MA342R—Covering Spaces and Fundamental Groups
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2. Winding Numbers of Closed Curves in the Plane

2.1. Paths in the Complex Plane

Let D be a subset of the complex plane $\mathbb{C}$. We define a *path* in D to be a continuous complex-valued function $\gamma\colon [a, b] \to D$ defined over some closed interval $[a, b]$. We shall denote the range $\gamma([a, b])$ of the function γ defining the path by $[\gamma]$. Now it follows from the Heine-Borel Theorem (Theorem 1.37) that the closed bounded interval $[a, b]$ is compact. Moreover continuous functions map compact sets to compact sets (see Lemma 1.39). It follows that $[\gamma]$ is a closed bounded subset of the complex plane.

Lemma 2.1

Let $\gamma \colon [a, b] \to \mathbb{C}$ be a path in the complex plane, and let w be a complex number that does not lie on the path γ. Then there exists some positive real number ε_0 such that $|\gamma(t) - w| \geq \varepsilon_0 > 0$ for all $t \in [a, b]$.

Proof

The closed unit interval $[a, b]$ is a closed bounded subset of $\mathbb{R}$. Now any continuous real-valued function on a compact set is bounded above and below on that set (Lemma 1.40). Therefore there exists some positive real number M such that $|\gamma(t) - w|^{-1} \leq M$ for all $t \in [a, b]$. Let $\varepsilon_0 = M^{-1}$. Then the positive real number ε_0 has the required property. ∎

Definition

A path $\gamma\colon [a,b] \to \mathbb{C}$ in the complex plane is said to be *closed* if $\gamma(a) = \gamma(b)$.

Remark

The use of the technical term *closed* as in the above definition has no relation to the notions of open and closed sets.) Thus a *closed path* is a path that returns to its starting point.

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path in the complex plane. We say that a complex number w *lies on* the path γ if $w \in [\gamma]$, where $[\gamma] = \gamma([a,b])$.

2.2. The Exponential Map

The exponential map $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined on the complex plane so that

$$\exp(x + iy) = e^x \cos y + i\, e^x \sin y$$

for all real numbers x and y, where $i^2 = -1$. Then

$$\exp(x + iy) = u(x, y) + i\, v(x, y)$$

where

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y$$

for all real numbers x and y. The functions $u\colon \mathbb{R}^2 \to \mathbb{R}$ and $v\colon \mathbb{R}^2 \to \mathbb{R}$ satisfy the partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

These partial differential equations are the *Cauchy-Riemann equations* that are satisfied by the real and imaginary parts of a function of a complex variable if and only if that function is holomorphic.

Lemma 2.2

The exponential map $\exp\colon \mathbb{C} \to \mathbb{C}$ *satisfies the identities* $\exp(z + w) = \exp(z)\exp(w)$ *and* $\exp(-z) = \exp(z)^{-1}$ *for all complex numbers* z *and* w*.*

Proof

Let $z = x + iy$ and $w = u + iv$, where x, y, u and v are real numbers and $i^2 = -1$. Then

$$\begin{aligned}
\exp(z+w) &= e^{x+u}\left(\cos(y+v) + i\sin(y+v)\right) \\
&= e^x e^u \big(\cos y\, \cos v - \sin y\, \sin v \\
&\qquad + i \sin y\, \cos v + i \cos y\, \sin v\big) \\
&= e^x e^u (\cos y + i \sin y)(\cos v + i \sin v) \\
&= \exp(z)\, \exp(w).
\end{aligned}$$

Applying this result with $w = -z$, we see that $\exp(z)\exp(-z) = \exp(0) = 1$, and therefore $\exp(-z) = \exp(z)^{-1}$, as required. ∎

Lemma 2.3

Let z and w be complex numbers. Then $\exp(z) = \exp(w)$ if and only if $w = z + 2\pi i n$ for some integer n.

Proof

Suppose that $w = z + 2\pi i n$ for some integer n. Then

$$\begin{aligned} \exp(w) &= \exp(z)\exp(2\pi i n) = \exp(z)(\cos 2\pi n + i \sin 2\pi n) \\ &= \exp(z). \end{aligned}$$

Conversely suppose that $\exp(w) = \exp(z)$. Let $w - z = u + iv$, where u and v are real numbers. Then

$$e^u(\cos v + i \sin v) = \exp(w - z) = \exp(w)\exp(z)^{-1} = 1.$$

Taking the modulus of both sides, we see that $e^u = 1$, and thus $u = 0$. Also $\cos v = 1$ and $\sin v = 0$, and therefore $v = 2\pi n$ for some integer n. The result follows. ∎

Remark

The infinite series $\displaystyle\sum_{n=0}^{+\infty} \frac{z^n}{n!}$ converges absolutely for all complex numbers z. Standard theorems concerning power series then ensure that the infinite series converges uniformly in z over any closed disk of positive radius about zero in the complex plane. A standard theorem of analysis regarding Cauchy products of absolutely convergent infinite series then ensures that

$$\left(\sum_{n=0}^{+\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{+\infty} \frac{w^n}{n!}\right) = \left(\sum_{n=0}^{+\infty} \frac{(z+w)^n}{n!}\right)$$

for all complex numbers z.

It follows that if $z = x + iy$, where x and y are real numbers and $i^2 = -1$, then

$$\begin{aligned}\sum_{n=0}^{+\infty} \frac{z^n}{n!} &= \left(\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right)\left(\sum_{k=0}^{+\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i \sum_{k=0}^{+\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}\right)\\ &= e^x(\cos y + i \sin y)\end{aligned}$$

for all real numbers x and y. Thus

$$\exp z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$$

for all complex numbers z.

Lemma 2.4

Let w be a non-zero complex number, and let

$$D_{w,|w|} = \{z \in \mathbb{C} : |z - w| < |w|\}.$$

Then there exists a continuous function $F_w \colon D_{w,|w|} \to \mathbb{C}$ with the property that $\exp(F_w(z)) = z$ for all $z \in D_{w,|w|}$.

Proof

Let $U = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$, and let $\log\colon U \to \mathbb{C}$ be the "principal branch" of the logarithm function, defined so that $\log(re^{i\theta}) = \log r + i\theta$ for all real numbers r and θ satisfying $r > 0$ and $\pi < \theta < \pi$. Then the function $\log\colon U \to \mathbb{C}$ is continuous, and $\exp(\log z) = z$ for all $z \in U$. Let ζ be a complex number satisfying $\exp \zeta = w$. Then $z/w \in U$ for all $z \in D_{w,|w|}$. Let $F_w\colon D_{w,|w|} \to \mathbb{C}$ be defined so that $F_w(z) = \zeta + \log(z/w)$ for all $z \in D_{w,|w|}$. Then

$$\exp(F_w(z)) = \exp(\zeta)\exp(\log(z/w)) = w(z/w) = z$$

for all $z \in D(w, |w|)$, as required. $\blacksquare$

2.3. Path-Lifting with respect to the Exponential Map

Theorem 2.5

Let $\gamma\colon [a,b] \to \mathbb{C} \setminus \{0\}$ *be a path in the set* $\mathbb{C} \setminus \{0\}$ *of non-zero complex numbers. Then there exists a path* $\tilde{\gamma}\colon [a,b] \to \mathbb{C}$ *in the complex plane which satisfies* $\exp(\tilde{\gamma}(t)) = \gamma(t)$ *for all* $t \in [a,b]$.

Proof

The complex number $\gamma(t)$ is non-zero for all $t \in [a,b]$, and therefore there exists some positive number ε_0 such that $|\gamma(t)| \geq \varepsilon_0$ for all $t \in [a,b]$. (Lemma 2.1). Now any continuous complex-valued function on a closed bounded interval is uniformly continuous. (This follows, for example, from Theorem 1.48.) Therefore there exists some positive real number δ such that $|\gamma(t) - \gamma(s)| < \varepsilon_0$ for all $s, t \in [a,b]$ satisfying $|t - s| < \delta$.

Let m be a positive integer satisfying $m > |b - a|/\delta$, and let $t_j = a + j(b - a)/m$ for $j = 0, 1, 2, \ldots, m$. Then $|t_j - t_{j-1}| < \delta$ for $j = 1, 2, \ldots, m$. It follows from this that

$$|\gamma(t) - \gamma(t_j)| < \varepsilon_0 \leq |\gamma(t_j)|$$

for all $t \in [t_{j-1}, t_j]$, and thus

$$\gamma([t_{j-1}, t_j]) \subset D_{\gamma(t_j), |\gamma(t_j)|}$$

for $j = 1, 2, \ldots, n$, where

$$D_{w,|w|} = \{z \in \mathbb{C} : |z - w| < |w|\}$$

for all $w \in \mathbb{C}$.

Now there exist continuous functions $F_j \colon D_{\gamma(t_j),|\gamma(t_j)|} \to \mathbb{C}$ with the property that $\exp(F_j(z)) = z$ for all $z \in D_{\gamma(t_j),|\gamma(t_j)|}$ (see Lemma 2.4). Let $\tilde{\gamma}_j(t) = F_j(\gamma(t))$ for all $t \in [t_{j-1}, t_j]$. Then, for each integer j between 1 and m, the function $\tilde{\gamma}_j \colon [t_{j-1}, t_j] \to \mathbb{C}$ is continuous, and is thus a path in the complex plane with the property that $\exp(\tilde{\gamma}_j(t)) = \gamma(t)$ for all $t \in [t_{j-1}, t_j]$. Now

$$\exp(\tilde{\gamma}_j(t_j)) = \gamma(t_j) = \exp(\tilde{\gamma}_{j+1}(t_j))$$

for each integer j between 1 and $m - 1$. The periodicity properties of the exponential function (Lemma 2.3) therefore ensure that there exist integers $k_1, k_2, \ldots, k_{m-1}$ such that $\tilde{\gamma}_{j+1}(t_j) = \tilde{\gamma}_j(t_j) + 2\pi i k_j$ for $j = 1, 2, \ldots, m - 1$. Then

$$\tilde{\gamma}_{j+1}(t_j) - 2\pi i \sum_{r=1}^{j} k_r = \tilde{\gamma}_j(t_j) - 2\pi i \sum_{r=1}^{j-1} k_r$$

for $j = 1, 2, \ldots, m-1$. The Pasting Lemma (Lemma 1.24) then ensures the existence of a continuous function $\tilde{\gamma}\colon [a, b] \to \mathbb{C}$ defined so that $\tilde{\gamma}(t) = \tilde{\gamma}_1(t)$ whenever $t \in [a, t_1]$, and

$$\tilde{\gamma}(t) = \tilde{\gamma}_j(t) - 2\pi i \sum_{r=1}^{j-1} k_r$$

whenever $t \in [t_{j-1}, t_j]$ for some integer j between 2 and m. Moreover $\exp(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [a, b]$. We have thus proved the existence of a path $\tilde{\gamma}$ in the complex plane with the required properties. ∎