MA342R—Covering Spaces and Fundamental Groups
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3.3. Simply-Connected Topological Spaces

Definition

A topological space X is said to be *simply-connected* if it is path-connected, and any continuous map $f\colon \partial D \to X$ mapping the boundary circle ∂D of a closed disc D into X can be extended continuously over the whole of the disk.

Example

$\mathbb{R}^n$ is simply-connected for all n. Indeed any continuous map $f\colon \partial D \to \mathbb{R}^n$ defined over the boundary ∂D of the closed unit disk D can be extended to a continuous map $F\colon D \to \mathbb{R}^n$ over the whole disk by setting $F(r\mathbf{x}) = rf(\mathbf{x})$ for all $\mathbf{x} \in \partial D$ and $r \in [0, 1]$.

Let E be a topological space that is homeomorphic to the closed disk D, and let $\partial E = h(\partial D)$, where ∂D is the boundary circle of the disk D and $h\colon D \to E$ is a homeomorphism from D to E. Then any continuous map $g\colon \partial E \to X$ mapping ∂E into a simply-connected space X extends continuously to the whole of E. Indeed there exists a continuous map $F\colon D \to X$ which extends $g \circ h\colon \partial D \to X$, and the map $F \circ h^{-1}\colon E \to X$ then extends the map g.

Theorem 3.4

A path-connected topological space X is simply-connected if and only if $\pi_1(X, x)$ is trivial for all $x \in X$.

Proof

Suppose that the space X is simply-connected. Let $\gamma\colon [0, 1] \to X$ be a loop based at some point x of X. Now the unit square is homeomorphic to the unit disk, and therefore any continuous map defined over the boundary of the square can be continuously extended over the whole of the square. It follows that there exists a continuous map $F\colon [0, 1] \times [0, 1] \to X$ such that $F(t, 0) = \gamma(t)$ and $F(t, 1) = x$ for all $t \in [0, 1]$, and $F(0, \tau) = F(1, \tau) = x$ for all $\tau \in [0, 1]$. Thus $\gamma \simeq \varepsilon_x \operatorname{rel}\{0, 1\}$, where ε_x is the constant loop at x, and hence $[\gamma] = [\varepsilon_x]$ in $\pi_1(X, x)$. This shows that $\pi_1(X, x)$ is trivial.

Conversely suppose that X is path-connected and $\pi_1(X, x)$ is trivial for all $x \in X$. Let $f\colon \partial D \to X$ be a continuous function defined on the boundary circle ∂D of the closed unit disk D in $\mathbb{R}^2$. We must show that f can be extended continuously over the whole of D. Let $x = f(1, 0)$. There exists a continuous map $G\colon [0, 1] \times [0, 1] \to X$ such that $G(t, 0) = f(\cos(2\pi t), \sin(2\pi t))$ and $G(t, 1) = x$ for all $t \in [0, 1]$ and $G(0, \tau) = G(1, \tau) = x$ for all $\tau \in [0, 1]$, since $\pi_1(X, x)$ is trivial. Moreover $G(t_1, \tau_1) = G(t_2, \tau_2)$ whenever $q(t_1, \tau_1) = q(t_2, \tau_2)$, where

$$q(t, \tau) = \bigl((1 - \tau)\cos(2\pi t) + \tau, (1 - \tau)\sin(2\pi t)\bigr)$$

for all $t, \tau \in [0, 1]$. It follows that there is a well-defined function $F\colon D \to X$ such that $F \circ q = G$.

However $q\colon [0,1] \times [0,1] \to D$ is a continuous surjection from a compact space to a Hausdorff space and is therefore an identification map. It follows that $F\colon D \to X$ is continuous (since a basic property of identification maps ensures that a function $F\colon D \to X$ is continuous if and only if $F \circ q\colon [0,1] \times [0,1] \to X$ is continuous). Moreover $F\colon D \to X$ extends the map f. We conclude that the space X is simply-connected, as required. ∎

One can show that, if two points x_1 and x_2 in a topological space X can be joined by a path in X then $\pi_1(X, x_1)$ and $\pi_1(X, x_2)$ are isomorphic. On combining this result with Theorem 3.4, we see that a path-connected topological space X is simply-connected if and only if $\pi_1(X, x)$ is trivial for some $x \in X$.

Theorem 3.5

Let X be a topological space, and let U and V be open subsets of X, with $U \cup V = X$. Suppose that U and V are simply-connected, and that $U \cap V$ is non-empty and path-connected. Then X is itself simply-connected.

Proof

We must show that any continuous function $f\colon \partial D \to X$ defined on the unit circle ∂D can be extended continuously over the closed unit disk D. Now the preimages $f^{-1}(U)$ and $f^{-1}(V)$ of U and V are open in ∂D (since f is continuous), and $\partial D = f^{-1}(U) \cup f^{-1}(V)$. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that any arc in ∂D whose length is less than δ is entirely contained in one or other of the sets $f^{-1}(U)$ and $f^{-1}(V)$.

Choose points $z_1, z_2, \ldots, z_n$ around ∂D such that the distance from z_i to z_{i+1} is less than δ for $i = 1, 2, \ldots, n-1$ and the distance from z_n to z_1 is also less than δ. Then, for each i, the short arc joining z_{i-1} to z_i is mapped by f into one or other of the open sets U and V.

Let x_0 be some point of $U \cap V$. Now the sets U, V and $U \cap V$ are all path-connected. Therefore we can choose paths $\alpha_i\colon [0,1] \to X$ for $i = 1, 2, \ldots, n$ such that $\alpha_i(0) = x_0$, $\alpha_i(1) = f(z_i)$, $\alpha_i([0,1]) \subset U$ whenever $f(z_i) \in U$, and $\alpha_i([0,1]) \subset V$ whenever $f(z_i) \in V$. For convenience let $\alpha_0 = \alpha_n$.

Now, for each i, consider the sector T_i of the closed unit disk bounded by the line segments joining the centre of the disk to the points z_{i-1} and z_i and by the short arc joining z_{i-1} to z_i. Now this sector is homeomorphic to the closed unit disk, and therefore any continuous function mapping the boundary ∂T_i of T_i into a simply-connected space can be extended continuously over the whole of T_i. In particular, let F_i be the function on ∂T_i defined by

$$F_i(z) = \begin{cases} f(z) & \text{if } z \in T_i \cap \partial D, \\ \alpha_{i-1}(t) & \text{if } z = tz_{i-1} \text{ for some } t \in [0,1], \\ \alpha_i(t) & \text{if } z = tz_i \text{ for some } t \in [0,1], \end{cases}$$

Note that $F_i(\partial T_i) \subset U$ whenever the short arc joining z_{i-1} to z_i is mapped by f into U, and $F_i(\partial T_i) \subset V$ whenever this short arc is mapped into V.

Now U and V are both simply-connected. It follows that each of the functions F_i can be extended continuously over the whole of the sector T_i. Moreover the functions defined in this fashion on each of the sectors T_i agree with one another wherever the sectors intersect, and can therefore be pieced together to yield a continuous map defined over the the whole of the closed disk D which extends the map f, as required. ∎

Example

The n-dimensional sphere S^n is simply-connected for all $n > 1$, where $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$. Indeed let $U = \{\mathbf{x} \in S^n : x_{n+1} > -\frac{1}{2}\}$ and $V = \{\mathbf{x} \in S^n : x_{n+1} < \frac{1}{2}\}$. Then U and V are homeomorphic to an n-dimensional ball, and are therefore simply-connected. Moreover $U \cap V$ is path-connected, provided that $n > 1$. It follows that S^n is simply-connected for all $n > 1$.