MA342R—Covering Spaces and Fundamental Groups
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Proposition 4.1

Let $p\colon \tilde{X} \to X$ be a covering map. Then $p(V)$ is open in X for every open set V in $\tilde{X}$.

Proof

Let V be open in $\tilde{X}$, and let $x \in p(V)$. Then $x = p(v)$ for some $v \in V$. Now there exists an open set U containing the point x which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains v; let $\tilde{U}$ be this open set, and let $N_x = p(V \cap \tilde{U})$. Now N_x is open in X, since $V \cap \tilde{U}$ is open in $\tilde{U}$ and $p|\tilde{U}$ is a homeomorphism from $\tilde{U}$ to U. Also $x \in N_x$ and $N_x \subset p(V)$. It follows that $p(V)$ is the union of the open sets N_x as x ranges over all points of $p(V)$, and thus $p(V)$ is itself an open set, as required. $\blacksquare$

Corollary 4.2

A bijective covering map is a homeomophism.

Proof

This result follows directly from Proposition 4.1 the fact that a continuous bijection is a homeomorphism if and only if it maps open sets to open sets. ∎

4.2. Uniqueness of Lifts into Covering Spaces

Definition

Let $p\colon \tilde{X} \to X$ be a covering map, let Z be a topological space, and let $f\colon Z \to X$ be a continuous map from Z to X. A continuous map $\tilde{f}\colon Z \to \tilde{X}$ is said to be a *lift* of $f\colon Z \to X$ to the covering space $\tilde{X}$ if $p \circ \tilde{f} = f$.

Much of the general theory of covering maps is concerned with the development of necessary and sufficient conditions to determine whether or not maps into the base space of a covering map can be lifted to the covering space.

We prove that any lift of a given map from a connected topological topological space into the base space of a covering map is determined by its value at a single point of its domain.

Proposition 4.3

Let $p\colon \tilde{X} \to X$ be a covering map, let Z be a connected topological space, and let $g\colon Z \to \tilde{X}$ and $h\colon Z \to \tilde{X}$ be continuous maps. Suppose that $p \circ g = p \circ h$ and that $g(z) = h(z)$ for at least one point z of Z. Then $g = h$.

Proof

Let $Z_0 = \{z \in Z : g(z) = h(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of Z. There exists an open set U in X containing the point $p(g(z))$ which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(z)$; let this open set be denoted by $\tilde{V}$. Let $N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. Then $g(z) = h(z)$, and therefore $\tilde{V} = \tilde{U}$. It follows from this that both g and h map the open set N_z into $\tilde{U}$. But $p \circ g = p \circ h$, and $p|\tilde{U}\colon \tilde{U} \to U$ is a homeomorphism. Therefore $g|N_z = h|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$, since $g(z) \neq h(z)$. But $g(N_z) \subset \tilde{U}$ and $h(N_z) \subset \tilde{V}$. Therefore $g(z') \neq h(z')$ for all $z' \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of Z is therefore both open and closed. Also Z_0 is non-empty by hypothesis. We deduce that $Z_0 = Z$, since Z is connected. Thus $g = h$, as required. ∎

Corollary 4.4

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Let Z be a connected topological space, and let $f: Z \to \tilde{X}$ be a continuous map. Suppose that $p(f(z)) = x_0$ for all $z \in Z$, where x_0 is some point of X. Then $f(z) = \tilde{x}_0$ for all $z \in Z$, where $\tilde{x}_0$ is some point of $\tilde{X}$ which satisfies $p(\tilde{x}_0) = x_0$.

Proof

Let z_0 be some point of Z. Let $\tilde{x}_0 = f(z_0)$, and let $c\colon Z \to \tilde{X}$ be the constant map defined by $c(z) = \tilde{x}_0$ for all $z \in Z$. Then $c(z_0) = f(z_0)$ and $p \circ c = p \circ f$. It follows from Theorem 4.3 that $f = c$, as required. ∎

4.3. The Path-Lifting Theorem

Theorem 4.5 (Path-Lifting Theorem)

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Let $\gamma\colon [a,b] \to X$ be a continuous map from the closed interval $[a,b]$ to X, and let w be a point of $\tilde{X}$ for which $p(w) = \gamma(a)$. Then there exists a unique continuous map $\tilde{\gamma}\colon [a,b] \to \tilde{X}$ for which $\tilde{\gamma}(a) = w$ and $p \circ \tilde{\gamma} = \gamma$.

Proof

Let S be the subset of $[a,b]$ defined as follows: an element c of $[a,b]$ belongs to S if and only if there exists a continuous map $\eta_c\colon [a,c] \to \tilde{X}$ such that $\eta_c(a) = w$ and $p(\eta_c(t)) = \gamma(t)$ for all $t \in [a,c]$. Note that S is non-empty, since a belongs to S. Let $s = \sup S$.

There exists an open neighbourhood U of $\gamma(s)$ which is evenly covered by the map p, since $p\colon \tilde{X} \to X$ is a covering map. It then follows from the continuity of the path γ that there exists some $\delta > 0$ such that $\gamma(J(s,\delta)) \subset U$, where

$$J(s,\delta) = \{t \in [a,b] : |t - s| < \delta\}.$$

Now $S \cap J(s,\delta)$ is non-empty, because s is the supremum of the set S. Choose some element c of $S \cap J(s,\delta)$. Then there exists a continuous map $\eta_c\colon [a,c] \to \tilde{X}$ such that $\eta_c(a) = w$ and $p(\eta_c(t)) = \gamma(t)$ for all $t \in [a,c]$. Now the open set U is evenly covered by the map p. Therefore $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains the point $\eta_c(c)$; let this open set be denoted by $\tilde{U}$.

There then exists a unique continuous map $\sigma\colon U \to \tilde{U}$ defined such that, for all $x \in U$, $\sigma(x)$ is the unique element of $\tilde{U}$ for which $p(\sigma(x)) = x$. Then $\sigma(\gamma(c)) = \eta_c(c)$.

Then, given any $d \in J(s, \delta)$, let $\eta_d\colon [a, d] \to \tilde{X}$ be the function from $[a, d]$ to $\tilde{X}$ defined so that

$$\eta_d(t) = \begin{cases} \eta_c(t) & \text{if } a \leq t \leq c; \\ \sigma(\gamma(t)) & \text{if } c \leq t \leq d. \end{cases}$$

Then $\eta_d(a) = w$ and $p(\eta_d(t)) = \gamma(t)$ for all $t \in [a, d]$. The restrictions of the function $\eta_d\colon [a, d] \to \tilde{X}$ to the intervals $[a, c]$ and $[c, d]$ are continuous. It follows from the Pasting Lemma (Lemma 1.24) that η_d is continuous on $[a, d]$. Thus $d \in S$. We conclude from this that $J(s, \delta) \subset S$. However s is defined to be the supremum of the set S. Therefore $s = b$, and b belongs to S. It follows that that there exists a continuous map $\tilde{\gamma}\colon [a, b] \to \tilde{X}$ for which $\tilde{\gamma}(a) = w$ and $p \circ \tilde{\gamma} = \gamma$, as required. ∎