MA342R—Covering Spaces and Fundamental Groups
School of Mathematics, Trinity College
Hilary Term 2017
Lecture 18 (February 24, 2017)

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4.4. The Homotopy-Lifting Theorem

Theorem 4.6 (Homotopy-Lifting Theorem)

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Let Z be a topological space, and let $F\colon Z \times [0,1] \to X$ and $g\colon Z \to \tilde{X}$ be continuous maps with the property that $p(g(z)) = F(z,0)$ for all $z \in Z$. Then there exists a unique continuous map $G\colon Z \times [0,1] \to \tilde{X}$ such that $G(z,0) = g(z)$ for all $z \in Z$ and $p \circ G = F$.

Proof

For each $z \in Z$, consider the path $\gamma_z \colon [0,1] \to Z$ defined by $\gamma_z(t) = F(z,t)$ for all $t \in [0,1]$. Note that $p(g(z)) = \gamma_z(0)$. It follows from the Path-Lifting Theorem (Theorem 4.5) that there exists a unique continuous path $\tilde{\gamma}_z \colon [0,1] \to \tilde{X}$ such that $\tilde{\gamma}_z(0) = g(z)$ for all $z \in Z$ and $p \circ \tilde{\gamma}_z = \gamma_z$. Let the function $G \colon Z \times [0,1] \to \tilde{X}$ be defined by $G(z,t) = \tilde{\gamma}_z(t)$ for all $z \in Z$ and $t \in [0,1]$. Then $G(z,0) = g(z)$ for all $z \in Z$ and

$$p(G(z,t)) = p(\tilde{\gamma}_z(t)) = \gamma_z(t) = F(z,t)$$

for all $z \in Z$ and $t \in [0,1]$. It remains to show that the function $G \colon Z \times [0,1] \to \tilde{X}$ is continuous and that it is unique.

Given any $z \in Z$, let S_z denote the set of all real numbers c belonging to the closed interval $[0, 1]$ which have the following property:

> *there exists an open set N in Z such that $z \in N$ and the function G is continuous on $N \times [0, c]$.*

Let s_z be the supremum $\sup S_z$ (i.e., the least upper bound) of the set S_z. We prove that s_z belongs to the set S_z and that $s_z = 1$.

Choose some $z \in Z$, and let $w \in \tilde{X}$ be given by $w = G(z, s_z)$. There exists an open neighbourhood U of $p(w)$ in X which is evenly covered by the map p. Thus $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains the point w; let this open set be denoted by $\tilde{U}$. Then there exists a unique continuous map $\sigma \colon U \to \tilde{U}$ defined such that, for all $x \in U$, $\sigma(x)$ is the unique element of $\tilde{U}$ for which $p(\sigma(x)) = x$. Then $\sigma(F(z, s_z)) = w$.

Now $F(z, s_z) = p(w)$. It follows from the continuity of the map F that there exists some positive real number δ and some open set M_1 in Z such that $z \in M_1$ and $F(M_1 \times J(s_z, \delta)) \subset U$, where

$$J(s_z, \delta) = \{t \in [0, 1] : s_z - \delta < t < s_z + \delta\}.$$

Now we can choose some c belonging to S_z which satisfies $s_z - \delta < c \leq s_z$, because s_z is the least upper bound of the set S_z. It then follows from the definition of the set S_z that there exists an open set M_2 in Z such that $z \in M_2$ and the function G is continuous on $M_2 \times [0, c]$. Let

$$N = \{z' \in M_1 \cap M_2 : G(z', c) \in \tilde{U}\}.$$

Then $z \in N$, and the continuity of the function G on $M_2 \times [0, c]$ ensures that N is open in Z. Moreover the function G is continuous on $N \times [0, c]$ and $F(N \times J(s_z, \delta)) \subset U$.

Let $z' \in N$. Then $G(z', c) \in \tilde{U}$ and $p(G(z', c)) = F(z', c)$. It follows from the definition of the map $\sigma \colon U \to \tilde{X}$ that $G(z', c) = \sigma(F(z', c))$. Also the interval $J(s_z, \delta)$ is connected, and

$$p(G(z', t)) = F(z', t) = p(\sigma(F(z', t))$$

for all $t \in J(s_z, \delta)$. It follows from Theorem 4.3 that $G(z', t) = \sigma(F(z', t)$ for all $t \in J(s_z, \delta)$.

We have thus shown that the function G is equal to the continuous function $\sigma \circ F$ on $N \times J(s_z, \delta)$. The function G is therefore continuous on both $N \times [0, c]$ and $N \times [c, t]$ for all $t \in J(s_z, \delta)$ satisfying $t \geq c$. It then follows from the Pasting Lemma (Lemma 1.24) that the function G is continuous on $N \times [0, t]$ for all $t \in J(s_z, \delta)$, and thus $J(s_z, \delta) \subset S_z$. This however contradicts the definition of S_z unless $s_z \in S_z$ and $s_z = 1$. We conclude therefore that $1 \in S_z$, and thus there exists an open set N in Z such that $z \in N$ and $G|N \times [0, 1]$ is continuous.

We conclude from this that every point of $Z \times [0, 1]$ is contained in some open subset of $Z \times [0, 1]$ on which that function G is continuous. It follows that $G\colon Z \times [0, 1] \to \tilde{X}$ is continuous (see Proposition 1.23).

The uniqueness of the map $G\colon Z \times [0,1] \to \tilde{X}$ follows directly from the fact that for any $z \in Z$ there is a unique continuous path $\tilde{\gamma}_z\colon [0,1] \to \tilde{X}$ such that $\tilde{\gamma}_z(0) = g(z)$ and $p(\tilde{\gamma}_z(t)) = F(z,t)$ for all $t \in [0,1]$. ∎

4.5. Path-Lifting and the Fundamental Group

Let $p: \tilde{X} \to X$ be a covering map and let $\alpha: [0,1] \to X$ and $\beta: [0,1] \to X$ be paths in the base space X which both start at some point x_0 of X and finish at some point x_1 of X, so that

$$\alpha(0) = \beta(0) = x_0 \quad \text{and} \quad \alpha(1) = \beta(1) = x_1.$$

Let $\tilde{x}_0$ be some point of the covering space $\tilde{X}$ that projects down to x_0, so that $p(\tilde{x}_0) = x_0$. It follows from the Path-Lifting Theorem (Theorem 4.5) that there exist paths $\tilde{\alpha}: [0,1] \to \tilde{X}$ and $\tilde{\beta}: [0,1] \to \tilde{X}$ in the covering space $\tilde{X}$ that both start at $\tilde{x}_0$ and that are lifts of the paths α and β respectively.

Thus

$$\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0,$$

$$p(\tilde{\alpha}(t)) = \alpha(t) \quad \text{and} \quad p(\tilde{\beta}(t)) = \beta(t) \quad \text{for all } t \in [0,1].$$

These lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β are uniquely determined by their starting point $\tilde{x}_0$ (see Proposition 4.3).

Now, though the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β have been chosen such that they start at the same point $\tilde{x}_0$ of the covering space $\tilde{X}$, they need not in general end at the same point of $\tilde{X}$. However we shall prove that if $\alpha \simeq \beta \operatorname{rel} \{0, 1\}$, then the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β respectively that both start at some point $\tilde{x}_0$ of $\tilde{X}$ will both finish at some point $\tilde{x}_1$ of $\tilde{x}$, so that $\tilde{\alpha}(1) = \tilde{\beta}(1) = \tilde{x}_1$. This result is established in Proposition 4.7 below.

Proposition 4.7

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X, let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be paths in X, where $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, and let $\tilde{\alpha}\colon [0,1] \to \tilde{X}$ and $\tilde{\beta}\colon [0,1] \to \tilde{X}$ be paths in $\tilde{X}$ such that $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that $\tilde{\alpha}(0) = \tilde{\beta}(0)$ and that $\alpha \simeq \beta \; \mathrm{rel} \; \{0,1\}$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$ and $\tilde{\alpha} \simeq \tilde{\beta} \; \mathrm{rel} \; \{0,1\}$.

Proof

Let x_0 and x_1 be the points of X given by

$$x_0 = \alpha(0) = \beta(0), \qquad x_1 = \alpha(1) = \beta(1).$$

Now $\alpha \simeq \beta \text{ rel } \{0, 1\}$, and therefore there exists a homotopy $F\colon [0, 1] \times [0, 1] \to X$ such that

$$F(t, 0) = \alpha(t) \quad \text{and} \quad F(t, 1) = \beta(t) \quad \text{for all } t \in [0, 1],$$

and

$$F(0, \tau) = x_0 \quad \text{and} \quad F(1, \tau) = x_1 \quad \text{for all } \tau \in [0, 1].$$

It then follows from the Homotopy-Lifting Theorem (Theorem 4.6) that there exists a continuous map $G\colon [0,1] \times [0,1] \to \tilde{X}$ such that $p \circ G = F$ and $G(0,0) = \tilde{\alpha}(0)$. Then $p(G(0,\tau)) = x_0$ and $p(G(1,\tau)) = x_1$ for all $\tau \in [0,1]$. A straightforward application of Proposition 4.3 shows that any continuous lift of a constant path must itself be a constant path. Therefore $G(0,\tau) = \tilde{x}_0$ and $G(1,\tau) = \tilde{x}_1$ for all $\tau \in [0,1]$, where

$$\tilde{x}_0 = G(0,0) = \tilde{\alpha}(0), \qquad \tilde{x}_1 = G(1,0).$$

However

$$G(0,0) = G(0,1) = \tilde{x}_0 = \tilde{\alpha}(0) = \tilde{\beta}(0),$$

$$p(G(t,0)) = F(t,0) = \alpha(t) = p(\tilde{\alpha}(t))$$

and

$$p(G(t,1)) = F(t,1) = \beta(t) = p(\tilde{\beta}(t))$$

for all $t \in [0,1]$. It follows that the map that sends $t \in [0,1]$ to $G(t,0)$ is a lift of the path α that starts at $\tilde{x}_0$, and the map that sends $t \in [0,1]$ to $G(t,1)$ is a lift of the path β that also starts at $\tilde{x}_0$.

However Proposition 4.3 ensures that the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β are uniquely determined by their starting points. It follows that $G(t,0) = \tilde{\alpha}(t)$ and $G(t,1) = \tilde{\beta}(t)$ for all $t \in [0,1]$. In particular,

$$\tilde{\alpha}(1) = G(1,0) = \tilde{x}_1 = G(1,1) = \tilde{\beta}(1).$$

Moreover the map $G\colon [0,1] \times [0,1] \to \tilde{X}$ is a homotopy between the paths $\tilde{\alpha}$ and $\tilde{\beta}$ which satisfies $G(0,\tau) = \tilde{x}_0$ and $G(1,\tau) = \tilde{x}_1$ for all $\tau \in [0,1]$. It follows that $\tilde{\alpha} \simeq \tilde{\beta} \ \mathrm{rel}\ \{0,1\}$, as required. ∎

Proposition 4.8

Let $p\colon \tilde{X} \to X$ be a covering map, and let $\tilde{x}_0$ be a point of the covering space $\tilde{X}$. Then the homomorphism

$$p_{\#}\colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, p(\tilde{x}_0))$$

of fundamental groups induced by the covering map p is injective.

Proof

Let σ_0 and σ_1 be loops in $\tilde{X}$ based at the point $\tilde{x}_0$, representing elements $[\sigma_0]$ and $[\sigma_1]$ of $\pi_1(\tilde{X}, \tilde{x}_0)$. Suppose that $p_{\#}[\sigma_0] = p_{\#}[\sigma_1]$. Then $p \circ \sigma_0 \simeq p \circ \sigma_1 \text{ rel } \{0, 1\}$. Also $\sigma_0(0) = \tilde{x}_0 = \sigma_1(0)$. Therefore $\sigma_0 \simeq \sigma_1 \text{ rel } \{0, 1\}$, by Proposition 4.7, and thus $[\sigma_0] = [\sigma_1]$. We conclude that the homomorphism $p_{\#}\colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, p(\tilde{x}_0))$ is injective. ∎

Proposition 4.9

Let $p\colon \tilde{X} \to X$ be a covering map, let $\tilde{x}_0$ be a point of the covering space $\tilde{X}$, and let γ be a loop in X based at $p(\tilde{x}_0)$. Then $[\gamma] \in p_\#\big(\pi_1(\tilde{X}, \tilde{x}_0)\big)$ if and only if there exists a loop $\tilde{\gamma}$ in $\tilde{X}$, based at the point $\tilde{x}_0$, such that $p \circ \tilde{\gamma} = \gamma$.

Proof

If $\gamma = p \circ \tilde{\gamma}$ for some loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$ then $[\gamma] = p_\#[\tilde{\gamma}]$, and therefore $[\gamma] \in p_\#\bigl(\pi_1(\tilde{X}, \tilde{x}_0)\bigr)$.

Conversely suppose that $[\gamma] \in p_\#\bigl(\pi_1(\tilde{X}, \tilde{x}_0)\bigr)$. We must show that there exists some loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$ such that $\gamma = p \circ \tilde{\gamma}$. Now there exists a loop σ in $\tilde{X}$ based at the point $\tilde{x}_0$ such that $[\gamma] = p_\#([\sigma])$ in $\pi_1(X, p(\tilde{x}_0))$. Then $\gamma \simeq p \circ \sigma \text{ rel } \{0, 1\}$. It follows from the Path-Lifting Theorem for covering maps (Theorem 4.5) that there exists a unique path $\tilde{\gamma}\colon [0, 1] \to \tilde{X}$ in $\tilde{X}$ for which $\tilde{\gamma}(0) = \tilde{x}_0$ and $p \circ \tilde{\gamma} = \gamma$. It then follows from Proposition 4.7 that $\tilde{\gamma}(1) = \sigma(1)$ and $\tilde{\gamma} \simeq \sigma \text{ rel } \{0, 1\}$. But $\sigma(1) = \tilde{x}_0$. Therefore the path $\tilde{\gamma}$ is the required loop in $\tilde{X}$ based the point $\tilde{x}_0$ which satisfies $p \circ \tilde{\gamma} = \gamma$. ∎

Corollary 4.10

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X, let w_0 and w_1 be points of $\tilde{X}$ satisfying $p(w_0) = p(w_1)$, and let $\alpha\colon [0,1] \to \tilde{X}$ be a path in $\tilde{X}$ from w_0 to w_1. Suppose that $[p \circ \alpha] \in p_\#\big(\pi_1(\tilde{X}, w_0)\big)$. Then the path α is a loop in $\tilde{X}$, and thus $w_0 = w_1$.

Proof

It follows from Proposition 4.9 that there exists a loop β based at w_0 satisfying $p \circ \beta = p \circ \alpha$. Then $\alpha(0) = \beta(0)$. Now Proposition 4.3 ensures that the lift to $\tilde{X}$ of any path in X is uniquely determined by its starting point. It follows that $\alpha = \beta$. But then the path α must be a loop in $\tilde{X}$, and therefore $w_0 = w_1$, as required. ∎

Theorem 4.11

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Suppose that $\tilde{X}$ is path-connected and that X is simply-connected. Then the covering map $p\colon \tilde{X} \to X$ is a homeomorphism.

Proof

We show that the map $p\colon \tilde{X} \to X$ is a bijection. This map is surjective (since covering maps are by definition surjective). We must show that it is injective. Let w_0 and w_1 be points of $\tilde{X}$ with the property that $p(w_0) = p(w_1)$. Then there exists a path $\alpha\colon [0,1] \to \tilde{X}$ with $\alpha(0) = w_0$ and $\alpha(1) = w_1$, since $\tilde{X}$ is path-connected. Then $p \circ \alpha$ is a loop in X based at the point x_0, where $x_0 = p(w_0)$. However $\pi_1(X, p(w_0))$ is the trivial group, since X is simply-connected. It follows from Corollary 4.10 that the path α is a loop in $\tilde{X}$ based at w_0, and therefore $w_0 = w_1$. This shows that the the covering map $p\colon \tilde{X} \to X$ is injective.

Thus the map $p\colon \tilde{X} \to X$ is a bijection, and thus has a well-defined inverse $p^{-1}\colon X \to \tilde{X}$. But any bijective covering map is a homeomorphism (Corollary 4.2). The result follows. ∎