MA342R—Covering Spaces and Fundamental Groups
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5. Free Discontinuous Group Actions on Topological Spaces

5.1. Discontinuous Group Actions

Definition

Let G be a group, and let X be a set. The group G is said to *act* on the set X (on the left) if each element g of G determines a corresponding function $\theta_g \colon X \to X$ from the set X to itself, where

(i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

(ii) the function θ_e determined by the identity element e of G is the identity function of X.

Let G be a group acting on a set X. Given any element x of X, the *orbit* $[x]_G$ of x (under the group action) is defined to be the subset $\{\theta_g(x) : g \in G\}$ of X, and the *stabilizer* of x is defined to the the subgroup $\{g \in G : \theta_g(x) = x\}$ of the group G. Thus the orbit of an element x of X is the set consisting of all points of X to which x gets mapped under the action of elements of the group G. The stabilizer of x is the subgroup of G consisting of all elements of this group that fix the point x. The group G is said to act *freely* on X if $\theta_g(x) \neq x$ for all $x \in X$ and $g \in G$ satisfying $g \neq e$. Thus the group G acts freely on X if and only if the stabilizer of every element of X is the trivial subgroup of G.

Let e be the identity element of G. Then $x = \theta_e(x)$ for all $x \in X$, and therefore $x \in [x]_G$ for all $x \in X$, where $[x]_G = \{\theta_g(x) : g \in G\}$.

Let x and y be elements of G for which $[x]_G \cap [y]_G$ is non-empty, and let $z \in [x]_G \cap [y]_G$. Then there exist elements h and k of G such that $z = \theta_h(x) = \theta_k(y)$. Then $\theta_g(z) = \theta_{gh}(x) = \theta_{gk}(y)$, $\theta_g(x) = \theta_{gh^{-1}}(z)$ and $\theta_g(y) = \theta_{gk^{-1}}(z)$ for all $g \in G$, and therefore $[x]_G = [z]_G = [y]_G$. It follows from this that the group action partitions the set X into orbits, so that each element of X determines an orbit which is the unique orbit for the action of G on X to which it belongs. We denote by X/G the set of orbits for the action of G on X.

Now suppose that the group G acts on a topological space X. Then there is a surjective function $q: X \to X/G$, where $q(x) = [x]_G$ for all $x \in X$. This surjective function induces a quotient topology on the set of orbits: a subset U of X/G is open in this quotient topology if and only if $q^{-1}(U)$ is an open set in X (see Lemma 1.34). We define the *orbit space* X/G for the action of G on X to be the topological space whose underlying set is the set of orbits for the action of G on X, the topology on X/G being the quotient topology induced by the function $q: X \to X/G$. This function $q: X \to X/G$ is then an identification map: we shall refer to it as the *quotient map* from X to X/G.

We shall be concerned here with situations in which a group action on a topological space gives rise to a covering map. The relevant group actions are those where the group acts *freely and properly discontinuously* on the topological space.

Definition

Let G be a group with identity element e, and let X be a topological space. The group G is said to act *freely and properly discontinuously* on X if each element g of G determines a corresponding continuous map $\theta_g \colon X \to X$, where the following conditions are satisfied:

(i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

(ii) the continuous map θ_e determined by the identity element e of G is the identity map of X;

(iii) given any point x of X, there exists an open set U in X such that $x \in U$ and $\theta_g(U) \cap U = \emptyset$ for all $g \in G$ satisfying $g \neq e$.

Let G be a group which acts freely and properly discontinuously on a topological space X. Given any element g of G, the corresponding continuous function $\theta_g \colon X \to X$ determined by X is a homeomorphism. Indeed it follows from conditions (i) and (ii) in the above definition that $\theta_{g^{-1}} \circ \theta_g$ and $\theta_g \circ \theta_{g^{-1}}$ are both equal to the identity map of X, and therefore $\theta_g \colon X \to X$ is a homeomorphism with inverse $\theta_{g^{-1}} \colon X \to X$.

Remark

The terminology 'freely and properly discontinuously' is traditional, but is hardly ideal. The adverb 'freely' refers to the requirement that $\theta_g(x) \neq x$ for all $x \in X$ and for all $g \in G$ satisfying $g \neq e$. The adverb 'discontinuously' refers to the fact that, given any point x of G, the elements of the orbit $\{\theta_g(x) : g \in G\}$ of x are separated; it does not signify that the functions defining the action are in any way discontinuous or badly-behaved. The adverb 'properly' refers to the fact that, given any compact subset K of X, the number of elements of g for which $K \cap \theta_g(K) \neq \emptyset$ is finite.

Moreover the definitions of *properly discontinuous actions* in textbooks and in sources of reference are not always in agreement: some say that an action of a group G on a topological space X (where each group element determines a corresponding homeomorphism of the topological space) is *properly discontinuous* if, given any $x \in X$, there exists an open set U in X such that the number of elements g of the group for which $g(U) \cap U \neq \emptyset$ is finite; others say that the action is *properly discontinuous* if it satisfies the conditions given in the definition above for a group acting freely and properly discontinuously on the set. William Fulton, in his textbook *Algebraic topology: a first course* (Springer, 1995), introduced the term 'evenly' in place of 'freely and properly discontinuously', but this change in terminology does not appear to have been generally adopted.