# MA342R—Covering Spaces and Fundamental Groups
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## 5.2. Orbit Spaces

**Example**

The cyclic group $C_2$ of order 2 consists of a set $\{e, a\}$ with two elements $e$ and $a$, together with a group multiplication operation defined so that $e^2 = a^2 = e$ and $ea = ae = a$. The identity element of $C_2$ is thus $e$.

Let us represent the $n$-dimensional sphere $S^n$ as the unit sphere in $\mathbb{R}^{n+1}$ centred on the origin. Let $\theta_e \colon S^n \to S^n$ be the identity map of $S^n$ and let $\theta_a \colon S^n \to S^n$ be the antipodal map of $S^n$, defined such that $\theta_a(\mathbf{x}) = -\mathbf{x}$ for all $\mathbf{x} \in S^n$. Then the group $C_2$ acts on $S^n$ (on the left) so that elements $a$ and $e$ of $S^n$ correspond under this action to the homeomorphisms $\theta_e$ and $\theta_a$ respectively. Points $\mathbf{x}$ and $\mathbf{y}$ are said to be *antipodal* to one another if and only if $\mathbf{y} = -\mathbf{x}$. Each orbit for the action of $C_2$ on $S^n$ thus consists of a pair of antipodal points on $S^n$.

Let $\mathbf{p}$ be an element of $S^n$, and let

$$U = \{\mathbf{x} \in S^n : \mathbf{x} \,.\, \mathbf{p} > 0\}.$$

Then $U$ is open in $S^n$ and $\mathbf{p} \in U$. Also

$$\theta_a(U) = \{\mathbf{x} \in S^n : \mathbf{x} \,.\, \mathbf{p} < 0\},$$

and therefore $U \cap \theta_a(U) = \emptyset$. It follows that the group $C_2$ acts freely and properly discontinuously on $S^n$.

Distinct points of $S^n$ belong to the same orbit under the action of $C_2$ on $S^n$ if and only if the line in $\mathbb{R}^{n+1}$ passing through those points also passes through the origin. It follows that lines in $\mathbb{R}^{n+1}$ that pass through the origin are in one-to-one correspondence with orbits for the action of $C_2$ on $S^n$. The orbit space $S^n/C_2$ thus represents the set of lines through the origin in $\mathbb{R}^{n+1}$. We define $n$-dimensional *real projective space* $\mathbb{R}P^n$ to be the topological space whose elements are the lines in $\mathbb{R}^{n+1}$ passing through the origin, with the topology obtained on identifying $\mathbb{R}P^n$ with the orbit space $S^n/C_2$. The quotient map $q\colon S^n \to \mathbb{R}P^n$ then sends each point $\mathbf{x}$ of $S^n$ to the orbit consisting of the two points $\mathbf{x}$ and $-\mathbf{x}$. Thus each pair of antipodal points on the $n$-dimenionsional sphere $S^n$ determines a single point of $n$-dimensional real projective space $\mathbb{R}P^n$.

**Proposition 5.1**

*Let $G$ be a group acting freely and properly discontinuously on a topological space $X$, let $X/G$ denote the resulting orbit space, and let $q\colon X \to X/G$ be the quotient map that sends each element of $X$ to its orbit under the action of the group $G$. Let $p\colon X \to Y$ be a continuous surjective map from $X$ to a topological space $Y$. Suppose that elements $x$ and $x'$ of $X$ satisfy $p(x) = p(x')$ if and only if $q(x) = q(x')$. Suppose also $p(V)$ is open in $Y$ for every open set $V$ in $X$. Then the surjective continuous map $p\colon X \to Y$ induces a homeomorphism $h\colon X/G \to Y$ between the topological spaces $X/G$ and $Y$, where $h(q(x)) = p(x)$ for all $x \in X$.*

**Proof**

The function $h\colon X/G \to Y$ is continuous because $p\colon X \to Y$ is continuous and $q\colon X \to Y$ is a quotient map (see Lemma 1.35). Moreover it is surjective because $p\colon X \to Y$ is a surjection, and it is injective because elements $x$ and $x'$ satisfy $p(x) = p(x')$ if and only if $q(x) = q(x')$. It follows that $h\colon X/G \to Y$ is a bijection.

Let $W$ be an open set in $X/G$. It follows from the definition of the quotient topology that $q^{-1}(W)$ is open in $X$. The map $p$ maps open sets to open sets. Therefore $p(q^{-1}(W))$ is open in $Y$. But $p(q^{-1}(W)) = h(W)$. Thus $h(W)$ is open in $Y$ for every open set $W$ in $X$, and therefore $h^{-1}\colon Y \to X/W$ is continuous. Thus the continuous bijection $h\colon X/G \to Y$ is a homeomorphism, as required. ∎

## Corollary 5.2

*Let the group $\mathbb{Z}$ act on the real line $\mathbb{R}$ by translation, where the action sends each integer $n$ to the translation $\theta_n\colon \mathbb{R} \to \mathbb{R}$ defined such that $\theta_n(t) = t + n$ for all real numbers $t$. Let $\mathbb{R}/\mathbb{Z}$ denote the orbit space for this action, and let $q\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map that sends each real number to its orbit under the action of the group $\mathbb{Z}$. Let $S^1$ denote the unit circle centred on the origin in $\mathbb{R}^2$, let $p\colon \mathbb{R} \to S^1$ be defined such that*

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

*for all real numbers $t$, and let $h\colon \mathbb{R}/\mathbb{Z} \to S^1$ be the map defined such that $h(q(t)) = p(t)$ for all real numbers $t$. Then $h\colon \mathbb{R}/\mathbb{Z} \to S^1$ is a homeomorphism.*

**Proof**

The map $p\colon \mathbb{R} \to S^1$ maps open sets to open sets. The result therefore follows directly on applying Proposition 5.1. ∎

**Corollary 5.3**

*Let the group $\mathbb{Z}$ act by translation on the complex plane $\mathbb{C}$, where the action sends each integer $n$ to the translation $\theta_n \colon \mathbb{C} \to \mathbb{C}$ defined such that $\theta_n(z) = z + n$ for all complex numbers $z$, where $i^2 = -1$. Let $\mathbb{C}/\mathbb{Z}$ denote the orbit space for this action, and let $q \colon \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ be the quotient map that sends each complex number to its orbit under the action of the group $\mathbb{Z}$. Let $p \colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be defined such that $p(z) = \exp(2\pi i z)$ for all complex numbers $z$, and let $h \colon \mathbb{C}/\mathbb{Z} \to \mathbb{C} \setminus \{0\}$ be the map defined such that $h(q(z)) = p(z)$ for all complex numbers $z$. Then $h \colon \mathbb{C}/\mathbb{Z} \to \mathbb{C} \setminus \{0\}$ is a homeomorphism.*

**Proof**

We show that the map $p\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ maps open sets to open sets. Let $V$ be an open set in $\mathbb{C}$, and let $u$ and $v$ be real numbers for which $u + iv \in V$. Then there exist real numbers $\theta_1$, $\theta_2$ and positive real numbers $r_1$ and $r_2$ satisfying the inequalities

$$\theta_1 < 2\pi u < \theta_2 \text{ and } \log r_1 < -2\pi v < \log r_2$$

where $\theta_1$ and $\theta_2$ are close enough to $2\pi u$ and $\log r_1$ and $\log r_2$ are close enough to $-2\pi v$ to ensure that $s + it \in V$ for all real numbers $s$ and $t$ that satisfy the inequalities

$$\theta_1 < 2\pi s < \theta_2 \text{ and } \log r_1 < -2\pi t < \log r_2.$$

It then follows that $u + iv \in N$ and $N \subset p(V)$, where

$$N = \{re^{i\theta} : r_1 < r < r_2 \text{ and } \theta_1 < \theta < \theta_2\}.$$

Now $N$ is an open set in $\mathbb{C} \setminus \{0\}$. It follows that $p(V)$ is a neighbourhood of $p(u + iv)$. We have now shown that the set $p(V)$ is a neighbourhood of each of its points. It follows that $p(V)$ is open in $\mathbb{C} \setminus \{0\}$. We conclude therefore that the map $p\colon \mathbb{C} \setminus \mathbb{C} \setminus \{0\}$ maps open sets to open sets. It then follows directly from Proposition 5.1 that $h\colon \mathbb{C}/\mathbb{Z} \to \mathbb{C} \setminus \{0\}$ is a homeomorphism. ∎

**Proposition 5.4**

*Let $G$ be a group acting freely and properly discontinuously on a topological space $X$, let $X/G$ denote the resulting orbit space, and let $q\colon X \to X/G$ be the quotient map that sends each element of $X$ to its orbit under the action of the group $G$. Let $p\colon X \to Y$ be a continuous surjective map from $X$ to a Hausdorff topological space $Y$. Suppose that elements $x$ and $x'$ of $X$ satisfy $p(x) = p(x')$ if and only if $q(x) = q(x')$. Suppose also that there exists a compact subset $K$ of $X$ that intersects every orbit for the action of $G$ on $X$. Then the surjective continuous map $p\colon X \to Y$ induces a homeomorphism $h\colon X/G \to Y$ between the topological spaces $X/G$ and $Y$, where $h(q(x)) = p(x)$ for all $x \in X$.*

**Proof**

The function $h\colon X/G \to Y$ is continuous because $X$ is continuous and $q\colon X \to Y$ is a quotient map (see Lemma 1.35). Moreover it is surjective because $p\colon X \to Y$ is a surjection, and it is injective because elements $x$ and $x'$ satisfy $p(x) = p(x')$ if and only if $q(x) = q(x')$. It follows that $h\colon X/G \to Y$ is a bijection.

The orbit space $X/G$ is compact, because it is the image $q(K)$ of the compact set $K$ under the continuous map $q\colon X \to X/G$. (see Lemma 1.39). Thus $h\colon X/G \to Y$ is a continuous bijection from a compact topological space to a Hausdorff space. This map is therefore a homeomorphism (see Theorem 1.45). ∎

**Example**

Let the group $\mathbb{Z}$ of integers under addition act by translation on the real line $\mathbb{R}$ by translation so that, under this action, an integer $n$ corresponds to the homeomorphism $\theta_n \colon \mathbb{R} \to \mathbb{R}$ defined such that $\theta_n(t) = t + n$ for all real numbers $t$. Let $q \colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map onto the orbit space, and let $p \colon \mathbb{R} \to S^1$ be defined such that

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

for all real numbers $t$, and let $h \colon \mathbb{R}/\mathbb{Z} \to S^1$ be the map defined such that $h(q(t)) = p(t)$ for all real numbers $t$.

Now $S^1$ is a Hausdorff space, as it is a subset of the metric space $\mathbb{R}^2$. Also the map $p \colon \mathbb{R} \to S^1$ is surjective. Real numbers $t_1$ and $t_2$ satisfy $p(t_1) = p(t_2)$ if and only if $t_1 = t_2 + n$ for some integer $n$. It follows that $p(t_1) = p(t_2)$ if and only if $q(t_1) = q(t_2)$. The compact subset $[0, 1]$ of $\mathbb{R}$ intersects every orbit for the action of $\mathbb{Z}$ on $\mathbb{R}$. It therefore follows from Proposition 5.4 that $h \colon \mathbb{R}/\mathbb{Z} \to S^1$ is a homeomorphism. (This result was also shown to follow from the fact that $p \colon \mathbb{R} \to S^1$ maps open sets to open sets: see Corollary 5.2.)

**Example**

Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be defined so that

$$f(s,t) = ((2+\cos 2\pi t)\cos 2\pi s,\ \ (2+\cos 2\pi t)\sin 2\pi s,\ \ \sin 2\pi t)$$

for all $(s,t) \in \mathbb{R}^2$, and let $Y = f(\mathbb{R}^2)$. Then $Y$ is a torus in $\mathbb{R}^3$ that bounds the 'solid doughnut' consisting of those points of $\mathbb{R}^3$ whose distance from the circle in the plane $z = 0$ of radius 2 centred on the origin is less than one. Points $(s_1, t_1)$ and $(s_2, t_2)$ of $\mathbb{R}^2$ satisfy $f(s_1, t_1) = f(s_2, t_2)$ if and only if $s_1 - t_1$ and $s_2 - t_2$ are integers. Let the group $\mathbb{Z}^2$ act on $\mathbb{R}^2$ by translation, so that, under this action, an element $(m, n)$ of $\mathbb{Z}^2$ corresponds to the homeomorphism $\theta_{(m,n)}: \mathbb{R}^2 \to \mathbb{R}^2$ from $\mathbb{R}^2$ to itself defined so that $\theta_{(m,n)}(s,t) = (s+m, t+n)$ for all $(s,t) \in \mathbb{R}^2$.

Let $\delta$ be a real number satisfying $0 < \delta \leq \frac{1}{2}$, and, for all $(s, t) \in \mathbb{R}^2$, let $B((s, t), \delta)$ denote the open disk in $\mathbb{R}^2$ of radius $\delta$ centred on the point $(s, t)$. Then

$$B((s + m, t + n), \delta) \cap B((s, t), \delta) = \emptyset$$

for all integers $m$ and $n$ for which $(m, n) \neq (0, 0)$. It follows that the group $\mathbb{Z}^2$ acts freely and properly discontinuously on $\mathbb{R}^2$ by translation. Let $\mathbb{R}^2/\mathbb{Z}^2$ be the orbit space determined by this action, let $q\colon \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ be the quotient map sending each point of $\mathbb{R}^2$ to its orbit under the action of $\mathbb{Z}^2$, and let $h\colon \mathbb{R}^2/\mathbb{Z}^2 \to Y$ be the function from $\mathbb{R}^2/\mathbb{Z}^2$ to the surface $Y$ defined so that $h(q(s, t)) = f(s, t))$ for all $(s, t) \in \mathbb{R}^2$.

Now the unit square $[0, 1] \times [0, 1]$ is a compact subset of $\mathbb{R}^2$ that intersects every orbit for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. It follows directly from Proposition 5.4 that $h\colon \mathbb{R}^2/\mathbb{Z}^2 \to Y$ is a homeomorphism. Thus the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ represents a 2-dimensional torus.

**Proposition 5.5**

*Let $G$ be a group acting freely and properly discontinuously on a topological space $X$. Then the quotient map $q\colon X \to X/G$ from $X$ to the corresponding orbit space $X/G$ is a covering map.*

**Proof**

The quotient map $q\colon X \to X/G$ is surjective. Let $V$ be an open set in $X$. Then $q^{-1}(q(V))$ is the union $\bigcup_{g \in G} \theta_g(V)$ of the open sets $\theta_g(V)$ as $g$ ranges over the group $G$, since $q^{-1}(q(V))$ is the subset of $X$ consisting of all elements of $X$ that belong to the orbit of some element of $V$. But any union of open sets in a topological space is an open set. We conclude therefore that if $V$ is an open set in $X$ then $q(V)$ is an open set in $X/G$.

Let $x$ be a point of $X$. Then there exists an open set $U$ in $X$ such that $x \in U$ and $\theta_g(U) \cap U = \emptyset$ for all $g \in G$ satisfying $g \neq e$. Now $q^{-1}(q(U)) = \bigcup_{g \in G} \theta_g(U)$. We claim that the sets $\theta_g(U)$ are disjoint. Let $g$ and $h$ be elements of $G$. Suppose that $\theta_g(U) \cap \theta_h(U) \neq \emptyset$. Then $\theta_{h^{-1}}(\theta_g(U) \cap \theta_h(U)) \neq \emptyset$. But $\theta_{h^{-1}} \colon X \to X$ is a bijection, and therefore

$$\theta_{h^{-1}}(\theta_g(U) \cap \theta_h(U)) = \theta_{h^{-1}}(\theta_g(U)) \cap \theta_{h^{-1}}(\theta_h(U)) = \theta_{h^{-1}g}(U) \cap U,$$

and therefore $\theta_{h^{-1}g}(U) \cap U \neq \emptyset$. It follows that $h^{-1}g = e$, where $e$ denotes the identity element of $G$, and therefore $g = h$. Thus if $g$ and $h$ are elements of $g$, and if $g \neq h$, then $\theta_g(U) \cap \theta_h(U) = \emptyset$. We conclude therefore that the preimage $q^{-1}(q(U))$ of $q(U)$ is the disjoint union of the sets $\theta_g(U)$ as $g$ ranges over the group $G$. Moreover each these sets $\theta_g(U)$ is an open set in $X$.

Now $U \cap [u]_G = \{u\}$ for all $u \in U$, since $[u]_G = \{\theta_g(u) : g \in G\}$ and $U \cap \theta_g(U) = \emptyset$ when $g \neq e$. Thus if $u$ and $v$ are elements of $U$, and if $q(u) = q(v)$ then $[u]_G = [v]_G$ and therefore $u = v$. It follows that the restriction $q|U \colon U \to X/G$ of the quotient map $q$ to $U$ is injective, and therefore $q$ maps $U$ bijectively onto $q(U)$. But $q$ maps open sets onto open sets, and any continuous bijection that maps open sets onto open sets is a homeomorphism. We conclude therefore that the restriction of $q \colon X \to X/G$ to the open set $U$ maps $U$ homeomorphically onto $q(U)$. Moreover, given any element $g$ of $G$, the quotient map $q$ satisfies $q = q \circ \theta_{g^{-1}}$, and the homeomorphism $\theta_{g^{-1}}$ maps $\theta_g(U)$ homeomorphically onto $U$. It follows that the quotient map $q$ maps $\theta_g(U)$ homeomorphically onto $q(U)$ for all $g \in U$. We conclude therefore that $q(U)$ is an evenly covered open set in $X/G$ whose preimage $q^{-1}(q(U))$ is the disjoint union of the open sets $\theta_g(U)$ as $g$ ranges over the group $G$. It follows that the quotient map $q \colon X \to X/G$ is a covering map, as required. ■