MA342R—Covering Spaces and Fundamental Groups
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4. Covering Maps

4.1. Evenly-Covered Open Sets and Covering Maps

Definition

Let X and $\tilde{X}$ be topological spaces and let $p\colon \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p\colon \tilde{X} \to X$ is said to be a *covering map* if $p\colon \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

If $p\colon \tilde{X} \to X$ is a covering map, then we say that $\tilde{X}$ is a *covering space* of X.

Example

Let S^1 be the unit circle in $\mathbb{R}^2$. Then the map $p\colon \mathbb{R} \to S^1$ defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

is a covering map. Indeed let $\mathbf{n}$ be a point of S^1. Consider the open set U in S^1 containing $\mathbf{n}$ defined by $U = S^1 \setminus \{-\mathbf{n}\}$. Now $\mathbf{n} = (\cos 2\pi t_0, \sin 2\pi t_0)$ for some $t_0 \in \mathbb{R}$. Then $p^{-1}(U)$ is the union of the disjoint open sets J_n for all integers n, where

$$J_n = \{t \in \mathbb{R} : t_0 + n - \tfrac{1}{2} < t < t_0 + n + \tfrac{1}{2}\}.$$

Each of the open sets J_n is mapped homeomorphically onto U by the map p. This shows that $p\colon \mathbb{R} \to S^1$ is a covering map.

Example

Let $p_{\exp}\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be the map from the complex plane $\mathbb{C}$ to the open subset $\mathbb{C} \setminus \{0\}$ of $\mathbb{C}$ defined such that $p_{\exp}(z) = \exp(z)$ for all complex numbers z. We show that $p_{\exp}(z)$ is a covering map.

Given any real number s, let

$$L_s = \{-re^{is} : r \in \mathbb{R} \text{ and } r \geq 0\}.$$

Then L_s is a ray in the complex plane starting at zero and passing through $-\cos s - i \sin s$. Moreover every complex number belonging to the complement $\mathbb{C} \setminus L_s$ of the ray L_s in $\mathbb{C}$ can be expressed uniquely in the form re^{it}, where r and t are real numbers satisfying $r > 0$ and $s - \pi < t < s + \pi$.

Let

$$W_s = \{w \in \mathbb{C} : s - \pi < \mathrm{Im}[w] < s + \pi\},$$

where $\mathrm{Im}[w]$ denotes the imaginary part of w for all complex numbers w, and let $F_s \colon \mathbb{C} \setminus L_s \to W_s$ be the complex-valued function on the open subset $\mathbb{C} \setminus L_s$ of the complex plane defined such that

$$F_s(re^{it}) = \log r + it$$

for all real numbers r and t satisfying $r > 0$ and $s - \pi < t < s + \pi$. Then $F_s \colon \mathbb{C} \setminus L_s \to W_s$ is a continuous map, $\exp(F_s(z)) = z$ for all $z \in \mathbb{C} \setminus L_s$ and $F_s(\exp(w)) = w$ for all $w \in W_s$. It follows that $F_s \colon \mathbb{C} \setminus L_s \to W_s$ is a homeomorphism between $\mathbb{C} \setminus L_s$ and W_s.

Let w be a complex number for which $\exp(w) \in \mathbb{C} \setminus L_s$. Then there exists a unique integer m such that $s + 2\pi m - \pi < \mathrm{Im}[w] < s + 2\pi m + \pi$. Then $w \in F_{s+m}(\exp w)$. It follows from this that, for each real number s, the preimage $p_{\exp}^{-1}(\mathbb{C} \setminus L_s)$ is the disjoint union of the sets $W_{s+2\pi m}$ as m ranges over the set $\mathbb{Z}$ of integers. Also $W_{s+2\pi m} \cap W_{s+2\pi n} = \emptyset$ when m and n are integers and $m \neq n$, and $p_{\exp}\colon \mathbb{C} \setminus \mathbb{C} \setminus \{0\}$ maps the open set $W_{s+2\pi m}$ homeomorphically onto $\mathbb{C} \setminus L_s$ for all integers m, where $p_{\exp}(w) = \exp(w)$ for all $w \in \mathbb{C}$. Thus $p_{\exp}\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering map.

Example

Let

$$\begin{aligned} X &= \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}, \\ \tilde{X} &= \{(x,y,z) \in \mathbb{R}^3 : (x,y) \neq (0,0), \\ & \qquad x = \sqrt{x^2+y^2}\cos 2\pi z \text{ and } y = \sqrt{x^2+y^2}\sin 2\pi z\}, \end{aligned}$$

and let $p\colon \tilde{X} \to X$ be defined so that $p(x,y,z) = (x,y)$ for all $(x,y,z) \in \tilde{X}$. Now $\exp(w) = T(p(h(w)))$ for all $w \in \mathbb{C}$, where

$$h(u+iv) = \left(e^u \cos v, e^u \sin v, \frac{v}{2\pi}\right)$$

for all real numbers u and v and $T(x,y) = x + iy$ for all $(x,y) \in X$.

Moreover $h\colon \mathbb{C} \to \tilde{X}$ is a homeomorphism whose inverse h^{-1} satisfies

$$h^{-1}(z) = \tfrac{1}{2}\log(x^2 + y^2) + 2\pi iz$$

for all $(x, y, z) \in \tilde{X}$.

The map $p\colon \tilde{X} \to X$ is a covering map. Indeed let

$$W_{s,m} = \{(x, y, z) \in \tilde{X} : s + m - \tfrac{1}{2} < z < s + m + \tfrac{1}{2}\}$$

and let $V_{s,m} = p(W_{s,0})$ for all real numbers s and integers m. Then $V_{s,0}$ is an open set in X, $p^{-1}(V_{s,0}) = \bigcup_{m\in\mathbb{Z}} W_{s,m}$ and p maps $W_{s,m}$ homeomorphically onto $V_{s,0}$ for all $s \in \mathbb{R}$ and $m \in \mathbb{Z}$.

The surface $\tilde{X}$ is a *helicoid* in $\mathbb{R}^3$.

Example

Consider the map $\alpha\colon (-2,2) \to S^1$, where $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in (-2,2)$. It can easily be shown that there is no open set U containing the point $(1,0)$ that is evenly covered by the map α. Indeed suppose that there were to exist such an open set U. Then there would exist some δ satisfying $0 < \delta < \frac{1}{2}$ such that $U_\delta \subset U$, where

$$U_\delta = \{(\cos 2\pi t, \sin 2\pi t) : -\delta < t < \delta\}.$$

The open set U_δ would then be evenly covered by the map α. However the connected components of $\alpha^{-1}(U_\delta)$ are $(-2,-2+\delta)$, $(-1-\delta,-1+\delta)$, $(-\delta,\delta)$, $(1-\delta,1+\delta)$ and $(2-\delta,2)$, and neither $(-2,-2+\delta)$ nor $(2-\delta,2)$ is mapped homeomorphically onto U_δ by α.

Example

Let $Z = \mathbb{C} \setminus \{1, -1\}$, let

$$\tilde{Z} = \{(z, w) \in \mathbb{C}^2 : w \neq 0 \text{ and } w^2 = z^2 - 1\},$$

and let $p \colon \tilde{Z} \to Z$ be defined such that $p(z, w) = z$ for all $(z, w) \in \tilde{Z}$. Let $(z_0, w_0) \in \tilde{Z}$, and let $z = z_0 + \zeta$. Then

$$\begin{aligned} z^2 - 1 &= z_0^2 - 1 + 2z_0\zeta + \zeta^2 = w_0^2 + 2z_0\zeta + \zeta^2 \\ &= w_0^2 \left(1 + \frac{2z_0\zeta + \zeta^2}{w_0^2}\right). \end{aligned}$$

Now the continuity at zero of the function sending each complex number ζ to $(2z_0\zeta + \zeta^2)/w_0^2$ ensures that there exists some positive real number δ such that

$$\left| \frac{2z_0\zeta + \zeta^2}{w_0^2} \right| < 1$$

whenever $|\zeta| < \delta$. Let $D(z_0, \delta)$ be the open disk of radius δ about z_0 in the complex plane, and let

$$F(z) = \frac{1}{2} \log \left(1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2} \right)$$

for all $z \in D(z_0, \delta)$, where $\log(re^{i\theta}) = \log r + i\theta$ for all real numbers r and θ satisfying $r > 0$ and $-\pi < \theta < \pi$. Then $F(z)$ is a continuous function of z on $D(z_0, \delta)$, and

$$\exp(F(z))^2 = 1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2} = \frac{z^2 - 1}{w_0^2}$$

for all $z \in D(z_0, \delta)$.

Let $(z, w) \in p^{-1}(D(z_0, \delta))$. Then $z \in D(z_0, \delta)$ and

$$w^2 = z^2 - 1 = \Big(w_0 \exp(F(z)) \Big)^2,$$

and therefore $w = \pm w_0 \exp(F(z))$. It follows that $p^{-1}(D(z_0, \delta)) = W_+ \cup W_-$ where

$$\begin{aligned} W_+ &= \{(z, w) \in \mathbb{C}^2 : z \in D(z_0, \delta) \text{ and } w = w_0 \exp(F(z))\}, \\ W_- &= \{(z, w) \in \mathbb{C}^2 : z \in D(z_0, \delta) \text{ and } w = -w_0 \exp(F(z))\}, \end{aligned}$$

Now

$$\mathrm{Re} \left[1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2} \right] > 0$$

for all $z \in D(z_0, \delta)$. It follows from the definition of $F(z)$ that

$$-\tfrac{1}{4}\pi < \mathrm{Im}[F(z)] < \tfrac{1}{4}\pi$$

for all $z \in D(z_0, \delta)$, and therefore

$$\mathrm{Re}[\exp(F(z))] = \exp(\mathrm{Re}[F(z)]) \cos(\mathrm{Im}[F(z)]) > 0$$

for all $z \in D(z_0, \delta)$. It follows that

$$\begin{aligned} W_+ &= \left\{(z, w) \in \tilde{Z} : z \in D(z_0, \delta) \text{ and } \mathrm{Re}\left[\frac{w}{w_0}\right] > 0\right\}, \\ &= \left\{(z, w) \in p^{-1}\left(D(z_0, \delta)\right) : \mathrm{Re}\left[\frac{w}{w_0}\right] > 0\right\}, \\ W_- &= \left\{(z, w) \in \tilde{Z} : z \in D(z_0, \delta) \text{ and } \mathrm{Re}\left[\frac{w}{w_0}\right] < 0\right\}, \\ &= \left\{(z, w) \in p^{-1}\left(D(z_0, \delta)\right) : \mathrm{Re}\left[\frac{w}{w_0}\right] < 0\right\}. \end{aligned}$$

Now $p^{-1}(D(z_0, \delta)$ is open in $\tilde{Z}$, because the it is the preimage of the open subset $D(z_0, \delta)$ of Z under the continuous map $p\colon \tilde{Z} \to Z$. Moreover the function mapping (z, w) to the real part of w/w_0 is continuous on $p^{-1}(D(z_0, \delta)$. It follows that W_+ and W_- are open in $\tilde{Z}$. Also $W_+ \cap W_- = \emptyset$, and the map $p\colon \tilde{Z} \to Z$ maps each of the sets W_+ and W_- homeomorphically onto Z, where $Z = \mathbb{C} \setminus \{1, -1\}$. It follows that the open disk $D(z_0, \delta)$ is evenly covered by the map $p\colon \tilde{Z} \to Z$. We have therefore shown that this map is a covering map.

Let $\tilde{f}(z, w) = w$ for all $(z, w) \in \tilde{Z}$. Then

$$\tilde{f}(\tilde{z})^2 = p(\tilde{z})^2 - 1$$

for all $\tilde{z} \in \tilde{Z}$. It follows that the function $\tilde{f}\colon \tilde{Z} \to \mathbb{C}$ represents in some sense the many-valued 'function' $\sqrt{z^2 - 1}$. However this function $\tilde{z}$ is not defined on the open subset Z of the complex plane, but is instead defined over a covering space $\tilde{Z}$ of this open set. This covering space is the *Riemann surface* for the 'function' $\sqrt{z^2 - 1}$. This method of representing many-valued 'functions' of a complex variable using single-valued functions defined over a covering space was initiated and extensively developed by Bernhard Riemann (1826–1866) in his doctoral thesis.