MA342R—Covering Spaces and Fundamental Groups
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7.5. Triangulated Closed Surfaces

Definition

A *topological closed surface* is a compact Hausdorff space that may be covered by open sets, where each of these open sets is homeomorphic to a open set in the Euclidean plane.

An open set in the Euclidean plane is a union of open disks in that plane. It follows that a compact Hausdorff space is a topological closed surface if and only if it can be covered by open sets, where each of these open sets is homeomorphic to a open disk in the Euclidean plane.

Proposition 7.6

Let K be a two-dimensional simplicial complex which satisfies the following two conditions:—

(i) *every edge belonging to K is an edge of exactly two triangles belonging to K;*

(ii) *given any vertex $\mathbf{v}$ belonging to K, the triangles that have $\mathbf{v}$ as vertex can be listed as a finite sequence $T_1, T_2, \ldots, T_m$, where $m > 1$, where T_i and T_{i-1} intersect along a common edge when $1 < i \leq m$, and where T_m and T_1 also intersect along a common edge.*

Then the polyhedron $|K|$ of K is a topological closed surface.

Proof

The polyhedron $|K|$ of the two-dimensional simplicial complex K is a compact Hausdorff space. We shall prove that the star neighbourhood of each point of $|K|$ is homeomorphic to an open disk.

Now suppose that the point $\mathbf{p}$ belongs to a triangle T of K with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ but does not lie on any edge of that triangle. Then the triangle T is the only member of the collection K of triangles, edges and vertices that contains the point $\mathbf{p}$. It follows that the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ consists of all points of the triangle T that do not lie on any edge of T. Thus $\mathrm{st}_K(\mathbf{p})$ is homeomorphic to the interior of a triangle in the Euclidean plane.

Next suppose that the point $\mathbf{p}$ belongs to an edge of K with vertices $\mathbf{v}$ and $\mathbf{w}$ but is not an endpoint of that edge. The edge is an edge of exactly two triangles belonging to K, because K represents a triangulated closed surface. Let these two triangles be $\mathbf{v}\,\mathbf{w}\,\mathbf{x}$ and $\mathbf{v}\,\mathbf{w}\,\mathbf{y}$. The conditions in the definition of two-dimensional complex ensure that the only members of the collection K that contain the point $\mathbf{p}$ are the edge $\mathbf{v}\,\mathbf{w}$ and the two triangles $\mathbf{v}\,\mathbf{w}\,\mathbf{x}$ and $\mathbf{v}\,\mathbf{w}\,\mathbf{y}$. It follows that the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ of the point $\mathbf{p}$ in $|K|$ consists of all points of the union of these two triangles that do not lie on any of the edges $\mathbf{v}\,\mathbf{x}$, $\mathbf{x}\,\mathbf{w}$, $\mathbf{w}\,\mathbf{y}$ and $\mathbf{y}\,\mathbf{v}$. It follows from this that $\mathrm{st}_K(\mathbf{p})$ is homeomorphic to the interior of a quadrilateral in the Euclidean plane.

Finally suppose that $\mathbf{v}$ is a vertex belonging to K. Then the triangles that have $\mathbf{v}$ as vertex can be listed as a finite sequence $T_1, T_2, \ldots, T_m$, where $m > 1$, where T_i and T_{i-1} intersect along a common edge when $1 < i \leq m$, and where T_m and T_1 also intersect along a common edge. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be the vertices of these triangles distinct from $\mathbf{v}$, ordered so that the triangles T_m and T_1 intersect along the edge $\mathbf{v}\,\mathbf{w}_1$ and the triangles T_i and T_{i-1} intersect along the edge $\mathbf{v}\,\mathbf{w}_i$ for $i < i \leq m$. Then T_i is the triangle $\mathbf{v}\,\mathbf{w}_i\,\mathbf{w}_{i+1}$ for $i = 1, 2, \ldots, m-1$, and T_m is the triangle $\mathbf{v}\,\mathbf{w}_m\,\mathbf{w}_1$.

The triangles of K that have $\mathbf{v}$ as a vertex are thus in the configuration depicted in Figure 1.

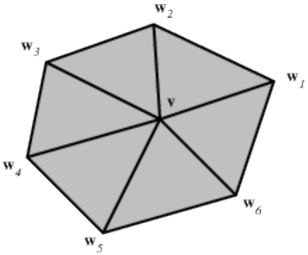

Figure 1

The union of these triangles $T_1, T_2, \ldots, T_m$ is then homeomorphic to a convex polygon in the Euclidean plane.

The union of those edges

$$\mathbf{w}_m\,\mathbf{w}_1,\quad \mathbf{w}_1\,\mathbf{w}_2,\quad \cdots\quad \mathbf{w}_{m-1}\,\mathbf{w}_m$$

of these triangles that do not have $\mathbf{v}$ as one endpoint corresponds under this homeomorphism to the boundary of the convex polygon, and therefore the star neighbourhood $\mathrm{st}_K(\mathbf{v})$ of $\mathbf{v}$ in $|K|$ is homeomorphic to the interior of a convex polygon in the Euclidean plane.

We have thus shown that, given any point $\mathbf{p}$ of the polyhedron of K, the star neighbourhood of the point $\mathbf{p}$ is an open set in $|K|$ which is homeomorphic to the interior of a convex polygon in the Euclidean plane. The interior of such a polygon is homeomorphic to a disk. The result follows. ∎

Lemma 7.7

Let K be a two-dimensional simplicial complex which satisfies the two conditions listed in the statement of Proposition 7.6 that ensure that the polyhedron $|K|$ of K is a topological closed surface. Then this polyhedron is a connected topological space if and only if, given any two triangles σ and τ of K, we can find a sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$ of triangles of K with $\sigma = \sigma_1$ and $\tau = \sigma_k$, where σ_{i-1} and σ_i intersect in a common edge for $i = 2, 3, \ldots, k$.

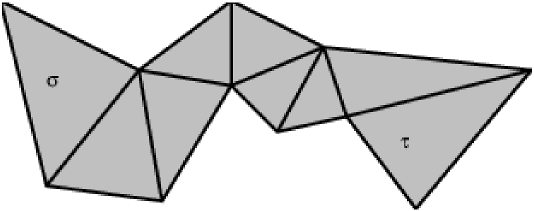

Figure 2

Proof

Let σ_0 be a triangle in K, and let F be the subset of the polyhedron $|K|$ of K which is the union of all triangles that can be joined to σ_0 by a finite sequence of triangles belonging to K, where successive triangles in this sequence intersect along a common edge. Then F is a finite union of triangles, and those trianges are closed subsets of $|K|$, and therefore F is itself a closed subset of $|K|$.

Let $\mathbf{p}$ be a point of F. If $\mathbf{p}$ does not lie on any edge belonging to K then the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ belongs to just one triangle belonging to K, and moreover this triangle must then be a subset of F (or else the point $\mathbf{p}$ would not belong to F). Thus if $\mathbf{p} \in F$ does not like on any edge belonging to K then $\mathrm{st}_K(\mathbf{p}) \subset F$.

Next suppose that the point $\mathbf{p}$ of F lies on some edge belonging to K but is not an endpoint of that edge. Then the point $\mathbf{p}$ belongs to exactly two triangles of K that intersect along a common edge (because the two-dimensional simplicial complex represents a closed surface). At least one of these triangles must be contained in the set F (since $\mathbf{p} \in F$) and therefore both triangles are contained in F. But the star neighbourhood of the point $\mathbf{p}$ is contained in the union of those two triangles. Therefore $\mathrm{st}_K(\mathbf{p}) \subset F$ in this case also.

Finally suppose that the point $\mathbf{p}$ is a vertex of K. Then the requirement that the two-dimensional simplicial complex K represent a triangulated closed surface ensures that if at least one of the triangles belonging to K with a vertex at $\mathbf{p}$ is contained in F then every triangle belonging to K with a vertex at $\mathbf{v}$ must be contained in F. It follows that $\mathrm{st}_K(\mathbf{p}) \subset F$.

We have now shown that, given any point $\mathbf{p}$ of F, the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ of $\mathbf{p}$ in $|K|$ is a subset of F. But this star neighbourhood is an open subset of $|K|$ (see Lemma 7.5). Therefore the subset F of $|K|$ is both open and closed in $|K|$. Thus if the topological space $|K|$ is connected then $F = |K|$.

Every point of a topological space belongs to unique connected component which is the union of all connected subsets of the topological space that contain the given point. It follows that every triangle belonging to K is contained in a some connected component of $|K|$, and if two triangles belonging to K intersect along a common edge, or at a common vertex, then both belong to the same connected component of $|K|$. It follows that the set F is contained in some connected component of $|K|$. Thus if the topological space $|K|$ is not connected then F is a proper subset of $|K|$. We deduce that $F = |K|$ if and only if $|K|$ is a connected topological space. The result follows. ∎

Lemma 7.8

Let K be a triangulated closed surface whose polyhedron $|K|$ is a connected topological space. Then $|K|$ is homeomorphic to the topological space obtained from a filled polygon with an even number of edges by identifying edges in pairs (i.e., given any edge with endpoints $\mathbf{a}$ and $\mathbf{b}$, there exists exactly one other edge with endpoints $\mathbf{c}$ and $\mathbf{d}$ such that $(1-t)\mathbf{a} + t\mathbf{b}$ is identified with $(1-t)\mathbf{c} + t\mathbf{d}$ for all $t \in [0, 1]$).

Proof

Suppose that we have constructed some subcomplex L of K whose polyhedron is homeomorphic to the identification space obtained from a filled polygon P_L by identifying some of the edges of that polygon in pairs. Let $q_L \colon P_L \to |L|$ denote the identification map.

Suppose that e is an edge of P_L that is not identified to any other edge of L. Then e corresponds under the identification map to some edge e' of L. Moreover only one of the two triangles in K adjoining the edge e' belongs to L. Thus there is some triangle σ of $K \setminus L$ which has e' as one of its edges. Let M be the subcomplex of K obtained on adjoining to L the triangle σ, together with all its edges and vertices. We now extend the polygon P_L by attaching a triangle T along the free edge e to obtain a filled polygon P_M, where $P_M = P_L \cup T$ and $P_L \cap T = e$. We also extend the identification map $q_L \colon P_L \to |L|$ over this attached triangle to obtain an identification map $q_M \colon P_M \to |M|$, where $q_M|P_L = q_L$ and $q_M|T$ is a simplicial homeomorphism mapping the triangle T onto σ. Then the new identification map $q_M \colon P_M \to |M|$ also identifies some of the edges of the polygon P_M in pairs.

If we successively add triangles to build up a polygon in this fashion, we eventually obtain a subcomplex L of K whose polyhedron is homeomorphic to the identification space obtained from a filled polygon on identifying all of the edges of that polygon in pairs. But then, given any two triangles of K that intersect along a common edge, either both triangles belong to L, or else neither triangle belongs to L. It now follows from Lemma 7.7 that $L = K$, and thus the polyhedron of K is an identification space of the prescribed type. ∎