MA342R—Covering Spaces and Fundamental Groups
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3. The Fundamental Group of a Topological Space

3.1. Homotopies between Continuous Maps

Definition

Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps between topological spaces X and Y. The maps f and g are said to be *homotopic* if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. If the maps f and g are homotopic then we denote this fact by writing $f \simeq g$. The map H with the properties stated above is referred to as a *homotopy* between f and g.

Continuous maps f and g from X to Y are homotopic if and only if it is possible to 'continuously deform' the map f into the map g.

Definition

Let X and Y be topological spaces, and let A be a subset of X. Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps from X to some topological space Y, where $f|A = g|A$ (i.e., $f(a) = g(a)$ for all $a \in A$). We say that f and g are *homotopic relative to* A (denoted by $f \simeq g \ \mathrm{rel}\ A$) if and only if there exists a (continuous) homotopy $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$ and $H(a,t) = f(a) = g(a)$ for all $a \in A$.

Proposition 3.1

Let X and Y be topological spaces, and let A be a subset of X. The relation of being homotopic relative to the subset A is then an equivalence relation on the set of all continuous maps from X to Y.

Proof

Given $f\colon X \to Y$, let $H_0\colon X \times [0,1] \to Y$ be defined so that $H_0(x,t) = f(x)$ for all $x \in X$ and $t \in [0,1]$. Then $H_0(x,0) = H_0(x,1) = f(x)$ for all $x \in X$ and $H_0(a,t) = f(a)$ for all $a \in A$ and $t \in [0,1]$, and therefore $f \simeq f \ \mathrm{rel}\ A$. Thus the relation of homotopy relative to A is reflexive.

Let f and g be continuous maps from X to Y that satisfy $f(a) = g(a)$ for all $a \in A$. Suppose that $f \simeq g \mathrm{\ rel\ } A$. Then there exists a homotopy $H \colon X \times [0,1] \to Y$ with the properties that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$ and $H(a,t) = f(a) = g(a)$ for all $a \in A$ and $t \in [0,1]$. Let $K \colon X \times [0,1] \to Y$ be defined so that $K(x,t) = H(x,1-t)$ for all $t \in [0,1]$. Then K is a homotopy between g and f, and $K(a,t) = g(a) = f(a)$ for all $a \in A$ and $t \in [0,1]$. It follows that $g \simeq f \mathrm{\ rel\ } A$. Thus the relation of homotopy relative to A is symmetric.

Finally let f, g and h be continuous maps from X to Y with the property that $f(a) = g(a) = h(a)$ for all $a \in A$. Suppose that $f \simeq g \text{ rel } A$ and $g \simeq h \text{ rel } A$. Then there exist homotopies $H_1 \colon X \times [0,1] \to Y$ and $H_2 \colon X \times [0,1] \to Y$ satisfying the following properties:

$$\begin{aligned} H_1(x,0) &= f(x), \\ H_1(x,1) &= g(x) = H_2(x,0), \\ H_2(x,1) &= h(x) \end{aligned}$$

for all $x \in X$;

$$H_1(a,t) = H_2(a,t) = f(a) = g(a) = h(a)$$

for all $a \in A$ and $t \in [0,1]$.

Define $H\colon X \times [0,1] \to Y$ by

$$H(x,t) = \begin{cases} H_1(x,2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ H_2(x,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now $H|X \times [0,\frac{1}{2}]$ and $H|X \times [\frac{1}{2},1]$ are continuous. It follows from the Pasting Lemma (Lemma 1.24) that H is continuous on $X \times [0,1]$. Moreover $H(x,0) = f(x)$ and $H(x,1) = h(x)$ for all $x \in X$. Thus $f \simeq h \,\mathrm{rel}\, A$. Thus the relation of homotopy relative to the subset A of X is transitive. This relation has now been shown to be reflexive, symmetric and transitive. It is therefore an equivalence relation. ∎

Remark

Let X and Y be topological spaces, and let $H\colon X \times [0,1] \to Y$ be a function whose restriction to the sets $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ is continuous. Then the function H is continuous on $X \times [0,1]$. The Pasting Lemma (Lemma 1.24) was applied in the proof of Proposition 3.1 to justify this assertion. We consider in more detail how the Pasting Lemma guarantees the continuity of this function. Let $x \in X$. If $t \in [0,1]$ and $t \neq \frac{1}{2}$ then the point (x,t) is contained in an open subset of $X \times [0,1]$ over which the function H is continuous, and therefore the function H is continuous at (x,t). In order to complete the proof that the function H is continuous everywhere on $X \times [0,1]$ it suffices to verify continuity of H at $(x, \frac{1}{2})$, where $x \in X$.

Let V be an open set in Y for which $f(x, \frac{1}{2}) \in V$. Then the continuity of the restrictions of H to $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ ensures the existence of open sets W_1 and W_2 in $X \times [0, 1]$ such that $(x, \frac{1}{2}) \in W_1 \cap W_2$, $H(W_1 \cap (X \times [0, \frac{1}{2}])) \subset V$ and $H(W_2 \cap (X \times [\frac{1}{2}, 1])) \subset V$. Let $W = W_1 \cap W_2$. Then $H(W) \subset V$. This completes the verification that the function H is continuous at $(x, \frac{1}{2})$. The Pasting Lemma is a basic tool for establishing the continuity of functions occurring in algebraic topology that are similar in nature to the function $H\colon X \times [0, 1] \to Y$ considered in this discussion. The continuity of such functions can typically be established directly using arguments analogous to that employed here.

Corollary 3.2

Let X and Y be topological spaces. The homotopy relation $\simeq$ is an equivalence relation on the set of all continuous maps from X to Y.

Proof

This result follows on applying Proposition 3.1 in the case where homotopies are relative to the empty set. ∎

3.2. The Fundamental Group of a Topological Space

Definition

Let X be a topological space, and let x_0 and x_1 be points of X. A *path* in X from x_0 to x_1 is defined to be a continuous map $\gamma\colon [0,1] \to X$ for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$. A *loop* in X based at x_0 is defined to be a continuous map $\gamma\colon [0,1] \to X$ for which $\gamma(0) = \gamma(1) = x_0$.

We can concatenate paths. Let $\gamma_1\colon [0,1] \to X$ and $\gamma_2\colon [0,1] \to X$ be paths in some topological space X. Suppose that $\gamma_1(1) = \gamma_2(0)$. We define the *product path* $\gamma_1 . \gamma_2\colon [0,1] \to X$ by

$$(\gamma_1 . \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

If $\gamma\colon [0,1] \to X$ is a path in X then we define the *inverse path* $\gamma^{-1}\colon [0,1] \to X$ by $\gamma^{-1}(t) = \gamma(1-t)$. (Thus if γ is a path from the point x_0 to the point x_1 then γ^{-1} is the path from x_1 to x_0 obtained by traversing γ in the reverse direction.)

Let X be a topological space, and let $x_0 \in X$ be some chosen point of X. We define an equivalence relation on the set of all (continuous) loops based at the basepoint x_0 of X, where two such loops γ_0 and γ_1 are equivalent if and only if $\gamma_0 \simeq \gamma_1 \text{ rel } \{0, 1\}$. We denote the equivalence class of a loop $\gamma\colon [0, 1] \to X$ based at x_0 by $[\gamma]$. This equivalence class is referred to as the *based homotopy class* of the loop γ. The set of equivalence classes of loops based at x_0 is denoted by $\pi_1(X, x_0)$. Thus two loops γ_0 and γ_1 represent the same element of $\pi_1(X, x_0)$ if and only if $\gamma_0 \simeq \gamma_1 \text{ rel } \{0, 1\}$ (i.e., there exists a homotopy $F\colon [0, 1] \times [0, 1] \to X$ between γ_0 and γ_1 which maps $(0, \tau)$ and $(1, \tau)$ to x_0 for all $\tau \in [0, 1]$).

Theorem 3.3

Let X be a topological space, let x_0 be some chosen point of X, and let $\pi_1(X, x_0)$ be the set of all based homotopy classes of loops based at the point x_0. Then $\pi_1(X, x_0)$ is a group, the group multiplication on $\pi_1(X, x_0)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1 . \gamma_2]$ for all loops γ_1 and γ_2 based at x_0.

Proof

First we show that the group operation on $\pi_1(X, x_0)$ is well-defined. Let γ_1, γ_1', γ_2 and γ_2' be loops in X based at the point x_0. Suppose that $[\gamma_1] = [\gamma_1']$ and $[\gamma_2] = [\gamma_2']$. Let the map $F\colon [0,1] \times [0,1] \to X$ be defined by

$$F(t,\tau) = \begin{cases} F_1(2t,\tau) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ F_2(2t-1,\tau) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $F_1\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_1 and γ_1', $F_2\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_2 and γ_2', and where the homotopies F_1 and F_2 map $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Then F is itself a homotopy from $\gamma_1 \,.\, \gamma_2$ to $\gamma_1' \,.\, \gamma_2'$, and maps $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Thus $[\gamma_1 \,.\, \gamma_2] = [\gamma_1' \,.\, \gamma_2']$, showing that the group operation on $\pi_1(X, x_0)$ is well-defined.

Next we show that the group operation on $\pi_1(X, x_0)$ is associative. Let γ_1, γ_2 and γ_3 be loops based at x_0, and let $\alpha = (\gamma_1.\gamma_2).\gamma_3$. Then $\gamma_1.(\gamma_2.\gamma_3) = \alpha \circ \theta$, where

$$\theta(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t \leq \frac{1}{2}; \\ t - \frac{1}{4} & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}; \\ 2t - 1 & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$

Thus the map $G\colon [0,1] \times [0,1] \to X$ defined by $G(t, \tau) = \alpha((1-\tau)t + \tau\theta(t))$ is a homotopy between $(\gamma_1.\gamma_2).\gamma_3$ and $\gamma_1.(\gamma_2.\gamma_3)$, and moreover this homotopy maps $(0, \tau)$ and $(1, \tau)$ to x_0 for all $\tau \in [0,1]$. It follows that $(\gamma_1 \,.\, \gamma_2) \,.\, \gamma_3 \simeq \gamma_1 \,.\, (\gamma_2 \,.\, \gamma_3) \text{ rel } \{0,1\}$ and hence $([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]([\gamma_2][\gamma_3])$. This shows that the group operation on $\pi_1(X, x_0)$ is associative.

Let $\varepsilon\colon [0,1] \to X$ denote the constant loop at x_0, defined by $\varepsilon(t) = x_0$ for all $t \in [0,1]$. Then $\varepsilon \mathbin{.} \gamma = \gamma \circ \theta_0$ and $\gamma \mathbin{.} \varepsilon = \gamma \circ \theta_1$ for any loop γ based at x_0, where

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2t - 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$\theta_1(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

for all $t \in [0,1]$. But the continuous map $(t, \tau) \mapsto \gamma((1 - \tau)t + \tau\theta_j(t))$ is a homotopy between γ and $\gamma \circ \theta_j$ for $j = 0, 1$ which sends $(0, \tau)$ and $(1, \tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\varepsilon \mathbin{.} \gamma \simeq \gamma \simeq \gamma \mathbin{.} \varepsilon \ \mathrm{rel}\ \{0, 1\}$, and hence $[\varepsilon][\gamma] = [\gamma] = [\gamma][\varepsilon]$. We conclude that $[\varepsilon]$ represents the identity element of $\pi_1(X, x_0)$.

It only remains to verify the existence of inverses. Now the map $K\colon [0,1] \times [0,1] \to X$ defined by

$$K(t,\tau) = \begin{cases} \gamma(2\tau t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma(2\tau(1-t)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

is a homotopy between the loops ε and $\gamma \,.\, \gamma^{-1}$, and moreover this homotopy sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\varepsilon \simeq \gamma \,.\, \gamma^{-1} \,\mathrm{rel}\{0,1\}$, and thus $[\gamma][\gamma^{-1}] = [\gamma \,.\, \gamma^{-1}] = [\varepsilon]$. On replacing γ by γ^{-1}, we see also that $[\gamma^{-1}][\gamma] = [\varepsilon]$, and thus $[\gamma^{-1}] = [\gamma]^{-1}$, as required. $\blacksquare$

Let x_0 be a point of some topological space X. The group $\pi_1(X, x_0)$ is referred to as the *fundamental group* of X based at the point x_0.
Let $f: X \to Y$ be a continuous map between topological spaces X and Y, and let x_0 be a point of X. Then f induces a homomorphism $f_\#: \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$, where $f_\#([\gamma]) = [f \circ \gamma]$ for all loops $\gamma: [0, 1] \to X$ based at x_0. If x_0, y_0 and z_0 are points belonging to topological spaces X, Y and Z, and if $f: X \to Y$ and $g: Y \to Z$ are continuous maps satisfying $f(x_0) = y_0$ and $g(y_0) = z_0$, then the induced homomorphisms $f_\#: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_\#: \pi_1(Y, y_0) \to \pi_1(Z, z_0)$ satisfy $g_\# \circ f_\# = (g \circ f)_\#$. It follows easily from this that any homeomorphism of topological spaces induces a corresponding isomorphism of fundamental groups, and thus the fundamental group is a topological invariant.