

Module MA3484: Methods of Mathematical
Economics
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1 Introduction to Linear Programming

1.1 A Furniture Retailing Problem

A retail business is planning to devote a number of retail outlets to the sale of armchairs and sofas.

The retail prices of armchairs and sofas are determined by fierce competition in the furniture retailing business. Armchairs sell for €700 and sofas sell for €1000.

However

- the amount of floor space (and warehouse space) available for stocking the sofas and armchairs is limited;
- the amount of capital available for purchasing the initial stock of sofas and armchairs is limited;
- market research shows that the ratio of armchairs to sofas in stores should neither be too low nor too high.

Specifically:

- there are 1000 square metres of floor space available for stocking the initial purchase of sofas and armchairs;
- each armchair takes up 1 square metre;
- each sofa takes up 2 square metres;
- the amount of capital available for purchasing the initial stock of armchairs and sofas is €351,000;
- the wholesale price of an armchair is €400;
- the wholesale price of a sofa is €600;
- market research shows that between 4 and 9 armchairs should be in stock for each 3 sofas in stock.

We suppose that the retail outlets are stocked with x armchairs and y sofas.

The armchairs (taking up 1 sq. metre each) and the sofas (taking up 2 sq. metres each) cannot altogether take up more than 1000 sq. metres of floor space. Therefore

$$x + 2y \leq 1000 \quad (\text{Floor space constraint}).$$

The cost of stocking the retail outlets with armchairs (costing €400 each) and sofas (costing €600 each) cannot exceed the available capital of €351000. Therefore

$$4x + 6y \leq 3510 \quad (\text{Capital constraint}).$$

Consumer research indicates that x and y should satisfy

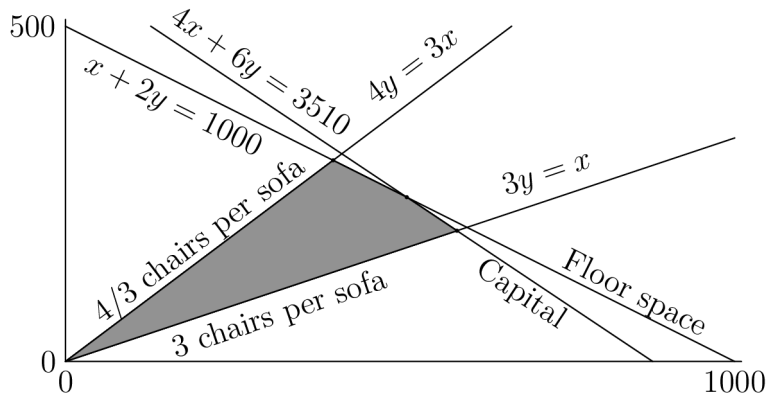
$$4y \leq 3x \leq 9y \quad (\text{Armchair/Sofa ratio}).$$

An ordered pair (x, y) of real numbers is said to specify a *feasible solution* to the linear programming problem if this pair of values meets all the relevant constraints.

An ordered pair (x, y) constitutes a feasible solution to the the Furniture Retailing problem if and only if all the following constraints are satisfied:

$$\begin{aligned} x - 3y &\leq 0; \\ 4y - 3x &\leq 0; \\ x + 2y &\leq 1000; \\ 4x + 6y &\leq 3510; \\ x &\geq 0; \\ y &\geq 0; \end{aligned}$$

The feasible region for the Furniture Retailing problem is depicted below:



We identify the *vertices* (or *corners*) of the feasible region for the Furniture Retailing problem. There are four of these:

- there is a vertex at $(0, 0)$;
- there is a vertex at $(400, 300)$ where the line $4y = 3x$ intersects the line $x + 2y = 1000$;

- there is a vertex at $(510, 245)$ where the line $x + 2y = 1000$ intersects the line $4x + 6y = 3510$;
- there is a vertex at $(585, 195)$ where the line $3y = x$ intersects the line $4x + 6y = 3510$.

These vertices are identified by inspection of the graph that depicts the constraints that determine the feasible region.

The furniture retail business obviously wants to confirm that the business will make a profit, and will wish to determine how many armchairs and sofas to purchase from the wholesaler to maximize expected profit.

There are fixed costs for wages, rental etc., and we assume that these are independent of the number of armchairs and sofas sold.

The *gross margin* on the sale of an armchair or sofa is the difference between the wholesale and retail prices of that item of furniture.

Armchairs cost €400 wholesale and sell for €700, and thus provide a gross margin of €300.

Sofas cost €600 wholesale and sell for €1000, and thus provide a gross margin of €400.

In a typical linear programming problem, one wishes to determine not merely *feasible* solutions to the problem. One wishes to determine an *optimal* solution that maximizes some *objective function* amongst all feasible solutions to the problem.

The objective function for the Furniture Retailing problem is the gross profit that would accrue from selling the furniture in stock. This gross profit is the difference between the cost of purchasing the furniture from the wholesaler and the return from selling that furniture.

This objective function is thus $f(x, y)$, where

$$f(x, y) = 300x + 400y.$$

We should determine the maximum value of this function on the feasible region.

Because the objective function $f(x, y) = 300x + 400y$ is linear in x and y , its maximum value on the feasible region must be achieved at one of the vertices of the region.

Clearly this function is not maximized at the origin $(0, 0)$!

Now the remaining vertices of the feasible region are located at $(400, 300)$, $(510, 245)$ and $(585, 195)$, and

$$\begin{aligned} f(400, 300) &= 240,000, \\ f(510, 245) &= 251,000, \end{aligned}$$

$$f(585, 195) = 253, 500.$$

It follows that the objective function is maximized at (585, 195).

The furniture retail business should therefore use up the available capital, stocking 3 armchairs for every sofa, despite the fact that this will not utilize the full amount of floor space available.

1.2 Linear Programming Problems in Von Neumann Maximizing Form

A linear programming problem may be presented in *Von Neumann maximizing form* as follows:

given real numbers c_i , $A_{i,j}$ and b_j for
 $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$,
 find real numbers x_1, x_2, \dots, x_n so as to
 maximize $c_1x_1 + c_2x_2 + \dots + c_nx_n$
 subject to constraints
 $x_j \geq 0$ for $j = 1, 2, \dots, n$, and
 $A_{i,1}x_1 + A_{i,2}x_2 + \dots + A_{i,n}x_n \leq b_i$ for $i = 1, 2, \dots, m$.

The furniture retailing problem may be presented in von Neumann maximizing form with $n = 2$, $m = 4$,

$$(c_1, c_2) = (300, 400),$$

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 4 \\ 1 & 2 \\ 4 & 6 \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1000 \\ 3510 \end{pmatrix}.$$

Here A represents the $m \times n$ whose coefficient in the i th row and j th column is $A_{i,j}$.

Linear programming problems may be presented in matrix form. We adopt the following notational conventions with regard to transposes, row and column vectors and vector inequalities:—

- vectors in \mathbb{R}^m and \mathbb{R}^n are represented as column vectors;

- we denote by M^T the $n \times m$ matrix that is the transpose of an $m \times n$ matrix M ;
- in particular, given $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, where b and c are represented as column vectors, we denote by b^T and c^T the corresponding row vectors obtained on transposing the column vectors representing b and c ;
- given vectors u and v in \mathbb{R}^n for some positive integer n , we write $u \leq v$ (and $v \geq u$) if and only if $u_j \leq v_j$ for $j = 1, 2, \dots, n$.

A linear programming problem in von Neumann maximizing form may be presented in matrix notation as follows:—

*Given an $m \times n$ matrix A with real coefficients,
and given column vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$,
find $x \in \mathbb{R}^n$ so as to
maximize $c^T x$
subject to constraints $Ax \leq b$ and $x \geq 0$.*