MA342R—Covering Spaces and Fundamental Groups
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David R. Wilkins

2.4. Winding Numbers

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane, and let w be a complex number that does not lie on γ. Then there exists a path $\tilde{\gamma}_w\colon [a,b] \to \mathbb{C}$ in the complex plane such that $\exp(\tilde{\gamma}_w(t)) = \gamma(t) - w$ for all $t \in [a,b]$ (Theorem 2.5). Now the definition of closed paths ensures that $\gamma(b) = \gamma(a)$. Also two complex numbers z_1 and z_2 satisfy $\exp z_1 = \exp z_2$ if and only if $(2\pi i)^{-1}(z_2 - z_1)$ is an integer (Lemma 2.3). It follows that there exists some integer $n(\gamma, w)$ such that $\tilde{\gamma}_w(b) = \tilde{\gamma}_w(a) + 2\pi i n(\gamma, w)$.

Now let $\varphi\colon [a,b] \to \mathbb{C}$ be any path with the property that $\exp(\varphi(t)) = \gamma(t) - w$ for all $t \in [a,b]$. Then the function sending $t \in [a,b]$ to $(2\pi i)^{-1}(\varphi(t) - \tilde{\gamma}_w(t))$ is a continuous integer-valued function on the interval $[a,b]$, and is therefore constant on this interval (Corollary 1.58). It follows that

$$\varphi(b) - \varphi(a) = \tilde{\gamma}_w(b) - \tilde{\gamma}_w(a) = 2\pi i n(\gamma, w).$$

It follows from this that the value of the integer $n(\gamma, w)$ depends only on the choice of γ and w, and is independent of the choice of path $\tilde{\gamma}_w$ satisfying $\exp(\tilde{\gamma}_w(t)) = \gamma(t) - w$ for all $t \in [a,b]$.

Definition

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane, and let w be a complex number that does not lie on γ. The *winding number* of γ about w is defined to be the unique integer $n(\gamma, w)$ with the property that $\varphi(b) - \varphi(a) = 2\pi i n(\gamma, w)$ for all paths $\varphi\colon [a,b] \to \mathbb{C}$ in the complex plane that satisfy $\exp(\varphi(t)) = \gamma(t) - w$ for all $t \in [a,b]$.

Example

Let n be an integer, and let $\gamma_n\colon [0,1] \to \mathbb{C}$ be the closed path in the complex plane defined by $\gamma_n(t) = \exp(2\pi i n t)$. Then $\gamma_n(t) = \exp(\varphi_n(t))$ for all $t \in [0,1]$ where $\varphi_n\colon [0,1] \to \mathbb{C}$ is the path in the complex plane defined such that $\varphi_n(t) = 2\pi i n t$ for all $t \in [0,1]$. It follows that $n(\gamma_n, 0) = (2\pi i)^{-1}(\varphi_n(1) - \varphi_n(0)) = n$.

Given a closed path γ, and given a complex number w that does not lie on γ, the winding number $n(\gamma, w)$ measures the number of times that the path γ winds around the point w of the complex plane in the anticlockwise direction.

Lemma 2.6 (Dog-Walking Lemma)

Let $\gamma_1\colon [a, b] \to \mathbb{C}$ and $\gamma_2\colon [a, b] \to \mathbb{C}$ be closed paths in the complex plane, and let w be a complex number that does not lie on γ_1. Suppose that $|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t) - w|$ for all $t \in [a, b]$. Then $n(\gamma_2, w) = n(\gamma_1, w)$.

Proof

Note that the inequality satisfied by the functions γ_1 and γ_2 ensures that w does not lie on the path γ_2. Let $\tilde{\gamma}_1\colon [0,1] \to \mathbb{C}$ be a path in the complex plane such that $\exp(\tilde{\gamma}_1(t)) = \gamma_1(t) - w$ for all $t \in [a, b]$, and let

$$\rho(t) = \frac{\gamma_2(t) - w}{\gamma_1(t) - w}$$

for all $t \in [a, b]$ Then

$$|\rho(t) - 1| = \left|\frac{\gamma_2(t) - \gamma_1(t)}{\gamma_1(t) - w}\right| < 1$$

for all $t \in [a, b]$.

Now it follows from Lemma 2.4 that there exists a continuous function $F\colon \{z \in \mathbb{C} : |z-1| < 1\} \to \mathbb{C}$ with the property that $\exp(F(z)) = z$ for all complex numbers z satisfying $|z-1| < 1$. Let $\tilde{\gamma}_2\colon [0,1] \to \mathbb{C}$ be the path in the complex plane defined such that $\tilde{\gamma}_2(t) = F(\rho(t)) + \tilde{\gamma}_1(t)$ for all $t \in [a,b]$. Then

$$\begin{aligned} \exp(\tilde{\gamma}_2(t)) &= \exp(F(\rho(t)))\exp(\tilde{\gamma}_1(t)) = \rho(t)(\gamma_1(t) - w) \\ &= \gamma_2(t) - w. \end{aligned}$$

Now $\rho(b) = \rho(a)$. It follows that

$$\begin{aligned} 2\pi i n(\gamma_2, w) &= \tilde{\gamma}_2(b) - \tilde{\gamma}_2(a) \\ &= F(\rho(b)) + \tilde{\gamma}_1(b) - F(\rho(a)) - \tilde{\gamma}_1(a) \\ &= \tilde{\gamma}_1(b) - \tilde{\gamma}_1(a) \\ &= 2\pi i n(\gamma_1, w), \end{aligned}$$

as required. ■

Remark

Imagine that you are exercising a dog in a park. You walk along a path close to the perimeter of the park that remains at all times at at least 200 metres from an oak tree in the centre of the park. Your dog runs around in your vicinity, but remains at all times within 100 metres of you. In order to leave the park you and your dog return to the point at which you entered the park. The Dog-Walking Lemma then ensures that the number of times that your dog went around the oak tree in the centre of the park is equal to the number of times that you yourself went around that tree.

Example

Let $\gamma\colon [0,1] \to \mathbb{C}$ be the closed curve in the complex plane defined such that

$$\begin{aligned}\gamma(t) &= 3\cos 6\pi t + 4i\sin 6\pi t + (\sin 16\pi t)(\sin 8\pi t) \\ &\quad - 2ie^{\cos 8\pi t - 1}\cos 8\pi t\end{aligned}$$

for all $t \in [0,1]$, where $i^2 = -1$. Let

$$\gamma_1(t) = 3\cos 6\pi t + 4i\sin 6\pi t$$

for all $t \in [0,1]$. Then $|\gamma_1(t)| \geq 3$ for all $t \in [0,1]$. Also $|\sin 16\pi t| \leq 1$ and $0 \leq e^{\cos 8\pi t - 1} \leq 1$ for all $t \in [0,1]$, and therefore

$$\begin{aligned}&\left|(\sin 16\pi t)(\sin 8\pi t) - 2ie^{\cos 8\pi t - 1}\cos 8\pi t\right|^2 \\ &\qquad\qquad \leq \quad \sin^2 8\pi t + 4\cos^2 8\pi t \leq 4\end{aligned}$$

for all $t \in [0, 1]$. It follows that

$$\begin{aligned}|\gamma(t) - \gamma_1(t)| &= \left|(\sin 16\pi t)(\sin 8\pi t) - 2ie^{\cos 8\pi t - 1}\cos 8\pi t\right| \\ &\leq 2 < |\gamma_1(t)|\end{aligned}$$

for all $t \in [0, 1]$. The Dog-Walking Lemma (Lemma 2.6) then ensures that $n(\gamma, 0) = n(\gamma_1, 0)$. Another application of the Dog-Walking Lemma then ensures that $n(\gamma_1, 0) = n(\gamma_2, 0)$, where

$$\gamma_2(t) = 3(\cos 6\pi t + i \sin 6\pi t)$$

for all $t \in [0, 1]$. Moreover $\gamma_2 = \exp \circ \tilde{\gamma}_2$ where $\tilde{\gamma}_2\colon [0, 1] \to \mathbb{C}$ is the path in $\mathbb{C}$ defined so that $\tilde{\gamma}_2(t) = \log 3 + 6\pi t$ for all $t \in [0, 1]$.

The definition of winding number ensures that

$$n(\gamma_2, 0) = (2\pi i)^{-1}(\tilde{\gamma}_2(1) - \tilde{\gamma}_2(0)) = 3.$$

Therefore $n(\gamma, 0) = 3$.

Lemma 2.7

Let $\gamma\colon [a, b] \to \mathbb{C}$ be a closed path in the complex plane and let W be the set $\mathbb{C} \setminus [\gamma]$ of all points of the complex plane that do not lie on the curve γ. Then the function that sends $w \in W$ to the winding number $n(\gamma, w)$ of γ about w is a continuous function on W.

Proof

Let $w \in W$. It then follows from Lemma 2.1 that there exists some positive real number ε_0 such that $|\gamma(t) - w| \geq \varepsilon_0 > 0$ for all $t \in [a, b]$. Let w_1 be a complex number satisfying $|w_1 - w| < \varepsilon_0$, and let $\gamma_1 \colon [a, b] \to \mathbb{C}$ be the closed path in the complex plane defined such that $\gamma_1(t) = \gamma(t) + w - w_1$ for all $t \in [a, b]$. Then $\gamma(t) - w_1 = \gamma_1(t) - w$ for all $t \in [a, b]$, and therefore $n(\gamma, w_1) = n(\gamma_1, w)$. Also $|\gamma_1(t) - \gamma(t)| < |\gamma(t) - w|$ for all $t \in [a, b]$. It follows from the Dog-Walking Lemma (Lemma 2.6) that $n(\gamma, w_1) = n(\gamma_1, w) = n(\gamma, w)$. This shows that the function sending $w \in W$ to $n(\gamma, w)$ is continuous on W, as required. ∎

Lemma 2.8

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane, and let R be a positive real number with the property that $|\gamma(t)| < R$ for all $t \in [a,b]$. Then $n(\gamma, w) = 0$ for all complex numbers w satisfying $|w| \geq R$.

Proof

Let $\gamma_0\colon [a,b] \to \mathbb{C}$ be the constant path defined by $\gamma_0(t) = 0$ for all $[a,b]$. If $|w| > R$ then $|\gamma(t) - \gamma_0(t)| = |\gamma(t)| < |w| = |\gamma_0(t) - w|$. It follows from the Dog-Walking Lemma (Lemma 2.6) that $n(\gamma, w) = n(\gamma_0, w) = 0$, as required. ∎

Proposition 2.9

Let $[a, b]$ and $[c, d]$ be closed bounded intervals, and, for each $s \in [c, d]$, let $\gamma_s \colon [a, b] \to \mathbb{C}$ be a closed path in the complex plane. Let w be a complex number that does not lie on any of the paths γ_s. Suppose that the function $H \colon [a, b] \times [c, d] \to \mathbb{C}$ is continuous, where $H(t, s) = \gamma_s(t)$ for all $t \in [a, b]$ and $s \in [c, d]$. Then $n(\gamma_c, w) = n(\gamma_d, w)$.

Proof

The rectangle $[a, b] \times [c, d]$ is a closed bounded subset of $\mathbb{R}^2$, and is therefore compact. It follows that the continuous function on the closed rectangle $[a, b] \times [c, d]$ that sends a point (t, s) of the rectangle to $|H(t, s) - w|^{-1}$ is a bounded function on $[a, b] \times [c, d]$ (see, for example, Lemma 1.40). Therefore there exists some positive number ε_0 such that $|H(t, s) - w| \geq \varepsilon_0 > 0$ for all $t \in [a, b]$ and $s \in [c, d]$.

Now any continuous complex-valued function on a closed bounded subset of a Euclidean space is uniformly continuous. (This follows, for example, on combining the results of Theorem 1.50 and Theorem 1.48.) Therefore there exists some positive real number δ such that $|H(t,s) - H(t,u)| < \varepsilon_0$ for all $t \in [a,b]$ and for all $s, u \in [c,d]$ satisfying $|s - u| < \delta$. Let $s_0, s_1, \ldots, s_m$ be real numbers chosen such that $c = s_0 < s_1 < \ldots < s_m = d$ and $|s_j - s_{j-1}| < \delta$ for $j = 1, 2, \ldots, m$. Then

$$\begin{aligned} |\gamma_{s_j}(t) - \gamma_{s_{j-1}}(t)| &= |H(t,s_j) - H(t,s_{j-1})| \\ &< \varepsilon_0 \leq |H(t,s_{j-1}) - w| = |\gamma_{s_{j-1}}(t) - w| \end{aligned}$$

for all $t \in [a,b]$, and for each integer j between 1 and m. It therefore follows from the Dog-Walking Lemma (Lemma 2.6) that $n(\gamma_{s_{j-1}}, w) = n(\gamma_{s_j}, w)$ for each integer j between 1 and m. But then $n(\gamma_c, w) = n(\gamma_d, w)$, as required. ∎

Definition

Let D be a subset of the complex plane, and let $\gamma\colon [a, b] \to D$ be a closed path in D. The closed path γ is said to be *contractible* in D if and only if there exists a continuous function $H\colon [a, b] \times [0, 1] \to D$ and an element z_0 of D such that $H(t, 0) = \gamma(t)$ and $H(t, 1) = z_0$ for all $t \in [a, b]$, and $H(a, s) = H(b, s)$ for all $s \in [0, 1]$.

Corollary 2.10

Let D be a subset of the complex plane, and let $\gamma\colon [a, b] \to D$ be a closed path in D. Suppose that γ is contractible in D. Then $n(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus D$, where $n(\gamma, w)$ denotes the winding number of γ about w.

Proof

The closed curve γ is contractible, and therefore there exists an element z_0 of D and a continuous function $H\colon [a,b] \times [0,1] \to D$ such that $H(t,0) = \gamma(t)$ and $H(t,1) = z_0$ for all $t \in [a,b]$, and $H(a,s) = H(b,s)$ for all $s \in [0,1]$. For each $s \in [0,1]$ let $\gamma_s\colon [a,b] \to D$ be the closed path in D defined such that $\gamma_s(t) = H(t,s)$ for all $t \in [a,b]$. Then γ_1 is a constant path, and therefore $n(\gamma_1, w) = 0$ for all points w that do not lie on γ_1. Let w be an element of $w \in \mathbb{C} \setminus D$. Then w does not lie on any of the paths γ_s. It follows from Proposition 2.9 that

$$n(\gamma, w) = n(\gamma_0, w) = n(\gamma_1, w) = 0,$$

as required. ∎

2.5. Simply-Connected Subsets of the Complex Plane

Definition

A subset D of the complex plane is said to be *path-connected* if, given any elements z_1 and z_2, there exists a path in D from z_1 and z_2.

Definition

A path-connected subset D of the complex plane is said to be *simply-connected* if every closed loop in D is contractible.

Definition

An subset D of the complex plane is said to be a *star-shaped* if there exists some complex number z_0 in D with the property that

$$\{(1-t)z + tz_0 : t \in [0,1]\} \subset D$$

for all $z \in D$. (Thus an open set in the complex plane is a star-shaped if and only if the line segment joining any point of D to z_0 is contained in D.)

Lemma 2.11

Star-shaped subsets of the complex plane are simply-connected.

Proof

Let D be a star-shaped subset of the complex plane. Then there exists some element z_0 of D such that the line segment joining z_0 to z is contained in D for all $z \in D$. The star-shaped set D is obviously path-connected. Let $\gamma\colon [a, b] \to D$ be a closed path in D, and let $H(t, s) = (1 - s)\gamma(t) + sz_0$ for all $t \in [a, b]$ and $s \in [0, 1]$. Then $H(t, s) \in D$ for all $t \in [a, b]$ and $s \in [0, 1]$, $H(t, 0) = \gamma(t)$ and $H(t, 1) = z_0$ for all $t \in [a, b]$. Also $\gamma(a) = \gamma(b)$, and therefore $H(a, s) = H(b, s)$ for all $s \in [0, 1]$. It follows that the closed path γ is contractible. Thus D is simply-connected. ∎

The following result is an immediate consequence of Corollary 2.10

Proposition 2.12

Let D be a simply-connected subset of the complex plane, and let γ be a closed path in D. Then $n(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus D$.