MA342R—Covering Spaces and Fundamental Groups School of Mathematics, Trinity College Hilary Term 2017 Lecture 12 (February 10, 2017)

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2.6. The Fundamental Theorem of Algebra

Theorem 2.13

(The Fundamental Theorem of Algebra) Let $P\colon \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial with complex coefficients. Then there exists some complex number z_0 such that $P(z_0) = 0$.

Proof

We shall prove that any polynomial that is everywhere non-zero must be a constant polynomial.

Let $P(z) = a_0 + a_1 z + \cdots + a_m z^m$, where $a_1, a_2, \ldots, a_m$ are complex numbers and $a_m \neq 0$. We write $P(z) = P_m(z) + Q(z)$, where $P_m(z) = a_m z^m$ and $Q(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}$. Let

$$R = \frac{|a_0| + |a_1| + \cdots + |a_m|}{|a_m|}.$$

If $|z| > R$ then $|z| \geq 1$, and therefore

$$\begin{aligned}
\left|\frac{Q(z)}{P_m(z)}\right| &= \frac{1}{|a_m z|}\left|\frac{a_0}{z^{m-1}} + \frac{a_1}{z^{m-2}} + \cdots + a_{m-1}\right| \\
&\leq \frac{1}{|a_m|\,|z|}\left(\left|\frac{a_0}{z^{m-1}}\right| + \left|\frac{a_1}{z^{m-2}}\right| + \cdots + |a_{m-1}|\right) \\
&\leq \frac{1}{|a_m|\,|z|}(|a_0| + |a_1| + \cdots + |a_{m-1}|) \leq \frac{R}{|z|} < 1.
\end{aligned}$$

It follows that $|P(z) - P_m(z)| < |P_m(z)|$ for all complex numbers z satisfying $|z| > R$.

For each non-zero real number r, let $\gamma_r\colon [0,1] \to \mathbb{C}$ and $\varphi_r\colon [0,1] \to \mathbb{C}$ be the closed paths defined such that $\gamma_r(t) = P(r\exp(2\pi i t))$ and $\varphi_r(t) = P_m(r\exp(2\pi i t)) = a_m r^m \exp(2\pi i m t)$ for all $t \in [0,1]$. If $r > R$ then $|\gamma_r(t) - \varphi_r(t)| < |\varphi_r(t)|$ for all $t \in [0,1]$. It then follows from the Dog-Walking Lemma (Lemma 2.6) that $n(\gamma_r, 0) = n(\varphi_r, 0) = m$ whenever $r > R$.

Now if the polynomial P is everywhere non-zero then it follows on applying Proposition 2.9 that the function sending each non-negative real number r to the winding number $n(\gamma_r, 0)$ of the closed path γ_r about zero is a continuous function on the set of non-negative real numbers. But any continuous integer-valued function on an interval is necessarily constant (see Corollary 1.58). It follows that $n(\gamma_r, 0) = n(\gamma_0, 0)$ for all positive real-numbers r. But γ_0 is the constant path defined by $\gamma_0(t) = P(0)$ for all $t \in [0, 1]$, and therefore $n(\gamma_0, 0) = 0$. It follows that is the polynomial P is everywhere non-zero then $n(\gamma_r, 0) = 0$ for all non-negative real numbers r. But we have shown that $n(\gamma_r, 0) = m$ for sufficiently large values of r, where m is the degree of the polynomial P. It follows that if the polynomial P is everywhere non-zero, then it must be a constant polynomial. The result follows. $\blacksquare$

2.7. The Kronecker Principle

The proof of the Fundamental Theorem of Algebra given above depends on the continuity of the polynomial P, together with the fact that the winding number $n(P \circ \sigma_r, 0)$ is non-zero for sufficiently large r, where σ_r denotes the circle of radius r about zero, described once in the anticlockwise direction. We can therefore generalize the proof of the Fundamental Theorem of Algebra in order to obtain the following result (sometimes referred to as the *Kronecker Principle*).

Proposition 2.14

Let $f: D \to \mathbb{C}$ be a continuous map defined on the closed unit disk D in $\mathbb{C}$, and let $w \in \mathbb{C} \setminus f(D)$. Then $n(f \circ \sigma, w) = 0$, where $\sigma: [0,1] \to \mathbb{C}$ is the parameterization of unit circle defined by $\sigma(t) = \exp(2\pi i t)$, and $n(f \circ \sigma, w)$ is the winding number of $f \circ \sigma$ about w.

Proof

Define $\gamma_s(t) = f(s \exp(2\pi i t))$ for all $t \in [0,1]$ and $s \in [0,1]$. Then none of the closed curves γ_s passes through w, and γ_0 is the constant curve with value $f(0)$. It follows from Proposition 2.9 that

$$n(f \circ \sigma, w) = n(\gamma_1, w) = n(\gamma_0, w) = 0,$$

as required. ∎

2.8. The Brouwer Fixed Point Theorem

We now use Proposition 2.14 to show that there is no continuous 'retraction' mapping the closed unit disk onto its boundary circle.

Corollary 2.15

There does not exist a continuous map $r\colon D \to \partial D$ with the property that $r(z) = z$ for all $z \in \partial D$, where ∂D denotes the boundary circle of the closed unit disk D.

Proof

Let $\sigma\colon [0,1] \to \mathbb{C}$ be defined by $\sigma(t) = \exp(2\pi i t)$. If a continuous map $r\colon D \to \partial D$ with the required property were to exist, then $r(z) \neq 0$ for all $z \in D$ (since $r(D) \subset \partial D$), and therefore $n(\sigma, 0) = n(r \circ \sigma, 0) = 0$, by Proposition 2.14. But $\sigma = \exp \circ \tilde{\sigma}$, where $\tilde{\sigma}(t) = 2\pi i t$ for all $t \in [0,1]$, and thus

$$n(\sigma, 0) = \frac{\tilde{\sigma}(1) - \tilde{\sigma}(0)}{2\pi i} = 1.$$

This shows that there cannot exist any continuous map r with the required property. ∎

Theorem 2.16

(The Brouwer Fixed Point Theorem in Two Dimensions) Let $f: D \to D$ be a continuous map which maps the closed unit disk D into itself. Then there exists some $z_0 \in D$ such that $f(z_0) = z_0$.

Proof

Suppose that there did not exist any fixed point z_0 of $f: D \to D$. Then one could define a continuous map $r: D \to \partial D$ as follows: for each $z \in D$, let $r(z)$ be the point on the boundary ∂D of D obtained by continuing the line segment joining $f(z)$ to z beyond z until it intersects ∂D at the point $r(z)$. Then $r: D \to \partial D$ would be a continuous map, and moreover $r(z) = z$ for all $z \in \partial D$. But Corollary 2.15 shows that there does not exist any continuous map $r: D \to \partial D$ with this property. We conclude that $f: D \to D$ must have at least one fixed point. ∎

Remark

The Brouwer Fixed Point Theorem is also valid in higher dimensions. This theorem states that any continuous map from the closed n-dimensional ball into itself must have at least one fixed point. The proof of the theorem for $n > 2$ is analogous to the proof for $n = 2$, once one has shown that there is no continuous map from the closed n-dimensional ball to its boundary which is the identity map on the boundary. However winding numbers cannot be used to prove this result, and thus more powerful topological techniques need to be employed.

2.9. The Borsuk-Ulam Theorem

Lemma 2.17

Let $f\colon S^1 \to \mathbb{C} \setminus \{0\}$ be a continuous function defined on S^1, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Suppose that $f(-z) = -f(z)$ for all $z \in S^1$. Then the winding number $n(f \circ \sigma, 0)$ of $f \circ \sigma$ about 0 is odd, where $\sigma\colon [0,1] \to S^1$ is given by $\sigma(t) = \exp(2\pi i t)$.

Proof

It follows from the Path Lifting Theorem (Theorem 2.5) that there exists a continuous path $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $\exp(\tilde{\gamma}(t)) = f(\sigma(t))$ for all $t \in [0,1]$. Now $f(\sigma(t+\frac{1}{2})) = -f(\sigma(t))$ for all $t \in [0, \frac{1}{2}]$, since $\sigma(t+\frac{1}{2}) = -\sigma(t)$ and $f(-z) = -f(z)$ for all $z \in \mathbb{C}$. Thus $\exp(\tilde{\gamma}(t+\frac{1}{2})) = \exp(\tilde{\gamma}(t) + \pi i)$ for all $t \in [0, \frac{1}{2}]$. It follows that $\tilde{\gamma}(t+\frac{1}{2}) = \tilde{\gamma}(t) + (2m+1)\pi i$ for some integer m. (The value of m for which this identity is valid does not depend on t, since every continuous function from $[0, \frac{1}{2}]$ to the set of integers is necessarily constant.) Hence

$$n(f\circ\sigma, 0) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i} = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(\frac{1}{2})}{2\pi i} + \frac{\tilde{\gamma}(\frac{1}{2}) - \tilde{\gamma}(0)}{2\pi i} = 2m+1.$$

Thus $n(f\circ\sigma, 0)$ is an odd integer, as required. ∎

We shall identify the space $\mathbb{R}^2$ with $\mathbb{C}$, identifying $(x, y) \in \mathbb{R}^2$ with the complex number $x + iy \in \mathbb{C}$ for all $x, y \in \mathbb{R}$. This is permissible, since we are interested in purely topological results concerning continuous functions defined on appropriate subsets of these spaces. Under this identification the closed unit disk D is given by

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

As usual, we define

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Lemma 2.18

Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. Then there exists some point $\mathbf{n}_0$ of S^2 with the property that $f(\mathbf{n}_0) = 0$.

Proof

Let $\varphi\colon D \to S^2$ be the map defined by

$$\varphi(x, y) = (x, y, +\sqrt{1 - x^2 - y^2}).$$

(Thus the map φ maps the closed disk D homeomorphically onto the upper hemisphere in $\mathbb{R}^3$.) Let $\sigma\colon [0, 1] \to S^2$ be the parameterization of the equator in S^2 defined by

$$\sigma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$$

for all $t \in [0, 1]$. Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. If $f(\sigma(t_0)) = 0$ for some $t_0 \in [0, 1]$ then the function f has a zero at $\sigma(t_0)$. It remains to consider the case in which $f(\sigma(t)) \neq 0$ for all $t \in [0, 1]$. In that case the winding number $n(f \circ \sigma, 0)$ is an odd integer, by Lemma 2.17, and is thus non-zero.

It follows from Proposition 2.14, applied to $f \circ \varphi \colon D \to \mathbb{R}^2$, that $0 \in f(\varphi(D))$, (since otherwise the winding number $n(f \circ \sigma, 0)$ would be zero). Thus $f(\mathbf{n}_0) = 0$ for some $\mathbf{n}_0 = \varphi(D)$, as required. $\blacksquare$

Theorem 2.19

(Borsuk-Ulam) Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists some point $\mathbf{n}$ of S^2 with the property that $f(-\mathbf{n}) = f(\mathbf{n})$.

Proof

This result follows immediately on applying Lemma 2.18 to the continuous function $g\colon S^2 \to \mathbb{R}^2$ defined by $g(\mathbf{n}) = f(\mathbf{n}) - f(-\mathbf{n})$. ∎

Remark

It is possible to generalize the Borsuk-Ulam Theorem to n dimensions. Let S^n be the unit n-sphere centered on the origin in $\mathbb{R}^n$. The Borsuk-Ulam Theorem in n-dimensions states that if $f: S^n \to \mathbb{R}^n$ is a continuous map then there exists some point $\mathbf{x}$ of S^n with the property that $f(\mathbf{x}) = f(-\mathbf{x})$.