Module MA3427: Algebraic Topology I Michaelmas Term 2010

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1 The Classification of Surfaces

1.1 Triangulated Closed Surfaces

Let **v** and **w** be distinct points of some Euclidean space \mathbb{R}^k . Then **v** and **w** are the vertices (or endpoints) of an edge E in \mathbb{R}^k , where

$$E = \{ r\mathbf{v} + s\mathbf{w} : s, t \in [0, 1] \text{ and } s + t = 1 \}.$$

We denote by $\mathbf{v} \mathbf{w}$ the edge in \mathbb{R}^k whose vertices (or endpoints) are the points \mathbf{v} and \mathbf{w} .

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be points of some Euclidean space \mathbb{R}^k which are not colinear. These points are then the vertices of a *triangle* T in \mathbb{R}^k , where

$$T = \{ r\mathbf{u} + s\mathbf{v} + t\mathbf{w}; r, s, t \in [0, 1] \text{ and } r + s + r = 1 \}.$$

We denote by $\mathbf{u} \mathbf{v} \mathbf{w}$ the triangle in \mathbb{R}^k whose vertices are the points \mathbf{u} , \mathbf{v} and \mathbf{w} and whose edges are $\mathbf{u} \mathbf{v}$, $\mathbf{v} \mathbf{w}$ and $\mathbf{w} \mathbf{u}$.

Any edge or triangle in a Euclidean space is a compact subset of that Euclidean space.

Definition A two-dimensional simplicial complex in a Euclidean space consists of a finite collection K of triangles, edges (which are line segments) and vertices (which are points) in that space which contains at least one triangle, and which satisfies the following conditions:

- (i) The edges and vertices of any triangle belonging to K themselves belong to K;
- (ii) The endpoints of any edge belonging to K are vertices belonging to K;
- (iii) if two distinct triangles belonging to K have a non-empty intersection, then that intersection is either a single common edge or a single common vertex of both triangles;
- (iv) if two distinct edges belonging to K have a non-empty intersection then that intersection is a common vertex (or endpoint) of both edges.

Definition Let K be a two-dimensional simplicial complex in some Euclidean space. The *polyhedron* |K| of K is the union of all the triangles, edges and vertices belonging to the collection K.

Lemma 1.1 The polyhedron of a two-dimensional simplicial complex is a compact Hausdorff space.

Proof The simplicial complex K is a finite collection of triangles, edges and vertices in some ambient Euclidean space, and each triangle, edge and vertex in the collection is a closed bounded subset of this ambient Euclidean space. Now a subset of a Euclidean space is compact if and only if it is both closed and bounded. It follows that each of the triangles, edges and vertices belonging to K is a compact subset of the ambient Euclidean space. Moreover it follows directly from the definition of compactness that any finite union of compact topological spaces is itself compact. Therefore the polyhedron |K|of K is a compact subset of the ambient Euclidean space. This ambient Euclidean space is a Hausdorff space (as it is a metric space, and all metric spaces are Hausdorff spaces), and any subset of a Hausdorff space is itself a Hausdorff space (with the subspace topology). Therefore the polyhedron |K|of K is a compact Hausdorff space, as required.

Definition Let \mathbf{p} be a point of the polyhedron |K| of the two-dimensional simplicial complex K. The star neighbourhood $\operatorname{st}_{K}(\mathbf{p})$ of the point \mathbf{p} in |K| is defined to be the subset of |K| whose complement is the union of all triangles, edges and vertices belonging to K that do not contain the point \mathbf{p} .

Lemma 1.2 Let K be a two-dimensional simplicial complex, and let \mathbf{p} be a point of K. Then the star neighbourhood $\operatorname{st}_{K}(\mathbf{p})$ of the point \mathbf{p} of |K| is an open subset of |K|, and moreover $\mathbf{p} \in \operatorname{st}_{K}(\mathbf{p})$.

Proof A two-dimensional simplicial complex is a finite collection of triangles, edges and vertices in some ambient Euclidean space. Each of those triangles, edges and vertices is a closed subset of the ambient Euclidean space, and therefore the union of any finite collection of such triangles, edges and vertices is a closed subset of the ambient Euclidean space.

Now, given any point \mathbf{p} of |K|, the complement $|K| \setminus \operatorname{st}_K(\mathbf{p})$ of the star neighbourhood $\operatorname{st}_K(\mathbf{p})$ of \mathbf{p} in |K| is by definition the union of all triangles, edges and vertices belonging to K that do not contain the point \mathbf{p} . It follows that $|K| \setminus \operatorname{st}_K(\mathbf{p})$ is closed in |K|, and $\mathbf{p} \notin |K| \setminus \operatorname{st}_K(\mathbf{p})$. Therefore $\operatorname{st}_K(\mathbf{p})$ is open in |K|, and $\mathbf{p} \in \operatorname{st}_K(\mathbf{p})$, as required.

1.2 Triangulated Closed Surfaces

Definition A *topological closed surface* is a compact Hausdorff space that may be covered by open sets, where each of these open sets is homeomorphic to a open set in the Euclidean plane.

An open set in the Euclidean plane is a union of open disks in that plane. It follows that a compact Hausdorff space is a topological closed surface if and only if it can be covered by open sets, where each of these open sets is homeomorphic to a open disk in the Euclidean plane.

Proposition 1.3 Let K be a two-dimensional simplicial complex which satisfies the following two conditions:—

- (i) every edge belonging to K is an edge of exactly two triangles belonging to K;
- (ii) given any vertex v belonging to K, the triangles that have v as vertex can be listed as a finite sequence T₁, T₂,..., T_m, where m > 1, where T_i and T_{i-1} intersect along a common edge when 1 < i ≤ m, and where T_m and T₁ also intersect along a common edge.

Then the polyhedron |K| of K is a topological closed surface.

Proof The polyhedron |K| of the two-dimensional simplicial complex K is a compact Hausdorff space. We shall prove that the star neighbourhood of each point of |K| is homeomorphic to an open disk.

Now suppose that the point \mathbf{p} belongs to a triangle T of K with vertices \mathbf{u}, \mathbf{v} and \mathbf{w} but does not lie on any edge of that triangle. Then the triangle T is the only member of the collection K of triangles, edges and vertices that contains the point \mathbf{p} . It follows that the star neighbourhood $\operatorname{st}_K(\mathbf{p})$ consists of all points of the triangle T that do not lie on any edge of T. Thus $\operatorname{st}_K(\mathbf{p})$ is homeomorphic to the interior of a triangle in the Euclidean plane.

Next suppose that the point \mathbf{p} belongs to an edge of K with vertices \mathbf{v} and \mathbf{w} but is not an endpoint of that edge. The edge is an edge of exactly two triangles belonging to K, because K represents a triangulated closed surface. Let these two triangles be $\mathbf{v} \mathbf{w} \mathbf{x}$ and $\mathbf{v} \mathbf{w} \mathbf{y}$. The conditions in the definition of two-dimensional complex ensure that the only members of the collection Kthat contain the point \mathbf{p} are the edge $\mathbf{v} \mathbf{w}$ and the two triangles $\mathbf{v} \mathbf{w} \mathbf{x}$ and $\mathbf{v} \mathbf{w} \mathbf{y}$. It follows that the star neighbourhood $\operatorname{st}_K(\mathbf{p})$ of the point \mathbf{p} in |K|consists of all points of the union of these two triangles that do not lie on any of the edges $\mathbf{v} \mathbf{x}$, $\mathbf{x} \mathbf{w}$, $\mathbf{w} \mathbf{y}$ and $\mathbf{y} \mathbf{v}$. It follows from this that $\operatorname{st}_K(\mathbf{p})$ is homeomorphic to the interior of a quadrilateral in the Euclidean plane.

Finally suppose that \mathbf{v} is a vertex belonging to K. Then the triangles that have \mathbf{v} as vertex can be listed as a finite sequence T_1, T_2, \ldots, T_m , where m > 1, where T_i and T_{i-1} intersect along a common edge when $1 < i \leq m$, and where T_m and T_1 also intersect along a common edge. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be the vertices of these triangles distinct from \mathbf{v} , ordered so that the triangles T_m and T_1 intersect along the edge $\mathbf{v} \mathbf{w}_1$ and the triangles T_i and T_{i-1} intersect along the edge $\mathbf{v} \mathbf{w}_i$ for $i < i \leq m$. Then T_i is the triangle $\mathbf{v} \mathbf{w}_i \mathbf{w}_{i+1}$ for i = 1, 2, ..., m - 1, and T_m is the triangle $\mathbf{v} \mathbf{w}_m \mathbf{w}_1$. The triangles of K that have \mathbf{v} as a vertex are thus in the configuration depicted in Figure 1. The union of these triangles $T_1, T_2, ..., T_m$ is then homeomorphic to a convex



Figure 1

polygon in the Euclidean plane. The union of those edges

 $\mathbf{w}_m \, \mathbf{w}_1, \ \mathbf{w}_1 \, \mathbf{w}_2, \ \cdots \ \mathbf{w}_{m-1} \, \mathbf{w}_m$

of these triangles that do not have \mathbf{v} as one endpoint corresponds under this homeomorphism to the boundary of the convex polygon, and therefore the star neighbourhood $\operatorname{st}_K(\mathbf{v})$ of \mathbf{v} in |K| is homeomorphic to the interior of a convex polygon in the Euclidean plane.

We have thus shown that, given any point \mathbf{p} of the polyhedron of K, the star neighbourhood of the point \mathbf{p} is an open set in |K| which is homeomorphic to the interior of a convex polygon in the Euclidean plane. The interior of such a polygon is homeomorphic to a disk. The result follows.

Lemma 1.4 Let K be a two-dimensional simplicial complex which satisfies the two conditions listed in the statement of Proposition 1.3 that ensure that the polyhedron |K| of K is a topological closed surface. Then this polyhedron is a connected topological space if and only if, given any two triangles σ and τ of K, we can find a sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$ of triangles of K with $\sigma = \sigma_1$ and $\tau = \sigma_k$, where σ_{i-1} and σ_i intersect in a common edge for $i = 2, 3, \ldots, k$.



Figure 2

Proof Let σ_0 be a triangle in K, and let F be the subset of the polyhedron |K| of K which is the union of all triangles that can be joined to σ_0 by a finite sequence of triangles belonging to K, where successive triangles in this sequence intersect along a common edge. Then F is a finite union of triangles, and those triangles are closed subsets of |K|, and therefore F is itself a closed subset of |K|.

Let \mathbf{p} be a point of F. If \mathbf{p} does not lie on any edge belonging to K then the star neighbourhood $\operatorname{st}_K(\mathbf{p})$ belongs to just one triangle belonging to K, and moreover this triangle must then be a subset of F (or else the point \mathbf{p} would not belong to F. Thus if $\mathbf{p} \in F$ does not like on any edge belonging to K then $\operatorname{st}_K(\mathbf{p}) \subset F$.

Next suppose that the point \mathbf{p} of F lies on some edge belonging to K but is not an endpoint of that edge. Then the point \mathbf{p} belongs to exactly two triangles of K that intersect along a common edge (because the twodimensional simplicial complex represents a closed surface). At least one of these triangles must be contained in the set F (since $\mathbf{p} \in F$) and therefore both triangles are contained in F. But the star neighbourhood of the point \mathbf{p} is contained in the union of those two triangles. Therefore $\mathrm{st}_K(\mathbf{p}) \subset F$ in this case also.

Finally suppose that the point \mathbf{p} is a vertex of K. Then the requirement that the two-dimensional simplicial complex K represent a triangulated closed surface ensures that if at least one of the triangles belonging to K with a vertex at \mathbf{p} is contained in F then every triangle belonging to K with a vertex at \mathbf{v} must be contained in F. It follows that $\operatorname{st}_K(\mathbf{p}) \subset F$.

We have now shown that, given any point \mathbf{p} of F, the star neighbourhood $\operatorname{st}_{K}(\mathbf{p})$ of \mathbf{p} in |K| is a subset of F. But this star neighbourhood is an open subset of |K| (see Lemma 1.2). Therefore the subset F of |K| is both open and closed in |K|. Thus if the topological space |K| is connected then F = |K|.

Every point of a topological space belongs to unique connected component which is the union of all connected subsets of the topological space that contain the given point. It follows that every triangle belonging to K is contained in a some connected component of |K|, and if two triangles belonging to K intersect along a common edge, or at a common vertex, then both belong to the same connected component of |K|. It follows that the set Fis contained in some connected component of |K|. Thus if the topological space |K| is not connected then F is a proper subset of |K|. We deduce that F = |K| if and only if |K| is a connected topological space. The result follows.

Lemma 1.5 Let K be a triangulated closed surface whose polyhedron |K| is

a connected topological space. Then |K| is homeomorphic to the topological space obtained from a filled polygon with an even number of edges by identifying edges in pairs (i.e., given any edge with endpoints **a** and **b**, there exists exactly one other edge with endpoints **c** and **d** such that $(1 - t)\mathbf{a} + t\mathbf{b}$ is identified with $(1 - t)\mathbf{c} + t\mathbf{d}$ for all $t \in [0, 1]$).

Proof Suppose that we have constructed some subcomplex L of K whose polyhedron is homeomorphic to the identification space obtained from a filled polygon P_L by identifying some of the edges of that polygon in pairs. Let $q_L: P_L \to L$ denote the identification map. Suppose that e is an edge of P_L that is not identified to any other edge of L. Then e corresponds under the identification map to some edge e' of P_L . Moreover only one of the two triangles in K adjoining the edge e' belongs to L. Thus there is some triangle σ of $K \setminus L$ which has e' as one of its edges. Let M be the subcomplex of K obtained on adjoining to L the triangle σ , together with all its edges and vertices.

We now extend the polygon P_L by attaching a triangle T along the free edge e to obtain a filled polygon P_M , where $P_M = P_L \cup T$ and $P_L \cap T = e$. We also extend the identification map $q_L: P_L \to |L|$ over this attached triangle to obtain an identification map $q_M: P_M \to |M|$, where $q_M |P_L = q_L$ and $q_M |T$ is a simplicial homeomorphism mapping the triangle T onto σ . Then the new identification map $q_M: P_M \to M$ also identifies some of the edges of the polygon P_M in pairs. Thus, by successively adding triangles in this fashion, we eventually obtain a subcomplex L of K whose polyhedron is homeomorphic to the identification space obtained from a filled polygon on identifying all of the edges of that polygon in pairs. But then, given any two triangles of K that intersect along a common edge, either both triangles belong to L, or else neither triangle belongs to L. It now follows from Lemma 1.4 that L = K, and thus the polyhedron of K is an identification space of the prescribed type.

1.3 The Topological Classification of Closed Surfaces

We wish to classify up to homeomorphism the identification spaces obtained from polygons by identifying edges in pairs.

Suppose that we are given a polygon with its edges identified in pairs. Choose an orientation (i.e., a 'direction') on each edge of the polygon in such a way that, for each pair of identified edges, the orientations on those edges correspond under the identification of these edges. Denote each pair of identified edges by some letter a, b, c, \ldots Suppose that we travel round the boundary of the polygon in the anticlockwise direction, starting at some

chosen vertex. We obtain a surface symbol consisting of a sequence of symbols taken from $a, a^{-1}, b, b^{-1}, c, c^{-1}$, ordered so as to represent the order in which the corresponding edges of the polygon are traversed (on travelling round the polygon in the anticlockwise direction), and where an edge represented by some letter x occurs in the surface symbol as 'x' if the chosen orientation on the edge agrees with the anticlockise orientation, or as ' x^{-1} ' if the chosen orientation. For example, the surface symbol $xyx^{-1}y^{-1}$ represents the torus (see Figure 3), and the surface symbol $xy^{-1}xy^{-1}$ represents the real projective plane (see Figure 4).



Lemma 1.5 shows that any connected triangulated closed surface can be described by such a surface symbol. This is a finite sequence of symbols of the form $a, a^{-1}, b, b^{-1}, c, c^{-1}, \ldots, a, b, c, \ldots$ representing some suitable list of 'letters' that label the pairs of identified edges of the polygon representing the surface. Each 'letter' x present occurs exactly twice in the surface symbol, either as 'x' or as 'x⁻¹'. Conversely any surface symbol of this form determines a scheme for identifying in pairs the edges of a suitable polygon to obtain a closed surface. We wish to determine necessary and sufficient conditions for determining whether or not the surfaces obtained in this way from two such surface symbols are homeomorphic.

Let us use capital letters A, B, C, \ldots to denote (possibly empty) sequences of symbols taken from the list $a, a^{-1}, b, b^{-1}, c, c^{-1}, \ldots$. Also if A is such a sequence of symbols, given by $A = a_1 a_2 \cdots a_n$, then we write A^{-1} for the sequence $a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$ (where $(x^{-1})^{-1} \equiv x$ for any letter x.) Using these conventions, we now state three rules which enable one to transform one surface symbol into another in such a way that the two surface symbols represent surfaces that are homeomorphic.

- Rule 1. cyclically permute the symbols occurring in the surface symbol,
- Rule 2. Replace ABxCDxE by $AyDB^{-1}yC^{-1}E$ (where y represents some letter not occurring in the first surface symbol).
- **Rule 3.** Replace $ABxCDx^{-1}E$ by $AyDCy^{-1}BE$.
- Rule 4. Replace $Axx^{-1}B$ or $Ax^{-1}xB$ by AB, provided that AB contains at least two letters (each occurring twice).

Lemma 1.6 The application of Rules 1–4 to a surface symbol gives a new surface symbol such that the surfaces determined by the two surface symbols are homeomorphic.

Proof Rule 1 corresponds to traversing the boundary of the polygon starting from a different vertex. Rules 2–4 are justified by the simple 'cut and paste' operations depicted in Figures 5, 6 and 7.

Rules 1–4 allow the reduction of surface symbols to certain standard forms. Now each letter x in a surface symbol occurs exactly twice; if the letter x either occurs both times as 'x' or else occurs both times as ' x^{-1} ', then we call the occurrence of the letter x in the surface symbol a *similar pair*; otherwise we call the occurrence of this letter a *reversed pair*. Two reversed pairs in some given surface symbol are said to *interlock* if they occur in the order $\cdots y \cdots z \cdots y^{-1} \cdots z^{-1} \cdots$ (after interchanging y and y^{-1} , or z and z^{-1} , if necessary).

Proposition 1.7 Any surface symbol can be reduced, by suitable applications of the transformations described in Rules 1–4 above and their inverses, to one of the following canonical forms:—

$$x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_g y_g x_g^{-1} y_g^{-1} \qquad (g \ge 1),$$
$$x_1 x_1 x_2 x_2 \cdots x_h x_h \qquad (h \ge 2),$$
$$x x^{-1} y y^{-1}, \qquad x x y y^{-1}.$$

Proof First we note that if C is any sequence of the form $x_1x_1x_2x_2...$, then a sequence of transformations

$$CDxExF \rightarrow CyD^{-1}yE^{-1}F \rightarrow CzzDE^{-1}F$$



of the type specified by Rule 2 will transform any surface symbol CDxExFwith a similar pair to one of the form $CzzDE^{-1}$. Repeated applications of this procedure reduce any surface symbol to one of the form AB, where A is of the form form $x_1x_1x_2x_2\cdots x_rx_r$ and B contains only reversed pairs (where either A or B may be empty).

One can now use Rule 3 in order to reduce a surface symbol of the form AB to one of the form ACD, where C is of the form

$$y_1 z_1 y_1^{-1} z_1^{-1} \cdots y_s z_s y_s^{-1} z_s^{-1},$$

and D contains only non-interlocking reversed pairs. Indeed if E is any surface symbol of the required form, then successive applications of Rule 3 show that any surface symbol of the form $EFaGbHa^{-1}Ib^{-1}J$ can be transformed to $Eefe^{-1}f^{-1}FIHGJ$ by the following sequence of transformations:

$$\begin{split} EFaGbHa^{-1}Ib^{-1}J &\to EcGbHc^{-1}FIb^{-1}J \\ &\to EcGdFIHc^{-1}d^{-1}J \\ &\to EeFIHGde^{-1}d^{-1}J \\ &\to Eefe^{-1}f^{-1}FIHGJ. \end{split}$$

The stated reduction of AB to ACD now follows by induction on the number of interlocking reversed pairs.

If A is non-empty then one can reduce a surface symbol of the form ACD (where A, C and D are as above) to one of the form ED, where E is of the form $x_1x_1x_2x_2\cdots$ and D contains only non-interlocking reversed pairs. This follows from successive applications of the following sequence of transformations (which are inverses of transformations of the type specified by Rule 2):

$$Fxxaba^{-1}b^{-1}G \to Fyb^{-1}a^{-1}ya^{-1}b^{-1}G \to Fyay^{-1}accG \leftarrow FyyddccG.$$

Now consider D, which consists only of non-interlocking reversed pairs. Let $\cdots x \cdots x^{-1} \cdots$ be the closest reversed pair occurring in D. Then x and x^{-1} must be adjacent (since otherwise D would contain two interlocking reversed pairs). We can therefore 'cancel' xx^{-1} , by Rule 4, provided that the resultant symbol always contains at least two letters. It follows that any surface symbol with more than two letters can be reduced to one or other of the first two canonical forms specified (with $g \ge 1$ or $h \ge 2$).

Finally consider surface symbols with two letters occurring. The procedures described above reduce such a surface symbol to one of the following forms: $xyx^{-1}y^{-1}$, xxyy, $xxyy^{-1}$, $xx^{-1}yy^{-1}$, $xyy^{-1}x^{-1}$. The first four of these are included in the list of canonical forms. The symbol $xyy^{-1}x^{-1}$ reduces to $zz^{-1}yy^{-1}$ on cyclically permuting the symbol (according to Rule 1) and replacing x^{-1} and x by z and z^{-1} respectively, as required.

Let M_g $(g \ge 1)$ be the space obtained from a regular 4g-sided polygon by identifying the edges according to the sequence $x_1y_1x_1^{-1}y_1^{-1}\cdots x_gy_gx_g^{-1}y_g^{-1}$ (see Figure 8), and let N_h $(h \ge 2)$ be defined similarly using $x_1x_1\cdots x_hx_h$ (see Figure 9). Also let M_0 and N_1 be surfaces whose surface symbols are $xx^{-1}yy^{-1}$ and $xxyy^{-1}$ respectively. We have so far proved that the polyhedron of any connected triangulated closed surface is homeomorphic to one of the spaces M_g $(g \ge 0)$ or N_h $(h \ge 1)$.





Figure 8: The surface M_g (g = 3)

Figure 9: The surface N_h (h = 4)