MA342R—Covering Spaces and Fundamental Groups
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3.4. The Fundamental Group of the Circle

Proposition 3.6

Let S^1 be the unit circle in the Euclidean plane, defined so that

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and let $\gamma\colon [a, b] \to S^1$ be a continuous map into S^1 defined on a closed bounded interval $[a, b]$. Then there exists a continuous real-valued function $\tilde{\gamma}\colon [a, b] \to \mathbb{R}$ on the interval $[a, b]$ with the property that

$$(\cos 2\pi\tilde{\gamma}(t), \sin 2\pi\tilde{\gamma}(t)) = \gamma(t)$$

for all $t \in [a, b]$.

Proof

Let $\gamma(t) = \big(\gamma_1(t), \gamma_2(t)\big)$ for all $t \in [a, b]$ and let $\eta\colon [a, b] \to \mathbb{C}$ be the continuous map into the complex plane defined such that $\eta(t) = \gamma_1(t) + i\gamma_2(t)$ for all $t \in [a, b]$, where $i^2 = -1$. Now $|\eta(t)| = 1$ for all $t \in [a, b]$. It follows from the path-lifting property of the exponential map (Theorem 2.5) that there exists a continuous map $\tilde{\eta}\colon [a, b] \to \mathbb{C}$ with the property that $\exp(\tilde{\eta}(t)) = \eta(t)$ for all $t \in [a, b]$. Moreover $\mathrm{Re}[\tilde{\eta}(t)] = 0$ for all $t \in [a, b]$ (where $\mathrm{Re}[\tilde{\eta}(t)]$ denotes the real part of $\tilde{\eta}(t)$), because $|\eta(t)| = 1$ for all $t \in [a, b]$. Therefore there exists a continuous map $\tilde{\gamma}\colon [a, b] \to \mathbb{R}$ such that $\tilde{\eta}(t) = 2\pi i\,\tilde{\gamma}(t)$ for all $t \in [a, b]$. Then

$$\begin{aligned} \cos 2\pi\tilde{\gamma}(t) + i\,\sin 2\pi\tilde{\gamma}(t) &= \exp(2\pi i\,\tilde{\gamma}(t)) = \exp(\tilde{\eta}(t)) \\ &= \eta(t) = \gamma_1(t) + i\,\gamma_2(t) \end{aligned}$$

for all $t \in [a, b]$. The result follows. ∎

Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

and let $p\colon \mathbb{R} \to S^1$ be defined so that $p(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in \mathbb{R}$. This function p has the following periodicity property:

> *real numbers s and t satisfy $p(s) = p(t)$ if and only if $s - t$ is an integer.*

It follows from Proposition 3.6 that, given any loop $\gamma\colon [0,1] \to S^1$ in the circle S^1, there exists a continuous real-valued function $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ with the property that $p \circ \tilde{\gamma} = \gamma$. Then $p(\tilde{\gamma}(1)) = p(\tilde{\gamma}(0))$. It follows from the periodicity property of the function p that $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ is an integer. We now that the value of this integer is determined by the loop γ, and does not depend on the choice of function $\tilde{\gamma}$, provided that $p \circ \tilde{\gamma} = \gamma$.

If $\eta\colon [0,1] \to \mathbb{R}$ is a continuous function with the property that $p \circ \eta = \gamma$ then $p \circ \eta = p \circ \tilde{\gamma}$ and therefore

$$\eta(t) - \tilde{\gamma}(t) \in \mathbb{Z}$$

for all $t \in [0,1]$. But $\eta(t) - \tilde{\gamma}(t)$ is a continuous function of t on $[0,1]$, and the connectedness of $[0,1]$ ensures that every continuous integer-valued function on $[0,1]$ is constant (Corollary 1.58). It follows that there exists some integer m with the property that $\eta(t) = \tilde{\gamma}(t) + m$ for all $t \in [0,1]$, where the value of m is independent of t. But then $\eta(1) - \eta(0) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$. It follows that the loop γ determines a well-defined integer $n(\gamma)$ characterized by the property that $n(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$ for all continuous real-valued functions $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ on $[0,1]$ that satisfy $p \circ \tilde{\gamma} = \gamma$.

Definition

Let $\gamma\colon [0,1] \to S^1$ be a loop in the circle S^1, where

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

The *winding number* $n(\gamma)$ of γ is defined to be unique integer characterized by the property that

$$n(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$$

for all continuous functions $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ that satisfy

$$(\cos 2\pi\tilde{\gamma}(t), \sin 2\pi\tilde{\gamma}(t)) = \gamma(t)$$

for all $t \in [0,1]$.

Proposition 3.7

Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

let $H\colon [0,1] \times [0,1] \to S^1$ *be a continuous map that satisfies* $H(0,\tau) = H(1,\tau)$ *for all* $\tau \in [0,1]$. *Also, for each* $\tau \in [0,1]$, *let* $n(\gamma_\tau)$ *be the winding number of the loop* γ_τ *in* S^1 *defined such that* $\gamma_\tau(t) = H(t,\tau)$ *for all* $t \in [0,1]$. *Then* $n(\gamma_0) = n(\gamma_1)$.

Proof

Let $G = T \circ H$, where $T\colon \mathbb{R}^2 \to \mathbb{C}$ is defined so that $T(x, y) = x + iy$ for all real numbers x and y. Then $G(t, \tau) = T \circ \gamma_\tau(t)$ for all $t \in [0, 1]$ and $\tau \in [0, 1]$. Moreover $n(\gamma_\tau) = n(T \circ \gamma_\tau, 0)$ for all $\tau \in [0, 1]$, where $n(T \circ \gamma_\tau, 0)$ denotes the winding number of the closed curve $T \circ \gamma_\tau$ around zero. It therefore follows from Proposition 2.9 that

$$n(\gamma_0) = n(T \circ \gamma_0, 0) = n(T \circ \gamma_1, 0) = n(\gamma_1),$$

as required. ∎

Corollary 3.8

Let S^1 be the unit circle in the Euclidean plane, defined so that

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and let $\mathbf{b}$ *be a point of* S^1*. Let* α *and* β *be loops in* S^1 *based at* $\mathbf{b}$*. Suppose that* $\alpha \simeq \beta \operatorname{rel} \{0, 1\}$*. Then* $n(\alpha) = n(\beta)$*, where* $n(\alpha)$ *and* $n(\beta)$ *denote the winding numbers of the loops* α *and* β *respectively.*

Proof

The loops α and β satisfy $\alpha \simeq \beta \operatorname{rel} \{0, 1\}$ if and only if there exists a homotopy $H\colon [0, 1] \times [0, 1] \to S^1$ with the following properties: $H(t, 0) = \alpha(t)$ and $H(t, 1) = \beta(t)$ for all $t \in [0, 1]$; $H(0, \tau) = H(1, \tau) = \mathbf{b}$ for all $\tau \in [0, 1]$. The result therefore follows directly from Proposition 3.7. ∎

Theorem 3.9

Let S^1 be the unit circle in the Euclidean plane, defined so that

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and let $\mathbf{b}$ be a point of S^1. Then the function sending each loop γ in S^1 based at $\mathbf{b}$ to its winding number $n(\gamma)$ induces an isomorphism from the fundamental group $\pi_1(S^1, \mathbf{b})$ of the circle S^1 to the group $\mathbb{Z}$ of integers.

Proof

Let $p\colon \mathbb{R} \to S^1$ denote the function from $\mathbb{R}$ to S^1 defined so that

$$p(t) = (cos 2\pi t, \sin 2\pi t)$$

for all real numbers t. Also, for each loop $\gamma\colon [0,1] \to S^1$ in S^1 based at $\mathbf{b}$ let $[\gamma]$ denote the element of the fundamental group $\pi_1(S^1, \mathbf{b})$ determined by γ, and let $n(\gamma)$ denote the winding number of γ. Every element of $\pi_1(S^1, \mathbf{b})$ is the based homotopy class $[\gamma]$ of some loop γ in S^1 based at $\mathbf{b}$. If $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ is a real-valued function for which $p \circ \tilde{\gamma} = \gamma$ then $n(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$.

Let α and β be loops in S^1 based at $\mathbf{b}$. Suppose that $[\alpha] = [\beta]$. Then $\alpha \simeq \beta \text{ rel } \{0,1\}$. It then follows from Corollary 3.8 that $n(\alpha) = n(\beta)$. It follows from this that there is a well-defined function $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ characterized by the property that $\lambda([\gamma]) = n(\gamma)$ for all loops γ in S^1 based at $\mathbf{b}$.

Next we show that the function $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is a homomorphism. Let $\alpha\colon [0,1] \to S^1$ and $\beta\colon [0,1] \to S^1$ be loops in S^1 based at $\mathbf{b}$. Then there exists a continuous real-valued function $\eta\colon [0,1] \to \mathbb{R}$ with the property that

$$p(\eta(t)) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

where $p(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in \mathbb{R}$ (see Proposition 3.6). Then $\alpha(t) = p(\eta(\frac{1}{2}t))$ for all $t \in [0,1]$. It follows from the definition of winding numbers that $n(\alpha) = \eta(\frac{1}{2}) - \eta(0)$. Also $\beta(t) = p(\eta(\frac{1}{2}(t+1)))$ for all $t \in [0,1]$, and therefore $n(\beta) = \eta(1) - \eta(\frac{1}{2})$. It follows that

$$n(\alpha) + n(\beta) = \eta(1) - \eta(0) = n(p \circ \eta) = n(\alpha \,.\, \beta),$$

where $\alpha \,.\, \beta$ is the concatenation of the loops α and β. It follows that

$$\lambda([\alpha]) + \lambda([\beta]) = n(\alpha) + n(\beta) = n(\alpha\,.\,\beta) = \lambda([\alpha\,.\,\beta]) = \lambda([\alpha][\beta]).$$

We conclude that $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is a homomorphism.

Next we show that $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is injective. Let α and β be loops in S^1 for which $n(\alpha) = n(\beta)$. Then there exist real-valued functions $\tilde{\alpha}\colon [0,1] \to \mathbb{R}$ and $\tilde{\beta}\colon [0,1] \to \mathbb{R}$ for which $\alpha = p \circ \tilde{\alpha}$ and $\beta = p \circ \tilde{\beta}$ (Proposition 3.6). Moreover

$$\tilde{\alpha}(1) - \tilde{\alpha}(0) = n(\alpha) = n(\beta) = \tilde{\beta}(1) - \tilde{\beta}(0).$$

Also $p(\tilde{\alpha}(0)) = \mathbf{b} = p(\tilde{\beta}(0))$, and therefore there exists some integer m for which $\tilde{\beta}(0) = \tilde{\alpha}(0) + m$. Then

$$\tilde{\beta}(1) = \tilde{\beta}(1) - \tilde{\beta}(0) + \tilde{\alpha}(0) + m = \tilde{\alpha}(1) + m.$$

Let

$$F(t,\tau) = (1-\tau)\tilde{\alpha}(t) + \tau(\tilde{\beta}(t) - m).$$

Then $F(t,0) = \tilde{\alpha}(t)$ and $F(t,1) = \tilde{\beta}(t) - m$ for all $t \in [0,1]$. Also $F(0,\tau) = \tilde{\alpha}(0)$ and $F(1,\tau) = \tilde{\alpha}(1)$ for all $\tau \in [0,1]$. Let $H\colon [0,1] \times [0,1] \to S^1$ be defined so that $H(t,\tau) = p(F(t,\tau))$ for all $t \in [0,1]$ and $\tau \in [0,1]$. Then $H(t,0) = \alpha(t)$ and $H(t,1) = \beta(t)$ for all $t \in [0,1]$. Also $H(0,\tau) = H(1,\tau) = \mathbf{b}$ for all $\tau \in [0,1]$. It follows that $\alpha \simeq \beta \text{ rel } \{0,1\}$ and therefore $[\alpha] = [\beta]$ in $\pi_1(X,\mathbf{b})$. We conclude therefore that $\lambda\colon \pi_1(S^1,\mathbf{b}) \to \mathbb{Z}$ is injective.

Let m be an integer, let t_0 be a real number for which $p(t_0) = \mathbf{b}$, and let $\gamma(t) = p(t_0 + mt)$ for all $t \in [0,1]$. Then $\gamma\colon [0,1] \to S^1$ is a loop in S^1 based at $\mathbf{b}$, and $\lambda([\gamma]) = n(\gamma) = m$. We conclude that $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is surjective. We have now shown that the function λ is a homomorphism that is both injective and surjective. It follows that $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is an isomorphism. This completes the proof. ∎

Proposition 3.10

Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$. *Then* $\pi_1(X, (1,0)) \cong \mathbb{Z}$.

Proof

Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1,$$

let $i\colon S^1 \to X$ be the inclusion map, and let $r\colon X \to S^1$ be the radial projection map, defined such that

$$r(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

for all $(x, y) \in X$. Now the composition map $r \circ i$ is the identity map of S^1. Let

$$u(x, y, \tau) = \frac{1 - \tau}{\sqrt{x^2 + y^2}} + \tau$$

for all $(x, y) \in X$ and $\tau \in [0, 1]$. Then the function $F\colon X \times [0, 1] \to X$ that sends $((x, y), \tau) \in X \times [0, 1]$ to $(u(x, y, \tau)x, u(x, y, \tau)y)$ is a homotopy between the composition map $i \circ r$ and the identity map of the punctured plane X. Moreover $F((x, y), \tau) = (x, y)$ for all $(x, y) \in S^1$ and $\tau \in [0, 1]$.

Let $\gamma\colon [0, 1] \to X$ be a loop in X based at $(1, 0)$ and let $H\colon [0, 1] \times [0, 1] \to X$ be defined so that $H(t, \tau) = F(\gamma(t), \tau)$ for all $t \in [0, 1]$ and $\tau \in [0, 1]$. Then $H(t, 0) = r(\gamma(t))$ and $H(t, 1) = \gamma(t)$ for all $t \in [0, 1]$, and $H(0, \tau) = H(1, \tau) = (1, 0)$ for all $\tau \in [0, 1]$, and therefore $i \circ r \circ \gamma \simeq \gamma \text{ rel } \{0, 1\}$.

Now the continuous maps $i\colon S^1 \to X$ and $r\colon X \to S^1$ induce well-defined homomorphisms $i_\#\colon \pi_1(S^1,(1,0)) \to \pi_1(X,(1,0))$ and $r_\#\colon \pi_1(X,(1,0)) \to \pi_1(S^1,(1,0))$, where $i_\#[\eta] = [i \circ \eta]$ for all loops η in S^1 based at $(1,0)$ and $r_\#[\gamma] = [r \circ \gamma]$ for all loops γ in X based at $(1,0)$. Moreover

$$i_\#(r_\#([\gamma]) = i_\#([r \circ \gamma]) = [i \circ r \circ \gamma] = [\gamma]$$

for all loops γ in X based at $(1,0)$, and

$$r_\#(i_\#([\eta])r_\#[i \circ \eta] = [r \circ i \circ \eta] = [\eta]$$

for all loops η in S^1 based at $(1,0)$. It follows that the homomorphism $i_\#\colon \pi_1(S^1,(1,0)) \to \pi_1(X,(1,0))$ is an isomorphism whose inverse is the homomorphism $r_\#\colon \pi_1(X,(1,0)) \to \pi_1(S^1,(1,0))$, and therefore

$$\pi_1(X,(1,0)) \cong \pi_1(S^1,(1,0)) \cong \mathbb{Z},$$

as required. ∎

Example

Let D be the closed unit disk in $\mathbb{R}^2$ and let ∂D be its boundary circle, where

$$\begin{aligned} D^2 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}, \\ \partial D^2 = S^1 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \end{aligned}$$

let $i\colon \partial D \to D$ be the inclusion map, and let $\mathbf{b} = (1,0)$. Suppose there were to exist a continuous map $r\colon D \to \partial D$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial D$. Then $r \circ i\colon \partial D \to \partial D$ would be the identity map of the unit circle ∂D. It would then follow that $r_\# \circ i_\#$ would be the identity isomorphism of $\pi_1(\partial D, \mathbf{b})$, where $i_\#\colon \pi_1(\partial D, \mathbf{b}) \to \pi_1(D,)$ and $r_\#\colon \pi_1(D, \mathbf{b}) \to \pi_1(\partial D,)$ denote the homomorphisms of fundamental groups induced by $i\colon \partial D \to D$ and $r\colon D \to \partial D$ respectively.

But $\pi_1(D, \mathbf{b})$ is the trivial group, because D is a convex set in $\mathbb{R}^2$, and $\pi_1(\partial D, \mathbf{b}) \cong \mathbb{Z}$ (Theorem 3.9). It follows that the identity homomorphism of $\pi_1(D, \mathbf{b})$ cannot be expressed as a composition of two homomorphisms $\theta \circ \varphi$ where θ is a homomorphism from $\pi_1(\partial D, \mathbf{b})$ to $\pi_1(D, \mathbf{b})$ and φ is a homomorphism from $\pi_1(D, \mathbf{b})$ to $\pi_1(\partial D, \mathbf{b})$. Therefore there cannot exist any continous map $r \colon D \to \partial D$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial D$. This result has already been established (see Corollary 2.15). Moreover the result is used to establish the Brouwer Fixed Point Theorem in the two-dimensional case (Theorem 2.16) which ensures that every continuous map from the two-dimensional closed disk D^2 to itself has a fixed point.