Module MA3484: Transportation Problem
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Review of Linear Algebra

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A Review of Linear Algebra

A.1 Real Vector Spaces

Definition A *real vector space* consists of a set V on which there is defined an operation of vector addition, yielding an element $\mathbf{v}+\mathbf{w}$ of V for each pair $\mathbf{v}, \mathbf{w}$ of elements of V, and an operation of multiplication-by-scalars that yields an element $\lambda\mathbf{v}$ of V for each $\mathbf{v} \in V$ and for each real number λ. The operation of vector addition is required to be commutative and associative. There must exist a zero element $\mathbf{0}_V$ of V that satisfies $\mathbf{v} + \mathbf{0}_V = \mathbf{v}$ for all $\mathbf{v} \in V$, and, for each $\mathbf{v} \in V$ there must exist an element $-\mathbf{v}$ of V for which $\mathbf{v}+(-\mathbf{v}) = \mathbf{0}_V$. The following identities must also be satisfied for all $\mathbf{v}, \mathbf{w} \in V$ and for all real numbers λ and μ:

$$(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}, \quad \lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w},$$

$$\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}, \quad 1\mathbf{v} = \mathbf{v}.$$

Let n be a positive integer. The set $\mathbb{R}^n$ consisting of all n-tuples of real numbers is then a real vector space, with addition and multiplication-by-scalars defined such that

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$$

and

$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}$ and for all real numbers λ.

The set $M_{m,n}(\mathbb{R})$ of all $m \times n$ matrices is a real vector space with respect to the usual operations of matrix addition and multiplication of matrices by real numbers.

A.2 Linear Dependence and Bases

Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ of a real vector space V are said to be *linearly dependent* if there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$, not all zero, such that

$$\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \cdots + \lambda_m\mathbf{u}_m = \mathbf{0}_V.$$

If elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ of real vector space V are not linearly dependent, then they are said to be *linearly independent*.

Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of a real vector space V are said to *span* V if, given any element $\mathbf{v}$ of V, there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \cdots + \lambda_n\mathbf{u}_n$.

A vector space is said to be *finite-dimensional* if there exists a finite subset of V whose members span V.

Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of a finite-dimensional real vector space V are said to constitute a *basis* of V if they are linearly independent and span V.

Lemma A.1 *Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of a real vector space V constitute a basis of V if and only if, given any element $\mathbf{v}$ of V, there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that*

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n.$$

Proof Suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a basis of V. Let $\mathbf{v}$ be an element V. The requirement that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span V ensures that there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$v = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n.$$

If $\mu_1, \mu_2, \ldots, \mu_n$ are real numbers for which

$$v = \mu_1 \mathbf{u}_1 + \mu_2 \mathbf{u}_2 + \cdots + \mu_n \mathbf{u}_n,$$

then

$$(\mu_1 - \lambda_1)\mathbf{u}_1 + (\mu_2 - \lambda_2)\mathbf{u}_2 + \cdots + (\mu_n - \lambda_n)\mathbf{u}_n = \mathbf{0}_V.$$

It then follows from the linear independence of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ that $\mu_i - \lambda_i = 0$ for $i = 1, 2, \ldots, n$, and thus $\mu_i = \lambda_i$ for $i = 1, 2, \ldots, n$. This proves that the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are uniquely-determined.

Conversely suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a list of elements of V with the property that, given any element $\mathbf{v}$ of V, there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$v = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n.$$

Then $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span V. Moreover we can apply this criterion when $\mathbf{v} = 0$. The uniqueness of the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ then ensures that if

$$\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n = \mathbf{0}_V$$

then $\lambda_i = 0$ for $i = 1, 2, \ldots, n$. Thus $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent. This proves that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a basis of V, as required. ∎

Proposition A.2 *Let V be a finite-dimensional real vector space, let*

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$$

be elements of V that span V, and let K be a subset of $\{1, 2, \ldots, n\}$. Suppose either that $K = \emptyset$ or else that those elements $\mathbf{u}_i$ for which $i \in K$ are linearly independent. Then there exists a basis of V whose members belong to the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ which includes all the vectors $\mathbf{u}_i$ for which $i \in K$.

Proof We prove the result by induction on the number of elements in the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of vectors that span V. The result is clearly true when $n = 1$. Thus suppose, as the induction hypothesis, that the result is true for all lists of elements of V that span V and that have fewer than n members.

If the elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent, then they constitute the required basis. If not, then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n = \mathbf{0}_V.$$

Now there cannot exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all zero, such that both $\sum_{i=1}^{n} \lambda_i \mathbf{u}_i = \mathbf{0}_V$ and also $\lambda_i = 0$ whenever $i \neq K$. Indeed, in the case where $K = \emptyset$, this conclusion follows from the requirement that the real numbers λ_i cannot all be zero, and, in the case where $K \neq \emptyset$, the conclusion follows from the linear independence of those $\mathbf{u}_i$ for which $i \in K$. Therefore there must exist some integer i satisfying $1 \leq i \leq n$ for which $\lambda_i \neq 0$ and $i \notin K$. Without loss of generality, we may suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are ordered so that $n \notin K$ and $\lambda_n \neq 0$. Then

$$\mathbf{u}_n = -\sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda_n} \mathbf{u}_i.$$

Let $\mathbf{v}$ be an element of V. Then there exist real numbers $\mu_1, \mu_2, \ldots, \mu_n$ such that $\mathbf{v} = \sum_{i=1}^{n} \mu_i \mathbf{u}_i$, because $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span V. But then

$$\mathbf{v} = \sum_{i=1}^{n-1} \left(\mu_i - \frac{\mu_n \lambda_i}{\lambda_n} \right) \mathbf{u}_i.$$

We conclude that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$ span the vector space V. The induction hypothesis then ensures that there exists a basis of V consisting of members of this list that includes the linearly independent elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$, as required. ∎

Corollary A.3 *Let V be a finite-dimensional real vector space, and let*

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$$

be elements of V that span the vector space V. Then there exists a basis of V whose elements are members of the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Proof This result is a restatement of Proposition A.2 in the special case where the set K in the statement of that proposition is the empty set. ∎

Proposition A.4 *Let V be a finite-dimensional real vector space with basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, let $\mathbf{w}$ be an element of V, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the unique real numbers for which $\mathbf{w} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$. Suppose that $\lambda_j \neq 0$ for some integer j between 1 and n. Then the element $\mathbf{w}$ of V and those elements $\mathbf{u}_i$ of the given basis for which $i \neq j$ together constitute a basis of V.*

Proof We result follows directly when $n = 1$. Thus it suffices to prove the result when $n > 1$. We may suppose, without loss of generality, that the basis elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are ordered so that $j = n$. We must then show that $\mathbf{w}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$ is a basis of V. Now

$$\mathbf{w} = \sum_{i=1}^{n-1} \lambda_i \mathbf{u}_i + \lambda_n \mathbf{u}_n,$$

where $\lambda_n \neq 0$, and therefore

$$\mathbf{u}_n = \frac{1}{\lambda_n} \mathbf{w} - \sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda_n} \mathbf{u}_{n-1}.$$

Let $\mathbf{v}$ be an element of V. Then there exist real numbers $\mu_1, \mu_2, \ldots, \mu_n$ such that $\mathbf{v} = \sum_{i=1}^{n} \mu_i \mathbf{u}_i$. Then

$$\mathbf{v} = \frac{\mu_n}{\lambda_n} \mathbf{w} + \sum_{i=1}^{n-1} \left(\mu_i - \frac{\lambda_i \mu_n}{\lambda_n} \right) \mathbf{u}_i.$$

We conclude from this that the vectors $\mathbf{w}, \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$ span the vector space V.

Now let $\rho_0, \rho_1, \ldots, \rho_{n-1}$ be real numbers with the property that

$$\rho_0 \mathbf{w} + \sum_{i=1}^{n-1} \rho_i \mathbf{u}_i = \mathbf{0}_V.$$

Then

$$\sum_{i=1}^{n-1} (\rho_i + \rho_0 \lambda_i) \mathbf{u}_i + \rho_0 \lambda_n \mathbf{u}_n = \mathbf{0}_V.$$

It then follows from the linear independence of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ that $\rho_i + \rho_0 \lambda_i = 0$ for $i = 1, 2, \ldots, n-1$ and $\rho_0 \lambda_n = 0$. But $\lambda_n \neq 0$. It follows that $\rho_0 = 0$. But then $\rho_i = -\rho_0 \lambda_i = 0$ for $i = 1, 2, \ldots, n-1$. This proves that $\mathbf{w}, \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$ are linearly independent. These vectors therefore constitute a basis of the vector space V, as required. ∎

Proposition A.5 *Let V be a finite-dimensional real vector space. Suppose that elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of V span the vector space V and that elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ of V are linearly independent. Then $m \leq n$, and there exists a basis of V consisting of the elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ together with not more than $n - m$ elements belonging to the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.*

Proof If the elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ spanning V are not linearly independent then it follows from Corollary A.3 that we may remove elements from this list so as to obtain a basis for the vector space V. We may therefore assume, without loss of generality, that the elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ constitute a basis of V with n elements.

Suppose that $m \geq 1$. It then follows from Proposition A.4 that there exists a basis of V consisting of $\mathbf{w}_1$ together with $n - 1$ members of the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Suppose that, for some integer k satisfying $1 \leq k < m$ and $k < n$, there exist distinct integers $j_1, j_2, \ldots, j_k$ between 1 and n such that the elements $\mathbf{w}_i$ for $1 \leq i \leq k$ together with the elements $\mathbf{u}_i$ for $i \notin \{j_1, j_2, \ldots, j_k\}$ together constitute a basis of the vector space V. Then there exist real numbers $\rho_1, \rho_2, \ldots, \rho_k$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\mathbf{w}_{k+1} = \sum_{s=1}^{k} \rho_s \mathbf{w}_s + \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$$

and

$$\lambda_i = 0 \text{ for } i = j_1, j_2, \ldots, j_k.$$

If it were the case that $\lambda_i = 0$ for all integers i satisfying $1 \leq i \leq n$ then $\mathbf{w}_{k+1}$ would be expressible as a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$, and therefore the elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k+1}$ of V would be linearly dependent. But these elements are linearly independent. It follows that $\lambda_{j_{k+1}} \neq 0$ for some integer j_{k+1} between 1 and n. Moreover the integers $j_1, j_2, \ldots, j_{k+1}$ are then distinct, and it follows from Proposition A.4 that the elements $\mathbf{w}_i$ for $1 \leq i \leq k+1$ together with the elements $\mathbf{u}_i$ for $i \notin \{j_1, j_2, \ldots, j_{k+1}\}$ together constitute a basis of the vector space V.

It then follows by repeated applications of this result that if m_0 is the minimum of m and n then there exists a basis of V consisting of the elements $\mathbf{w}_i$ for $1 \leq i \leq m_0$ together with $n - m_0$ members of the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

If it were the case that $n < m$ then the n elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ would be a basis of V, and thus the elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ would not be linearly independent. Therefore $n \geq m$, and there exists a basis of V consisting of the elements $\mathbf{w}_i$ for $1 \leq i \leq m$ together with $n - m$ members of the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, as required. ∎

Corollary A.6 *Any two bases of a finite-dimensional real vector space contain the same number of elements.*

Proof It follows from Proposition A.5 that the number of members in a list of linearly independent elements of a finite-dimensional real vector space V cannot exceed the number of members in a list of elements of V that spans V. The members of a basis of V are linearly independent and also span V. Therefore the number of members of one basis of V cannot exceed the number of members of another. The result follows. ∎

Definition The *dimension* of a finite-dimensional real vector space V is the number of members of any basis of V.

The dimension of a real vector space V is denoted by $\dim V$.

It follows from Corollary A.3 that every finite-dimensional real vector space V has a basis. It follows from Corollary A.6 that any two bases of that vector space have the same number of elements. These results ensure that every finite-dimensional real vector space has a well-defined dimension that is equal to the number of members of any basis of that vector space.

A.3 Subspaces of Real Vector Spaces

Definition Let V be a finite-dimensional vector space. A subset U of V is said to be a *subspace* of V if the following two conditions are satisfied:—

- $\mathbf{v} + \mathbf{w} \in U$ for all $\mathbf{v}, \mathbf{w} \in U$;
- $\lambda\mathbf{v} \in U$ for all $\mathbf{v} \in U$ and for all real numbers λ.

Every subspace of a real vector space is itself a real vector space.

Proposition A.7 *Let V be a finite-dimensional vector space, and let U be a subspace of V. Then U is itself a finite-dimensional vector space, and* $\dim U \leq \dim V$.

Proof It follows from Proposition A.5 the number of members of any list of linearly independent elements of U cannot exceed the dimension $\dim V$ of the real vector space V. Let m be the maximum number of members in any list of linearly independent elements of U, and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be a list consisting of m linearly independent elements of U. We claim that this list constitutes a basis of U.

Let $\mathbf{v} \in U$. Then the maximality of m ensures that the members of the list $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ must be linearly dependent. Therefore there exist a real number ρ and real numbers $\lambda_1, \ldots, \lambda_m$, where these real numbers ρ and λ_i are not all zero, such that

$$\rho\mathbf{v} + \sum_{i=1}^{m} \lambda_i \mathbf{w}_i = \mathbf{0}_V.$$

But then $\rho \neq 0$, because otherwise the elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ would be linearly dependent. It then follows that

$$\mathbf{v} = -\sum_{i=1}^{m} \frac{\lambda_i}{\rho} \mathbf{w}_i.$$

This shows that the linearly independent elements $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ of U span U and therefore constitute a basis of U. Thus U is a finite-dimensional vector space, and $\dim U = m$. But $m \leq n$. It follows that $\dim U \leq \dim V$, as required. ∎

A.4 Linear Transformations

Definition Let V and W be real vector spaces. A function $\theta: V \to W$ from V to W is said to be a *linear transformation* if it satisfies the following two conditions:—

- $\theta(\mathbf{v} + \mathbf{w}) = \theta(\mathbf{v}) + \theta(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V$;
- $\theta(\lambda\mathbf{v}) = \lambda\theta(\mathbf{v})$ for all $\mathbf{v} \in V$ and for all real numbers λ.

Definition The *image* of a linear transformation $\theta: V \to W$ between real vector spaces V and W is the subspace $\theta(V)$ of W defined such that

$$\theta(V) = \{\theta(\mathbf{v}) : \mathbf{v} \in V\}.$$

Definition The *rank* of a linear transformation $\theta: V \to W$ between real vector spaces V and W is the dimension of the image $\theta(V)$ of θ.

A linear transformation $\theta: V \to W$ is *surjective* if and only if $\theta(V) = W$. Thus the linear transformation $\theta: V \to W$ is surjective if and only if its rank is equal to the dimension $\dim W$ of the codomain W.

Definition The *kernel* of a linear transformation $\theta: V \to W$ between real vector spaces V and W is the subspace $\ker\theta$ of V defined such that

$$\ker\theta = \{\mathbf{v} \in V : \theta(\mathbf{v}) = 0\}.$$

Definition The *nullity* of a linear transformation $\theta: V \to W$ between real vector spaces V and W is the dimension of the kernel $\ker\theta$ of θ.

A linear transformation $\theta: V \to W$ is *injective* if and only if $\ker\theta = \{\mathbf{0}_V\}$. Indeed let $\mathbf{v}$ and $\mathbf{v}'$ be elements of V satisfying $\theta(\mathbf{v}) = \theta(\mathbf{v}')$. Tnen

$$\theta(\mathbf{v} - \mathbf{v}') = \theta(\mathbf{v}) - \theta(\mathbf{v}') = \mathbf{0}_W,$$

and therefore $\mathbf{v} - \mathbf{v}' \in \ker\theta$. It follows that if $\ker\theta = \{\mathbf{0}_W\}$ and if elements $\mathbf{v}$ and $\mathbf{v}'$ of V satisfy $\theta(\mathbf{v}) = \theta(\mathbf{v}')$ then $\mathbf{v} - \mathbf{v}' = \mathbf{0}_V$, and therefore $\mathbf{v} = \mathbf{v}'$. Thus if $\ker\theta = \{\mathbf{0}_V\}$ then the linear transformation $\theta: V \to W$ is injective. The converse is immediate. It follows that $\theta: V \to W$ is injective if and only if $\ker\theta = \{\mathbf{0}_W\}$.

A linear transformation $\theta: V \to W$ between vector spaces V and W is an *isomorphism* if and only if it is both injective and surjective.

Proposition A.8 *Let V and W be finite-dimensional real vector spaces, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V, let $\theta: V \to W$ be a linear transformation from V to W, and let $\theta(V)$ be the image of this linear transformation. Let $I = \{1, 2, \ldots, n\}$, and let K and L be a subsets of I satisfying $K \subset L$ that satisfy the following properties:—*

- *the elements $\theta(\mathbf{u}_i)$ for which $i \in K$ are linearly independent;*
- *the elements $\theta(\mathbf{u}_i)$ for which $i \in L$ span the vector space $\theta(V)$.*

Then there exists a subset B of I satisfying $K \subset B \subset L$ such that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis for the vector space $\theta(V)$.

Proof The elements $\theta(\mathbf{u}_i)$ for which $i \in L$ span the real vector space $\theta(V)$. The result therefore follows immediately on applying Proposition A.2. ∎

Lemma A.9 *Let V and W be finite-dimensional real vector spaces, and let $\theta: V \to W$ be a linear transformation from V to W. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V, let $I = \{1, 2, \ldots, n\}$, and let B be a subset of I. Suppose that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis of the image $\theta(V)$ of the linear transformation θ. Then, for each $j \in I \setminus B$, there exist uniquely-determined real numbers $\kappa_{i,j}$ for all $i \in B$ such that*

$$\mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i \in \ker \theta.$$

Proof The elements $\theta(\mathbf{u}_i)$ of $\theta(V)$ for which $i \in B$ constitute a basis of $\theta(V)$. Therefore, for each $j \in I \setminus B$, the element $\theta(\mathbf{u}_j)$ may be expressed as a linear combination $\sum\limits_{i \in B} \kappa_{i,j} \theta(\mathbf{u}_i)$ of the basis elements. Moreover the linear independence of the basis elements ensures that the real numbers $\kappa_{i,j}$ that occur as coefficients in this expression of $\theta(\mathbf{u}_j)$ as a linear combination of basis elements are uniquely determined. But then

$$\theta\left(\mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i\right) = \theta(\mathbf{u}_j) - \sum_{i \in B} \kappa_{i,j} \theta(\mathbf{u}_i) = \mathbf{0}_W,$$

and thus $\mathbf{u}_j - \sum\limits_{i \in B} \kappa_{i,j} \mathbf{u}_i \in \ker \theta$, as required. ∎

Proposition A.10 *Let V and W be finite-dimensional real vector spaces, and let $\theta: V \to W$ be a linear transformation from V to W. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V, and let B be a subset of I, where $I = \{1, 2, \ldots, n\}$, with the property that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis of the image $\theta(V)$ of the linear transformation θ. Let*

$$\mathbf{g}_j = \mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i,$$

for all $j \in I \setminus B$, where $\kappa_{i,j}$ are the unique real numbers for which

$$\mathbf{u}_j - \sum_{i \in B} \kappa_{i,j} \mathbf{u}_i \in \ker \theta.$$

Then the elements $\mathbf{u}_i$ for $i \in B$ and $\mathbf{g}_j$ for $j \in I \setminus B$ together constitute a basis for the vector space V.

Proof Let λ_i for $i \in B$ and μ_j for $j \in I \setminus B$ are real numbers with the property that

$$\sum_{i \in B} \lambda_i \mathbf{u}_i + \sum_{j \in I \setminus B} \mu_j \mathbf{g}_j = \mathbf{0}_V.$$

Then

$$\sum_{i\in B}\left(\lambda_i - \sum_{j\in I\setminus B}\kappa_{i,j}\mu_j\right)\mathbf{u}_i + \sum_{j\in I\setminus B}\mu_j\mathbf{u}_j = \mathbf{0}_V.$$

It then follows from the linear independence of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ that $\lambda_i - \sum_{j\in I\setminus B}\kappa_{i,j}\mu_j = 0$ for all $i \in B$ and $\mu_j = 0$ for all $j \in I \setminus B$. But then $\lambda_i = 0$ for all $i \in B$. This shows that the elements $\mathbf{u}_i$ for $i \in B$ and $\mathbf{g}_j$ for $j \in I \setminus B$ are linearly independent.

Let $\mathbf{v} \in V$. Then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \sum_{i=1}^{n}\lambda_i\mathbf{u}_i$. But then

$$\mathbf{v} = \sum_{i\in B}\left(\lambda_i + \sum_{j\in I\setminus B}\kappa_{i,j}\lambda_i\right)\mathbf{u}_i + \sum_{j\in I\setminus B}\lambda_j\mathbf{g}_j.$$

It follows that the elements $\mathbf{u}_i$ for $i \in B$ and $\mathbf{g}_j$ for $j \in I \setminus B$ span the vector space V. We have shown that these elements are linearly independent. It follows that they constitute a basis for the vector space V, as required. ∎

Corollary A.11 *Let V and W be finite-dimensional real vector spaces, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V, let $\theta: V \to W$ be a linear transformation from V to W, let B be a subset of I, where $I = \{1, 2, \ldots, n\}$, with the property that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis of the image $\theta(V)$ of the linear transformation θ, and let*

$$\mathbf{g}_j = \mathbf{u}_j - \sum_{i\in B}\kappa_{i,j}\mathbf{u}_i,$$

for all $j \in I \setminus B$, where $\kappa_{i,j}$ are the unique real numbers for which $\mathbf{u}_j - \sum_{i\in B}\kappa_{i,j}\mathbf{u}_i \in \ker\theta$. Then the elements $\mathbf{g}_j$ for $j \in I \setminus B$ constitute a basis for $\ker\theta$.

Proof We have shown that the elements $\mathbf{u}_i$ for $i \in B$ and $\mathbf{g}_j$ for $j \in I \setminus B$ together constitute a basis for the vector space B (Proposition A.10). It follows that the elements $\mathbf{g}_j$ for which $j \in I \setminus B$ are linearly independent.

Let $\mathbf{v} \in \ker\theta$. Then there exist real numbers λ_i for $i \in B$ and μ_j for $j \in I \setminus B$ such that

$$\mathbf{v} = \sum_{i\in B}\lambda_i\mathbf{u}_i + \sum_{j\in I\setminus B}\mu_i\mathbf{g}_j.$$

Now $\theta(\mathbf{g}_j) = \mathbf{0}_W$ for all $j \in I \setminus B$, because $\mathbf{g}_j \in \ker\theta$. Also $\theta(\mathbf{v}) = \mathbf{0}_W$, because $\mathbf{v} \in \ker\theta$. It follows that

$$\mathbf{0}_W = \theta(\mathbf{v}) = \sum_{i\in B} \lambda_i \theta(\mathbf{u}_i).$$

However the subset B of I has the property that the elements $\theta(\mathbf{u}_i)$ for $i \in B$ constitute a basis of the vector space $\theta(V)$. It follows that $\lambda_i = 0$ for all $i \in B$. Thus

$$\mathbf{v} = \sum_{j\in I\setminus B} \mu_i \mathbf{g}_j.$$

This proves that the elements $\mathbf{g}_j$ for $j \in I \setminus B$ span the kernel $\ker\theta$ of the linear transformation $\theta\colon V \to W$. This elements have been shown to be linearly independent. It follows that they constitute a basis for $\ker\theta$, as required. ∎

Corollary A.12 *Let V and W be finite-dimensional vector spaces, let $\theta\colon V \to W$ be a linear transformation from V to W, and let* $\operatorname{rank}(\theta)$ *and* $\operatorname{nullity}(\theta)$ *denote the rank and nullity respectively of the linear transformation θ. Then*

$$\operatorname{rank}(\theta) + \operatorname{nullity}(\theta) = \dim V.$$

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of the vector space V. Then there exists a subset B of I, where $I = \{1, 2, \ldots, n\}$, with the property that the elements $\theta(\mathbf{u}_i)$ for which $i \in B$ constitute a basis of the image $\theta(V)$ of the linear transformation θ (see Proposition A.8). Let

$$\mathbf{g}_j = \mathbf{u}_j - \sum_{i\in B} \kappa_{i,j}\mathbf{u}_i,$$

for all $j \in I \setminus B$, where $\kappa_{i,j}$ are the unique real numbers for which

$$\mathbf{u}_j - \sum_{i\in B} \kappa_{i,j}\mathbf{u}_i \in \ker\theta.$$

Then the elements $\mathbf{g}_j$ for $j \in I \setminus B$ constitute a basis for $\ker\theta$.

Now the rank of the linear transformation θ is by definition the dimension of the real vector space $\theta(V)$, and is thus equal to the number of elements in any basis of that vector space. The elements $\theta(\mathbf{u}_i)$ for $i \in B$ constitute a basis of that vector space. Therefore $\operatorname{rank}(\theta) = |B|$, where $|B|$ denotes the number of integers belonging to the finite set B. Similarly the nullity of θ is by definition the dimension of the kernel $\ker\theta$ of θ. The elements $\mathbf{g}_j$ for

$j \in I \setminus B$ constitute a basis of $\ker \theta$. Therefore $\text{nullity}(\theta) = |I \setminus B|$, where $|I \setminus B|$ denotes the number of integers belonging to the finite set $I \setminus B$.

Now $|B| + |I \setminus B| = n$. It follows that

$$\text{rank}(\theta) + \text{nullity}(\theta) = n = \dim V,$$

as required. ∎

A.5 Dual Spaces

Definition Let V be a real vector space. A *linear functional* $\varphi: V \to \mathbb{R}$ on V is a linear transformation from the vector space V to the field $\mathbb{R}$ of real numbers.

Given linear functionals $\varphi: V \to \mathbb{R}$ and $\psi: V \to \mathbb{R}$ on a real vector space V, and given any real number λ, we define $\varphi + \psi$ and $\lambda\varphi$ to be the linear functionals on V defined such that $(\varphi+\psi)(\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v})$ and $(\lambda\varphi)(\mathbf{v}) = \lambda\,\varphi(\mathbf{v})$ for all $\mathbf{v} \in V$.

The set V^* of linear functionals on a real vector space V is itself a real vector space with respect to the algebraic operations of addition and multiplication-by-scalars defined above.

Definition Let V be a real vector space. The *dual space* V^* of V is the vector space whose elements are the linear functionals on the vector space V.

Now suppose that the real vector space V is finite-dimensional. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, where $n = \dim V$. Given any $\mathbf{v} \in V$ there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j$. It follows that there are well-defined functions $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ from V to the field $\mathbb{R}$ defined such that

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. These functions are linear transformations, and are thus linear functionals on V.

Lemma A.13 *Let V be a finite-dimensional real vector space, let*

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$$

be a basis of V, and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the linear functionals on V defined such that

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ constitute a basis of the dual space V^* of V. Moreover

$$\varphi = \sum_{i=1}^{n} \varphi(\mathbf{u_i}) \varepsilon_i$$

for all $\varphi \in V^*$.

Proof Let $\mu_1, \mu_2, \ldots, \mu_n$ be real numbers with the property that $\sum\limits_{i=1}^{n} \mu_i \varepsilon_i = \mathbf{0}_{V^*}$. Then

$$0 = \left(\sum_{i=1}^{n} \mu_i \varepsilon_i \right) (\mathbf{u}_j) = \sum_{i=1}^{n} \mu_i \varepsilon_i(\mathbf{u}_j) = \mu_j$$

for $j = 1, 2, \ldots, n$. Thus the linear functionals $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ on V are linearly independent elements of the dual space V^*.

Now let $\varphi \colon V \to \mathbb{R}$ be a linear functional on V, and let $\mu_i = \varphi(\mathbf{u}_i)$ for $i = 1, 2, \ldots, n$. Now

$$\varepsilon_i(\mathbf{u}_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

It follows that

$$\begin{aligned} \left(\sum_{i=1}^{n} \mu_i \varepsilon_i \right) \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i \lambda_j \varepsilon_i(\mathbf{u}_j) = \sum_{j=1}^{n} \mu_j \lambda_j \\ &= \sum_{j=1}^{n} \lambda_j \varphi(\mathbf{u}_j) = \varphi \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) \end{aligned}$$

for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

It follows that

$$\varphi = \sum_{i=1}^{n} \mu_i \varepsilon_i = \sum_{i=1}^{n} \varphi(\mathbf{u}_i) \varepsilon_i.$$

We conclude from this that every linear functional on V can be expressed as a linear combination of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Thus these linear functionals span V^*. We have previously shown that they are linearly independent. It follows that they constitute a basis of V^*. Moreover we have verified that $\varphi = \sum\limits_{i=1}^{n} \varphi(\mathbf{u}_i) \varepsilon_i$ for all $\varphi \in V^*$, as required. ∎

Definition Let V be a finite-dimensional real vector space, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V. The corresponding *dual basis* of the dual space V^* of V consists of the linear functionals $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ on V, where

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Corollary A.14 *Let V be a finite-dimensional real vector space, and let V^* be the dual space of V. Then* $\dim V^* = \dim V$.

Proof We have shown that any basis of V gives rise to a dual basis of V^*, where the dual basis of V has the same number of elements as the basis of V to which it corresponds. The result follows immediately from the fact that the dimension of a finite-dimensional real vector space is the number of elements in any basis of that vector space. ∎

Let V be a real-vector space, and let V^* be the dual space of V. Then V^* is itself a real vector space, and therefore has a dual space V^{**}. Now each element $\mathbf{v}$ of V determines a corresponding linear functional $E_{\mathbf{v}}: V^* \to \mathbb{R}$ on V^*, where $E_{\mathbf{v}}(\varphi) = \varphi(\mathbf{v})$ for all $\varphi \in V^*$. It follows that there exists a function $\iota: V \to V^{**}$ defined so that $\iota(\mathbf{v}) = E_{\mathbf{v}}$ for all $\mathbf{v} \in V$. Then $\iota(\mathbf{v})(\varphi) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in V$ and $\varphi \in V^*$.

Now

$$\iota(\mathbf{v} + \mathbf{w})(\varphi) = \varphi(\mathbf{v} + \mathbf{w}) = \varphi(\mathbf{v}) + \varphi(\mathbf{w}) = (\iota(\mathbf{v}) + \iota(\mathbf{w}))(\varphi)$$

and

$$\iota(\lambda\mathbf{v})(\varphi) = \varphi(\lambda\mathbf{v}) = \lambda\varphi(\mathbf{v}) = (\lambda\iota(\mathbf{v}))(\varphi)$$

for all $\mathbf{v}, \mathbf{w} \in V$ and $\varphi \in V^*$ and for all real numbers λ. It follows that $\iota(\mathbf{v} + \mathbf{w}) = \iota(\mathbf{v}) + \iota(\mathbf{w})$ and $\iota(\lambda\mathbf{v}) = \lambda\iota(\mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$ and for all real numbers λ. Thus $\iota: V \to V^{**}$ is a linear transformation.

Proposition A.15 *Let V be a finite-dimensional real vector space, and let $\iota: V \to V^{**}$ be the linear transformation defined such that $\iota(\mathbf{v})(\varphi) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in V$ and $\varphi \in V^*$. Then $\iota: V \to V^{**}$ is an isomorphism of real vector spaces.*

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the dual basis of V^*, where

$$\varepsilon_i(\mathbf{u}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and let $\mathbf{v} \in V$. Then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$.

Suppose that $\iota(\mathbf{v}) = \mathbf{0}_{V^{**}}$. Then $\varphi(\mathbf{v}) = E_{\mathbf{v}}(\varphi) = 0$ for all $\varphi \in V^*$. In particular $\lambda_i = \varepsilon_i(\mathbf{v}) = 0$ for $i = 1, 2, \ldots, n$, and therefore $\mathbf{v} = \mathbf{0}_V$. We conclude that $\iota: V \to V^{**}$ is injective.

Now let $F: V^* \to \mathbb{R}$ be a linear functional on V^*, let $\lambda_i = F(\varepsilon_i)$ for $i = 1, 2, \ldots, n$, let $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$, and let $\varphi \in V^*$. Then $\varphi = \sum_{i=1}^{n} \varphi(\mathbf{u}_i)\varepsilon_i$ (see Lemma A.13), and therefore

$$\begin{aligned} \iota(\mathbf{v})(\varphi) &= \varphi(\mathbf{v}) = \sum_{i=1}^{n} \lambda_i \varphi(\mathbf{u}_i) = \sum_{i=1}^{n} F(\varepsilon_i)\varphi(\mathbf{u}_i) \\ &= F\left(\sum_{i=1}^{n} \varphi(\mathbf{u}_i)\varepsilon_i\right) = F(\varphi). \end{aligned}$$

Thus $\iota(\mathbf{v}) = F$. We conclude that the linear transformation $\iota: V \to V^{**}$ is surjective. We have previously shown that this linear transformation is injective. There $\iota: V \to V^{**}$ is an isomorphism between the real vector spaces V and V^{**} as required. ∎

The following corollary is an immediate consequence of Proposition A.15.

Corollary A.16 *Let V be a finite-dimensional real vector space, and let V^* be the dual space of V. Then, given any linear functional $F: V^* \to \mathbb{R}$, there exists some $\mathbf{v} \in V$ such that $F(\varphi) = \varphi(\mathbf{v})$ for all $\varphi \in V^*$.*

Definition Let V and W be real vector spaces, and let $\theta: V \to W$ be a linear transformation from V to W. The *adjoint* $\theta^*: W^* \to V^*$ of the linear transformation $\theta: V \to W$ is the linear transformation from the dual space W^* of W to the dual space V^* of V defined such that $(\theta^*\eta)(\mathbf{v}) = \eta(\theta(\mathbf{v}))$ for all $\mathbf{v} \in V$ and $\eta \in W^*$.

A.6 Linear Transformations and Matrices

Let V and V' be finite-dimensional vector spaces, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, and let $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_{n'}$ be a basis of V'. Then every linear transformation $\theta: V \to V'$ can be represented with respect to these bases by an $n' \times n$ matrix, where $n = \dim V$ and $n' = \dim V'$. The basic formulae are presented in the following proposition.

Proposition A.17 *Let V and V' be finite-dimensional vector spaces, and let $\theta\colon V \to V'$ be a linear transformation from V to V'. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, and let $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_{n'}$ be a basis of V'. Let A be the $n' \times n$ matrix whose coefficients $(A)_{k,j}$ are determined such that $\theta(\mathbf{u}_j) = \sum_{k=1}^{m} (A)_{k,j} \mathbf{u}'_k$ for $k = 1, 2, \ldots, n'$. Then*

$$\theta\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right) = \sum_{k=1}^{m} \mu_k \mathbf{u}'_k,$$

where $\mu_k = \sum_{j=1}^{n} (A)_{k,j} \lambda_j$ for $k = 1, 2, \ldots, n'$.

Proof This result is a straightforward calculation, using the linearity of $\theta\colon V \to V'$. Indeed

$$\begin{aligned} \theta\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right) &= \sum_{j=1}^{n} \lambda_j \theta(\mathbf{u}_j) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n'} (A)_{k,j} \lambda_j \mathbf{u}'_k. \end{aligned}$$

It follows that $\theta\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right) = \sum_{k=1}^{n'} \mu_k \mathbf{u}'_k$, where $\mu_k = \sum_{j=1}^{n} (A)_{k,j} \lambda_j$ for $k = 1, 2, \ldots, n'$, as required. ∎

Corollary A.18 *Let V, V' and V'' be finite-dimensional vector spaces, and let $\theta\colon V \to V'$ be a linear transformation from V to V' and let $\psi\colon V' \to V''$ be a linear transformation from V' to V''. Let A and B be the matrices representing the linear transformations θ and ψ respectively with respect to chosen bases of V, V' and V''. Then the matrix representing the composition $\psi \circ \theta$ of the linear transformations θ and ψ is the product BA of the matrices representing those linear transformations.*

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, let $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_{n'}$ be a basis of V', and let $\mathbf{u}''_1, \mathbf{u}''_2, \ldots, \mathbf{u}''_{n''}$ be a basis of V''. Let A and B be the matrices whose coefficients $(A)_{k,j}$ and $(B)_{i,k}$ are determined such that $\theta(\mathbf{u}_j) = \sum_{k=1}^{n'} (A)_{k,j} \mathbf{u}'_k$ for $k = 1, 2, \ldots, n'$ and $\psi(\mathbf{u}'_k) = \sum_{i=1}^{p} (B)_{i,k} \mathbf{u}''_i$. Then

$$\psi\left(\theta\left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j\right)\right) = \sum_{i=1}^{p} \nu_i \mathbf{u}''_i,$$

where

$$\nu_i = \sum_{j=1}^{n} \left(\sum_{k=1}^{n'} (B)_{l,k} (A)_{k,j} \right) \lambda_j$$

for $l = 1, 2, \ldots, p$. Thus the composition $\psi \circ \theta$ of the linear transformations $\theta: V \to V'$ and $\psi: V' \to V''$ is represented by the product BA of the matrix B representing ψ and the matrix A representing A with respect to the chosen bases of V, V' and V'', as required. ∎

Lemma A.19 *Let V and W be finite-dimensional real vector spaces, and let $\theta: V \to W$ be a linear transformation from V to W. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the corresponding dual basis of the dual space V^* of V, let $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_{n'}$ be a basis of W, and let $\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_n$ be the corresponding dual basis of the dual space W^* of W. Then the matrix representing the adjoint $\theta^*: W^* \to V^*$ of $\theta: V \to W$ with respect to the dual bases of W^* and V^* is the transpose of the matrix representing $\theta: V \to W$ with respect to the chosen bases of V and W.*

Proof Let A be the $n' \times n$ matrix representing the linear transformation $\theta: V \to W$ with respect to the chosen bases. Then $\varphi(\mathbf{u}_j) = \sum_{i=1}^{n'} (A)_{i,j} \mathbf{u}'_i$ for $j = 1, 2, \ldots, n$. Let $\mathbf{v} \in V$ and $\eta \in W^*$, let $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$, let $\eta = \sum_{j=1}^{n'} c_i \varepsilon'_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $c_1, c_2, \ldots, c_{n'}$ are real numbers. Then

$$\begin{aligned}
(\theta^* \eta)(\mathbf{v}) &= \eta(\theta(\mathbf{v})) = \eta \left(\sum_{j=1}^{n} \lambda_j \theta(\mathbf{u}_j) \right) \\
&= \sum_{j=1}^{n} \lambda_j \eta((\theta(\mathbf{u}_j)) = \sum_{j=1}^{n} \lambda_j \eta \left(\sum_{i=1}^{n'} (A)_{i,j} \mathbf{u}'_i \right) \\
&= \sum_{i=1}^{n'} \sum_{j=1}^{n} (A)_{i,j} \lambda_j \eta(\mathbf{u}'_i) = \sum_{i=1}^{n'} \sum_{j=1}^{n} (A)_{i,j} \lambda_j c_i.
\end{aligned}$$

Thus if

$$\eta = \sum_{j=1}^{n'} c_i \varepsilon'_i,$$

where $c_1, c_2, \ldots, c_n$ are real numbers, then

$$\theta^* \eta = \sum_{i=1}^{n} h_j \varepsilon_j,$$

where

$$h_j = \sum_{i=1}^{n'} (A)_{i,j} c_i = \sum_{i=1}^{n'} (A^T)_{j,i} c_i$$

for $j = 1, 2, \ldots, n$, and where A^T is the transpose of the matrix A, defined so that $(A^T)_{j,i} = A_{i,j}$ for $i = 1, 2, \ldots, n'$ and $j = 1, 2, \ldots, n$. It follows from this that the matrix that represents the adjoint θ^* with respect to the dual bases on W^* and V^* is the transpose of the matrix A that represents θ with respect to the chosen bases on V and W, as required. ∎