,j} = z_j$ for $j \in J$, where $\overline{x}_{i,j} = 0$ whenever $(i, j) \notin B$ (see Proposition 3.4).

A feasible solution $(x_{i,j})$ is said to be a basic feasible solution associated with the basis B if and only if $x_{i,j} = 0$ for all $i \in I$ and $j \in J$ for which $(i, j) \notin B$.

Let $x_{i,j}$ be a non-negative real number for each $i \in I$ and $j \in J$. Suppose that $(x_{i,j})$ is a basic feasible solution to the Transportation Problem associated with basis B, where $B \subset I \times J$.

The cost associated with a feasible solution $(x_{i,j}$ is given by $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$, where the constants $c_{i,j}$ are real numbers for all $i \in I$ and $j \in J$. A feasible solution for the given instance of the Transportation Problems is an optimal solution if and only if it minimizes cost amongst all feasible solutions to the problem.

In order to test for optimality of a basic feasible solution $(x_{i,j})$ associated with a basis B, we determine real numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ with the property that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. (Proposition 3.10 below guarantees that, given any basis B, it is always possible to find the required quantities u_i and v_j.) Having calculated these quantities u_i and v_j we determine the values of $q_{i,j}$, where $q_{i,j} = c_{i,j} - v_j + u_i$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ whenever $(i, j) \in B$.

We claim that a basic feasible solution $(x_{i,j})$ associated with the basis B is optimal if and only if $q_{i,j} \geq 0$ for all $i \in I$ and $j \in J$. This is a consequence of the identity established in the following proposition.

Proposition 3.8 *Let $x_{i,j}$, $c_{i,j}$ and $q_{i,j}$ be real numbers defined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be real numbers.*

Suppose that

$$c_{i,j} = v_j - u_i + q_{i,j}$$

for $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$. *Then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \sum_{i=1}^{m} v_j d_j - \sum_{j=1}^{n} u_i s_i + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}x_{i,j},$$

where $s_i = \sum_{j=1}^{n} x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $d_j = \sum_{i=1}^{m} x_{i,j}$ *for* $j = 1, 2, \ldots, n$.

Proof The definitions of the relevant quantities ensure that

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} &= \sum_{i=1}^{m}\sum_{j=1}^{n} (v_j - u_i + q_{i,j})x_{i,j} \\
&= \sum_{j=1}^{n}\left(v_j \sum_{i=1}^{m} x_{i,j}\right) - \sum_{i=1}^{m}\left(u_i \sum_{j=1}^{n} x_{i,j}\right) \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}x_{i,j} \\
&= \sum_{i=1}^{m} v_j d_j - \sum_{j=1}^{n} u_i s_i + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}x_{i,j},
\end{aligned}$$

as required. ∎

Corollary 3.9 *Let* m *and* n *be integers, and let* $I = \{1, 2, \ldots, m\}$ *and* $J = \{1, 2, \ldots, n\}$. *Let* $x_{i,j}$ *and* $c_{i,j}$ *be real numbers defined for all* $i \in I$ *and* $j \in I$, *and let* $u_1, u_2, \ldots, u_m$ *and* $v_1, v_2, \ldots, v_n$ *be real numbers. Suppose that* $c_{i,j} = v_j - u_i$ *for all* $(i, j) \in I \times J$ *for which* $x_{i,j} \neq 0$. *Then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

where $s_i = \sum_{j=1}^{n} x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $d_j = \sum_{i=1}^{m} x_{i,j}$ *for* $j = 1, 2, \ldots, n$.

Proof Let $q_{i,j} = c_{i,j} + u_i - v_j$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ whenever $x_{i,j} \neq 0$. It follows from this that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}x_{i,j} = 0.$$

It then follows from Proposition 3.8 that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \sum_{i=1}^{m}\sum_{j=1}^{n}(v_j - u_i + q_{i,j})x_{i,j} = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

as required. ∎

Let m and n be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let the subset B of $I \times J$ be a basis for an instance of the Transportation Problem with m suppliers and n recipients. Let the cost of a feasible solution $(\overline{x}_{i,j})$ be $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j}$. Now $\sum_{j=1}^{n} \overline{x}_{i,j} = s_i$ and $\sum_{i=1}^{m} \overline{x}_{i,j} = d_j$, where the quantities s_i and d_j are determined by the specification of the problem and are the same for all feasible solutions of the problem. Let quantities u_i for $i \in I$ and v_j for $j \in J$ be determined such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$, and let $q_{i,j} = c_{i,j} + u_i - v_j$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ for all $(i, j) \in B$.

It follows from Proposition 3.8 that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} = \sum_{i=1}^{m} v_j d_j - \sum_{j=1}^{n} u_i s_i + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j}.$$

Now if the quantities $x_{i,j}$ for $i \in I$ and $j \in J$ constitute a basic feasible solution associated with the basis B then $x_{i,j} = 0$ whenever $(i, j) \notin B$. It follows that $\sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j} = 0$, and therefore

$$\sum_{i=1}^{m} v_j d_j - \sum_{j=1}^{n} u_i s_i = C,$$

where

$$C = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j}.$$

The cost $\overline{C}$ of the feasible solution $(\overline{x}_{i,j})$ then satisfies the equation

$$\overline{C} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} = C + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j}.$$

If $q_{i,j} \geq 0$ for all $i \in I$ and $j \in J$, then the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ is bounded below by the cost of the basic feasible solution $(x_{i,j})$. It follows that, in this case, the basic feasible solution $(x_{i,j})$ is optimal.

Suppose that (i_0, j_0) is an element of $I \times J$ for which $q_{i_0,j_0} < 0$. Then $(i_0, j_0) \notin B$. There is no basis for the Transportation Problem that includes the set $B \cup \{(i_0, j_0)\}$. A straightforward application of Lemma 3.5 establishes the existence of quantities $y_{i,j}$ for $i \in I$ and $j \in J$ such that $y_{i_0,j_0} = 1$ and $y_{i,j} = 0$ for all $i \in I$ and $j \in J$ for which $(i, j) \notin B \cup \{(i_0, j_0)\}$.

Let the $m \times n$ matrices X and Y be defined so that $(X)_{i,j} = x_{i,j}$ and $(Y)_{i,j} = y_{i,j}$ for all $i \in I$ and $j \in J$. Suppose that $x_{i,j} > 0$ for all $(i, j) \in B$. Then the components of X in the basis positions are strictly positive. It follows that, if λ is positive but sufficiently small, then the components of the matrix $X + \lambda Y$ in the basis positions are also strictly positive, and therefore the components of the matrix $X + \lambda Y$ are non-negative for all sufficiently small non-negative values of λ. There will then exist a maximum value λ_0 that is an upper bound on the values of λ for which all components of the matrix $X + \lambda Y$ are non-negative. It is then a straightforward exercise in linear algebra to verify that $X + \lambda_0 Y$ is another basic feasible solution associated with a basis that includes (i_0, j_0) together with all but one of the elements of the basis B. Moreover the cost of this new basic feasible solution is $C + \lambda_0 q_{i_0,j_0}$, where C is the cost of the basic feasible solution represented by the matrix X. Thus if $q_{i_0,j_0} < 0$ then the cost of the new basic feasible solution is lower than that of the basic feasible solution X from which it was derived.

Suppose that, for all basic feasible solutions of the given Transportation problem, the coefficients of the matrix specifying the basic feasible solution are strictly positive at the basis positions. Then a finite number of iterations of the procedure discussed above with result in an basic optimal solution of the given instance of the Transportation Problem. Such problems are said to be *non-degenerate*.

However if it turns out that a basic feasible solution $(x_{i,j})$ associated with a basis B satisfies $x_{i,j} = 0$ for some $(i, j) \in B$, then we are in a *degenerate* case of the Transportation Problem. The theory of degenerate cases of linear programming problems is discussed in detail in textbooks that discuss the details of linear programming algorithms.

We now establish the proposition that guarantees that, given any basis B, there exist quantities $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ such that the costs $c_{i,j}$ associated with the given instance of the Transportation Problem satisfy $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. This result is an essential component of the method described here for testing basic feasible solutions to determine whether or not they are optimal.

Proposition 3.10 *Let m and n be integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let B be a subset of $I \times J$ that is a basis for the transporta-*

tion problem with m suppliers and n recipients. For each $(i,j) \in B$ let $c_{i,j}$ be a corresponding real number. Then there exist real numbers u_i for $i \in I$ and v_j for $j \in J$ such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$. Moreover if $\overline{u}_i$ and $\overline{v}_j$ are real numbers for $i \in I$ and $j \in J$ that satisfy the equations $c_{i,j} = \overline{v}_j - \overline{u}_i$ for all $(i,j) \in B$, then there exists some real number k such that $\overline{u}_i = u_i + k$ for all $i \in I$ and $\overline{v}_j = v_j + k$ for all $j \in J$.

Proof Let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\},$$

let $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ be the linear transformations defined such that $\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$, let

$$M_B = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ whenever } (i,j) \notin B\},$$

and let C be the $m \times n$ matrix defined such that $(C)_{i,j} = c_{i,j}$ for all $i \in I$ and $j \in J$.

Now, given any element $(\mathbf{y}, \mathbf{z})$ of W, there exists a uniquely-determined $m \times n$ matrix X such that $\sum_{j=1}^{n} (X)_{i,j} = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} (X)_{i,j} = (\mathbf{z})_j$ for $j = 1, 2, \ldots, n$ and $(X)_{i,j} = 0$ unless $(i,j) \in B$ (see Proposition 3.4). Then X is the unique matrix belonging to M_B that satisfies $\rho(X) = \mathbf{y}$ and $\sigma(X) = \mathbf{z}$. We define

$$g(\mathbf{y}, \mathbf{z}) = \operatorname{trace}(C^T X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} (X)_{i,j}.$$

We obtain in this way a well-defined function $g\colon W \to \mathbb{R}$ characterized by the property that

$$g(\rho(X), \sigma(X)) = \operatorname{trace}(C^T X)$$

for all $X \in M_B$. Now $\rho(\lambda X) = \lambda \mathbf{y}$ and $\sigma(\lambda X) = \lambda \mathbf{z}$ for all real numbers λ. It follows that

$$g(\lambda(\mathbf{y}, \mathbf{z})) = \lambda g(\mathbf{y}, \mathbf{z})$$

for all real numbers λ. Also, given elements $(\mathbf{y}', \mathbf{z}')$ and $(\mathbf{y}'', \mathbf{z}'')$ of W, there exist unique matrices X' and X'' belonging to M_B such that $\rho(X') = \mathbf{y}'$,

$\rho(X'') = \mathbf{y}''$, $\sigma(X') = \mathbf{z}'$ and $\sigma(X'') = \mathbf{z}''$. Then $\rho(X' + X'') = \mathbf{y}' + \mathbf{y}''$ and $\sigma(X' + X'') = \mathbf{z}' + \mathbf{z}''$, and therefore

$$\begin{aligned} g((\mathbf{y}', \mathbf{z}') + (\mathbf{y}'', \mathbf{z}'')) &= \mathrm{trace}(C^T(X' + X'')) \\ &= \mathrm{trace}(C^T X') + \mathrm{trace}(C^T X'') \\ &= g(\mathbf{y}', \mathbf{z}') + g(\mathbf{y}'', \mathbf{z}''). \end{aligned}$$

It follows that the function $g: W \to \mathbb{R}$ is a linear transformation. It is thus a linear functional on the real vector space W.

For each integer i between 1 and m, let $\overline{\mathbf{b}}^{(i)}$ denote the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero, and, for each integer j between 1 and n, let $\mathbf{b}^{(j)}$ denote the vector in $\mathbb{R}^n$ whose jth component is equal to 1 and whose other components are zero. Then $(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0}) \in W$ for $i = 1, 2, \ldots, m$ and $(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) \in W$ for $j = 1, 2, \ldots, n$. We define $u_i = g(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0})$ for $i = 1, 2, \ldots, m$ and $v_j = g(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)})$ for $j = 1, 2, \ldots, n$. Then $u_1 = 0$ and

$$\begin{aligned} v_j - u_i &= g(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) - g(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0}) \\ &= g\left((\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) - (\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0})\right) \\ &= g(\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)}) \end{aligned}$$

for all $i \in I$ and $j \in J$.

If $(i, j) \in B$ then $\overline{\mathbf{b}}^{(i)} = \rho(E^{(i,j)})$ and $\mathbf{b}^{(j)} = \sigma(E^{(i,j)})$, where $E^{(i,j)}$ is the $m \times n$ matrix whose coefficient in the ith row and jth column is equal to 1 and whose other coefficients are zero. Moreover $E^{(i,j)} \in M_B$ for all $(i, j) \in B$. It follows from the definition of the linear functional g that

$$v_j - u_i = g(\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)}) = \mathrm{trace}(C^T E^{(i,j)}) = c_{i,j}$$

for all $(i, j) \in B$.

Now let $\overline{u}_i$ and $\overline{v}_j$ be real numbers for $i \in I$ and $j \in J$ that satisfy the equations $c_{i,j} = \overline{v}_j - \overline{u}_i$ for all $(i, j) \in B$. Let

$$\overline{g}(\mathbf{y}, \mathbf{z}) = \sum_{j=1}^{n} v_j (\mathbf{z})_j - \sum_{i=1}^{m} u_i (\mathbf{y})_i$$

for all $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$. Then

$$\overline{g}(\rho(E^{(i,j)}), \sigma(E^{(i,j)})) = \overline{g}(\overline{\mathbf{b}}^{(i)}, \mathbf{b}^{(j)}) = \overline{v}_j - \overline{u}_i$$

for all $(i, j) \in I \times J$. It follows that

$$\overline{g}(\rho(E^{(i,j)}), \sigma(E^{(i,j)})) = \overline{v}_j - \overline{u}_i = c_{i,j} = g(\rho(E^{(i,j)}), \sigma(E^{(i,j)})) \quad \text{for all } (i, j) \in B.$$

Now the matrices $E^{(i,j)}$ for all $(i,j) \in B$ constitute a basis of the vector space M_K. It follows that

$$\overline{g}(\rho(X), \sigma(X)) = g(\rho(X), \sigma(X))$$

for all $X \in M_B$. But every element of the vector space W is of the form $(\rho(X), \sigma(X))$ for some $X \in M_B$. (This follows Proposition 3.4, as discussed earlier in the proof.) Thus

$$\overline{g}(\mathbf{y}, \mathbf{z}) = g(\mathbf{y}, \mathbf{z})$$

for all $(\mathbf{y}, \mathbf{z}) \in W$. In particular

$$\overline{u}_i - \overline{u}_1 = \overline{g}(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0}) = g(\overline{\mathbf{b}}^{(1)} - \overline{\mathbf{b}}^{(i)}, \mathbf{0}) = u_i - u_1$$

for all $i \in I$, and

$$\overline{v}_j - \overline{u}_1 = \overline{g}(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) = g(\overline{\mathbf{b}}^{(1)}, \mathbf{b}^{(j)}) = v_j - u_1$$

for all $j \in J$. Let $k = \overline{u}_1 - u_1$. Then $\overline{u}_i = u_i + k$ for all $i \in I$ and $\overline{v}_j = v_j + k$ for all $j \in J$, as required. ∎