MA342R—Covering Spaces and Fundamental Groups
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7. The Topology of Closed Surfaces

7.1. Affine Independence

Definition

Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *affinely independent* (or *geometrically independent*) if the only solution of the linear system

$$\begin{cases} \sum_{j=0}^{q} s_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum_{j=0}^{q} s_j & = & 0 \end{cases}$$

is the trivial solution $s_0 = s_1 = \cdots = s_q = 0$.

Lemma 7.1

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be points of Euclidean space $\mathbb{R}^k$ of dimension k. Then the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent if and only if the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Proof

Suppose that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent. Let $s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equation

$$\sum_{j=1}^{q} s_j(\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

Then $\sum\limits_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum\limits_{j=0}^{q} s_j = 0$, where $s_0 = -\sum\limits_{j=1}^{q} s_j$, and therefore

$$s_0 = s_1 = \cdots = s_q = 0.$$

It follows that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Conversely, suppose that these displacement vectors are linearly independent. Let $s_0, s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equations $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$. Then $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$\mathbf{0} = \sum_{j=0}^{q} s_j \mathbf{v}_j = s_0 \mathbf{v}_0 + \sum_{j=1}^{q} s_j \mathbf{v}_j = \sum_{j=1}^{q} s_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows from the linear independence of the displacement vectors $\mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$ that

$$s_1 = s_2 = \cdots = s_q = 0.$$

But then $s_0 = 0$ also, because $s_0 = -\sum_{j=1}^{q} s_j$. It follows that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent, as required. ∎

It follows from Lemma 7.1 that any set of affinely independent points in $\mathbb{R}^k$ has at most $k+1$ elements. Moreover if a set consists of affinely independent points in $\mathbb{R}^k$, then so does every subset of that set.

7.2. Simplices in Euclidean Spaces

Definition

A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent points of $\mathbb{R}^k$. The points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are referred to as the *vertices* of the simplex. The non-negative integer q is referred to as the *dimension* of the simplex.

Example

A 0-simplex in a Euclidean space $\mathbb{R}^k$ is a single point of that space.

Example

A 1-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least one is a line segment in that space. Indeed let λ be a 1-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}$ and $\mathbf{w}$. Then

$$\begin{aligned}\lambda &= \{s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq s \leq 1,\ 0 \leq t \leq 1 \text{ and } s + t = 1\} \\ &= \{(1-t)\mathbf{v} + t\,\mathbf{w} : 0 \leq t \leq 1\},\end{aligned}$$

and thus λ is a line segment in $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$.

Example

A 2-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least two is a triangle in that space. Indeed let τ be a 2-simplex in $\mathbb{R}^k$ with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Then

$$\tau = \{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq r, s, t \leq 1 \text{ and } r + s + t = 1\}.$$

Let $\mathbf{x} \in \tau$. Then there exist $r, s, t \in [0, 1]$ such that $\mathbf{x} = r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w}$ and $r + s + t = 1$. If $r = 1$ then $\mathbf{x} = \mathbf{u}$. Suppose that $r < 1$. Then

$$\mathbf{x} = r\,\mathbf{u} + (1 - r)\Big((1 - p)\mathbf{v} + p\mathbf{w}\Big)$$

where $p = \dfrac{t}{1 - r}$. Moreover $0 < r \leq 1$ and $0 \leq p \leq 1$. Moreover the above formula determines a point of the 2-simplex τ for each pair of real numbers r and p satisfying $0 \leq r \leq 1$ and $0 \leq p \leq 1$.

Thus

$$\tau = \left\{ r\,\mathbf{u} + (1-r)\Bigl((1-p)\mathbf{v} + p\mathbf{w}\Bigr) : 0 \leq p, r \leq 1. \right\}.$$

Now the point $(1-p)\mathbf{v} + p\mathbf{w}$ traverses the line segment $\mathbf{v}\,\mathbf{w}$ from $\mathbf{v}$ to $\mathbf{w}$ as p increases from 0 to 1. It follows that τ is the set of points that lie on line segments with one endpoint at $\mathbf{u}$ and the other at some point of the line segment $\mathbf{v}\,\mathbf{w}$. This set of points is thus a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

A 3-dimensional simplex is a tetrahedron. Higher-dimensional simplices are the higher-dimensional analogues of points, line segments, triangles and tetrahedra.

7. The Topology of Closed Surfaces

7.3. Faces of Simplices

Definition

Let σ and τ be simplices in $\mathbb{R}^k$. We say that τ is a *face* of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a *proper face* if it is not equal to σ itself. An r-dimensional face of σ is referred to as an *r-face* of σ. A 1-dimensional face of σ is referred to as an *edge* of σ.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

7.4. Simplical Complexes in Euclidean Spaces

Definition

A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The *dimension* of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex. The union of all the simplices of K is a compact subset $|K|$ of $\mathbb{R}^k$ referred to as the *polyhedron* of K. (The polyhedron is compact since it is both closed and bounded in $\mathbb{R}^k$.)

Example

Let K_σ consist of some n-simplex σ together with all of its faces. Then K_σ is a simplicial complex of dimension n, and $|K_\sigma| = \sigma$.

Lemma 7.2

Let K be a simplicial complex, and let X be a subset of some Euclidean space. A function $f\colon |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.

Proof

Each simplex of the simplicial complex K is a closed subset of the polyhedron $|K|$ of the simplicial complex K. The numbers of simplices belonging to the simplicial complex is finite. The result therefore follows from a straightforward application of the Pasting Lemma (Lemma 1.24). ∎

Lemma 7.3

Let K be a finite collection of triangles, edges and points in some Euclidean space. Then K is a two-dimensional simplicial complex if and only if the following conditions are all satisfied:—

(i) *the edges and vertices of any triangle belonging to K themselves belong to K;*

(ii) *the endpoints of any edge belonging to K are vertices belonging to K;*

(iii) *if two distinct triangles belonging to K have a non-empty intersection, then that intersection is either a single common edge or a single common vertex of both triangles;*

(iv) if a triangle belonging to K intersects an edge belonging to K then either the edge is an edge of the triangle or else the intersection of the triangle and edge is a vertex of the triangle that is an endpoint of the edge;

(v) if two distinct edges belonging to K have a non-empty intersection then that intersection is a common vertex (or endpoint) of both edges;

1. (vi) if a vertex belongs to a triangle then it is a vertex of that triangle, and if a vertex belongs to an edge then it is an endpoint of that edge.

Proof

Consider a finite collection K of simplices of dimension two in a Euclidean space. The simplices belonging to K are points, line segments or triangles. Conditions (i) and (ii) in the statement of the lemma are equivalent to the condition that every face of a simplex belonging to the collection K must itself belong to that collection. Similarly conditions (iii), (iv), (v) and (vi) in the statement of the lemma are equivalent to the condition that any two simplices of K whose intersection is non-empty intersect in a common face. The result therefore follows from the definition of a simplicial complex, applied in the special case where the simplices of the complex are of dimension at most two. $\blacksquare$

Definition

A *two-dimensional simplicial complex* in a Euclidean space consists of a finite collection K of triangles, edges (which are line segments) and vertices (which are points) in that space which contains at least one triangle, and which satisfies the following conditions:

(i) The edges and vertices of any triangle belonging to K themselves belong to K;

(ii) The endpoints of any edge belonging to K are vertices belonging to K;

(iii) if two distinct triangles belonging to K have a non-empty intersection, then that intersection is either a single common edge or a single common vertex of both triangles;

(iv) if two distinct edges belonging to K have a non-empty intersection then that intersection is a common vertex (or endpoint) of both edges.

Definition

Let K be a two-dimensional simplicial complex in some Euclidean space. The *polyhedron* $|K|$ of K is the union of all the triangles, edges and vertices belonging to the collection K.

Lemma 7.4

The polyhedron of a two-dimensional simplicial complex is a compact Hausdorff space.

Proof

The simplicial complex K is a finite collection of triangles, edges and vertices in some ambient Euclidean space, and each triangle, edge and vertex in the collection is a closed bounded subset of this ambient Euclidean space. Now a subset of a Euclidean space is compact if and only if it is both closed and bounded. It follows that each of the triangles, edges and vertices belonging to K is a compact subset of the ambient Euclidean space. Moreover it follows directly from the definition of compactness that any finite union of compact topological spaces is itself compact. Therefore the polyhedron $|K|$ of K is a compact subset of the ambient Euclidean space. This ambient Euclidean space is a Hausdorff space (as it is a metric space, and all metric spaces are Hausdorff spaces), and any subset of a Hausdorff space is itself a Hausdorff space (with the subspace topology). Therefore the polyhedron $|K|$ of K is a compact Hausdorff space, as required. ∎

Definition

Let $\mathbf{p}$ be a point of the polyhedron $|K|$ of the two-dimensional simplicial complex K. The *star neighbourhood* $\mathrm{st}_K(\mathbf{p})$ of the point $\mathbf{p}$ in $|K|$ is defined to be the subset of $|K|$ whose complement is the union of all triangles, edges and vertices belonging to K that do not contain the point $\mathbf{p}$.

Lemma 7.5

Let K be a two-dimensional simplicial complex, and let $\mathbf{p}$ be a point of K. Then the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ of the point $\mathbf{p}$ of $|K|$ is an open subset of $|K|$, and moreover $\mathbf{p} \in \mathrm{st}_K(\mathbf{p})$.

Proof
A two-dimensional simplicial complex is a finite collection of triangles, edges and vertices in some ambient Euclidean space. Each of those triangles, edges and vertices is a closed subset of the ambient Euclidean space, and therefore the union of any finite collection of such triangles, edges and vertices is a closed subset of the ambient Euclidean space.

Now, given any point $\mathbf{p}$ of $|K|$, the complement $|K| \setminus \mathrm{st}_K(\mathbf{p})$ of the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ of $\mathbf{p}$ in $|K|$ is by definition the union of all triangles, edges and vertices belonging to K that do not contain the point $\mathbf{p}$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{p})$ is closed in $|K|$, and $\mathbf{p} \notin |K| \setminus \mathrm{st}_K(\mathbf{p})$. Therefore $\mathrm{st}_K(\mathbf{p})$ is open in $|K|$, and $\mathbf{p} \in \mathrm{st}_K(\mathbf{p})$, as required. $\blacksquare$