MA342R—Covering Spaces and Fundamental Groups
School of Mathematics, Trinity College
Hilary Term 2017
Lecture 9 (February 3, 2017)

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1.24. Path-Connected Topological Spaces

A concept closely related to that of connectedness is *path-connectedness*. Let x_0 and x_1 be points in a topological space X. A *path* in X from x_0 to x_1 is defined to be a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Definition

A topological space X is said to be *path-connected* if and only if, given any two points x_0 and x_1 of X, there exists a continuous map $\gamma\colon [0,1] \to X$ from the closed unit interval $[0,1]$ to the space X for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Proposition 1.64

Every path-connected topological space is connected.

Proof

Let X be a path-connected topological space, and let V and W be non-empty subsets of X that are open in X and satisfy $V \cap W = \emptyset$. We show that $V \cup W$ is a proper subset of X. Now X is path-connected. Therefore there exists a continuous map $\gamma\colon [0,1] \to X$ for which $\gamma(0) \in V$ and $\gamma(1) \in W$. Then the preimages $\gamma^{-1}(V)$ and $\gamma^{-1}(W)$ of V and W are open in $[0,1]$, because the map γ is continuous. Moreover $\gamma^{-1}(V)$ and $\gamma^{-1}(W)$ are non-empty and $\gamma^{-1}(V) \cap \gamma^{-1}(W) = \emptyset$. Now the interval $[0,1]$ is connected (Theorem 1.57). Therefore $\gamma^{-1}(V) \cup \gamma^{-1}(W)$ must be a proper subset of $[0,1]$ (see Lemma 1.53).

Let s be a real number satisfying $0 \leq s \leq 1$ that does not belong to either $\gamma^{-1}(V)$ or $\gamma^{-1}(W)$. Then $\gamma(s) \in X \setminus (V \cup W)$. Thus there cannot exist non-empty open subsets V and W of X for which both $V \cap W = \emptyset$ and $V \cup W = X$. It follows that X is connected (see Lemma 1.51). ∎

The topological spaces $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^n$ are all path-connected. Indeed, given any two points of one of these spaces, the straight line segment joining these two points is a continuous path from one point to the other. Also the n-sphere S^n is path-connected for all $n > 0$. We conclude that these topological spaces are connected.

Definition

A subset X of a real vector space is said to be *convex* if, given points $\mathbf{u}$ and $\mathbf{v}$ of X, the point $(1-t)\mathbf{u} + t\mathbf{v}$ belongs to X for all real numbers t satisfying $0 \leq t \leq 1$.

Corollary 1.65

All convex subsets of real vector spaces are connected.

Remark

Proposition 1.64 generalizes the Intermediate Value Theorem of real analysis. Indeed let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function on an interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. The range $f([a,b])$ is then a path-connected subset of $\mathbb{R}$. It follows from Proposition 1.64 that this set is connected. Let c be a real number that lies strictly between $f(a)$ and $f(b)$ and let

$$V = \{y \in f([a,b]) : y < c\} \quad \text{and} \quad W = \{y \in f([a,b]) : y > c\}.$$

Then V and W are non-empty open subsets of $f([a,b])$, and $V \cap W = \emptyset$. It follows from the connectness of $f([a,b])$ that $V \cup W$ must be a proper subset of $f([a,b])$ (see Lemma 1.53), and therefore $c \in f([a,b])$. Thus the range of the function f contains all real numbers between $f(a)$ and $f(b)$.

Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined so that

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and let

$$X = \{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$

We show that X is a connected set. Let

$$X_+ = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = f(x)\}$$

and

$$X_- = \{(x, y) \in \mathbb{R}^2 : x < 0 \text{ and } y = f(x)\}.$$

Now the restriction of the function f to the set of (strictly) positive real numbers is continuous on the set of positive real numbers. It follows from this that the set X_+ is path-connected. It then follows that the set X_+ is connected (see Proposition 1.64). The connectedness of X_+ can also be verified by noting that it is the image of the connected space $\{x \in \mathbb{R} : x > 0\}$ under a continuous map and is therefore itself connected (see Lemma 1.61). Similarly the set X_- is path-connected, and is therefore connected.

Let $\mathbf{p}_n = (1/\pi n, 0)$ for all natural numbers n. Then $\mathbf{p}_n \in X_+$ for all natural numbers n, and $\mathbf{p}_n \to (0, 0)$ as $n \to +\infty$. It follows that $(0, 0)$ belongs to the closure $\overline{X}_+$ of X_+ in X. Connected components of a topological space are closed (see Proposition 1.63). Thus the connected component of X that includes the connected subset X_+ also contains the point $(0, 0)$. Similarly the connected component of X that includes $X + -$ also contains the point $(0, 0)$. Therefore the unique connected component of X that contains the point $(0, 0)$ is the whole of X and thus X is a connected topological space.

However X is not a path-connected topological space. If $\gamma\colon [0,1] \to X$ is a continuous map from the closed unit interval $[0,1]$ into X, and if $\gamma(0) = (0,0)$, then $\gamma(t) = (0,0)$ for all $t \in [0,1]$. Indeed let

$$s = \sup\{t \in [0,1] : \gamma(t) = (0,0)\}.$$

It follows from the continuity of γ that $\gamma(s) = (0,0)$. There then exists some positive real number δ such that $|\gamma(t) - (0,0)| < \frac{1}{2}$ for all $t \in [0,1]$ satisfying $|t - s| < \delta$. But $\gamma([0,1] \cap [s, s+\delta))$ must also be a connected subset of X. It follows that $\gamma(t) = (0,0)$ for all $t \in [0,1]$ satisfying $s \leq t < s + \delta$, and therefore $s = 1$ and $\gamma(t) = (0,0)$ for all $t \in [0,1]$. (Essentially, the path γ cannot get from $(0,0)$ to any other point of X because continuity prevents from getting over intervening humps where the function f takes values such as ± 1.) We conclude that the connected topological space X is not path-connected.

1.25. Locally Path-Connected Topological Spaces

Definition

A topological space X is said to be *locally connected* if, given any point x of X, and given any open set N in X for which $x \in N$, there exists some connected open set V in X such that $x \in V$ and $V \subset N$.

Definition

A topological space X is said to be *locally path-connected* if, given any point x of X, and given any open set N in X for which $x \in N$, there exists some path-connected open set V in X such that $x \in V$ and $V \subset N$.

Every path-connected subset of a topological space is connected. (This follows directly from Proposition 1.64.) Therefore every locally path-connected topological space is locally connected.

Proposition 1.66

Let X be a connected, locally path-connected topological space. Then X is path-connected.

Proof

Choose a point x_0 of X. Let Z be the subset of X consisting of all points x of X with the property that x can be joined to x_0 by a path. We show that the subset Z is both open and closed in X.

Now, given any point x of X there exists a path-connected open set N_x in X such that $x \in N_x$. We claim that if $x \in Z$ then $N_x \subset Z$, and if $x \notin Z$ then $N_x \cap Z = \emptyset$.

Suppose that $x \in Z$. Then, given any point x' of N_x, there exists a path in N_x from x' to x. Moreover it follows from the definition of the set Z that there exists a path in X from x to x_0. These two paths can be concatenated to yield a path in X from x' to x_0, and therefore $x' \in Z$. This shows that $N_x \subset Z$ whenever $x \in Z$.

Next suppose that $x \notin Z$. Let $x' \in N_x$. If it were the case that $x' \in Z$, then we would be able to concatenate a path in N_x from x to x' with a path in X from x' to x_0 in order to obtain a path in X from x to x_0. But this is impossible, as $x \notin Z$. Therefore $N_x \cap Z = \emptyset$ whenever $x \notin Z$.

Now the set Z is the union of the open sets N_x as x ranges over all points of Z. It follows that Z is itself an open set. Similarly $X \setminus Z$ is the union of the open sets N_x as x ranges over all points of $X \setminus Z$, and therefore $X \setminus Z$ is itself an open set. It follows that Z is a subset of X that is both open and closed. Moreover $x_0 \in Z$, and therefore Z is non-empty. But the only subsets of X that are both open and closed are $\emptyset$ and X itself, since X is connected. Therefore $Z = X$, and thus every point of X can be joined to the point x_0 by a path in X. We conclude that X is path-connected, as required. ∎

1.26. Contractible and Locally Contractible Spaces

Definition

A topological space X is said to be *contractible* if there exists a point p of X and a continuous map $H\colon X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) = p$ for all $x \in X$.

Lemma 1.67

Convex sets in Euclidean spaces are contractible.

Proof

Let X be a convex set in a Euclidean space, and let $\mathbf{p}$ be a point of X. Let Let $H\colon X \times [0,1] \to X$ be defined such that $H(\mathbf{x}, t) = (1-t)\mathbf{x} + t\mathbf{p}$ for all $\mathbf{p} \in X$ and $t \in [0,1]$. Then $H(\mathbf{x}, 0) = \mathbf{x}$ and $H(\mathbf{x}, 1) = \mathbf{p}$ for all $\mathbf{x} \in X$. It follows that the convex set X is contractible, as required. ∎

Corollary 1.68

Open and closed balls in Euclidean spaces are contractible.

Lemma 1.69

Every contractible topological space is path-connected, and is therefore connected.

Proof

Let X be a contractible topological space. Then there exists a point p of X and a continuous map $H\colon X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) = p$ for all $x \in X$.

Let u and v be points of X, and let $\gamma\colon [0,1] \to X$ be defined such that

$$\gamma(t) = \begin{cases} H(u, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ H(v, 2-2t) & \text{if } \frac{1}{2} \leq t \leq 1; \end{cases}$$

(Note that the formulae defining $\gamma(t)$ for $t \leq \frac{1}{2}$ and for $t \geq \frac{1}{2}$ are consistent with each other, because $H(u, 2t) = p = H(v, 2-2t)$ when $t = \frac{1}{2}$.) Now the restrictions $\gamma|[0, \frac{1}{2}]$ and $\gamma|[\frac{1}{2}, 1]$ of the function γ to the closed intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are continuous on those intervals. It follows from the Pasting Lemma (Lemma 1.24) that the function $\gamma\colon [0,1] \to X$ is continuous. Moreover $\gamma(0) = u$ and $\gamma(1) = v$. We conclude from this that the topological space X is path-connected. It then follows from Proposition 1.64 that the topological spaces X is connected, as required. ∎

Example

The *Comb Space* X is defined so that

$$\begin{aligned} X &= \{(x,y) \in [0,1] \times [0,1] : \\ &\qquad y = 0 \text{ or } x = 0 \text{ or } x = n^{-1} \text{ for some } n \in \mathbb{N}\}. \end{aligned}$$

This topological space is the union of one horizontal line, from $(0,0)$ to $(1,0)$, and an infinite number of vertical lines. First we show that the Comb Space is contractible. Let $H\colon X \times [0,1] \to X$ be defined such that

$$H((x,y),t) = \begin{cases} (x,(1-2t)y) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ ((2-2t)x,0) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The Pasting Lemma (Lemma 1.24) ensures $H\colon X \times [0,1] \to X$ is a continuous map from $X \times [0,1]$ to X. Moreover $H((x,y),0) = (x,y)$ and $H((x,y),1) = (0,0)$ for all $(x,y) \in X$. Thus X is contractible.

Next we show that the Comb Space X is not locally connected. Let v be a real number satisfying $0 < v \leq 1$, and let

$$N = \{(x, y) \in X : y > 0\}.$$

Then N is an open set in X, and $(0, v) \in N$. Let W be an open set in X for which $(0, v) \in W$ and $W \subset N$. Then there exists a positive integer n large enough to ensure that $(n^{-1}, v) \in W$. Let q be a positive real number chosen to ensure that $1/q$ is not an integer and $0 < q < n^{-1}$, and let

$$\begin{aligned} V_1 &= \{(x, y) \in W : x < q\}, \\ V_2 &= \{(x, y) \in W : x > q\}. \end{aligned}$$

Now there is no point (x, y) of W for which $x = q$. It follows that V_1 and V_2 are non-empty open subsets of W for which $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = W$. It follows that W is not connected. We conclude from this that the Comb Space X is not locally connected.

Now suppose that we remove the point $(0, v)$ from the Comb Space X, where $0 < v < 1$. The resultant subset $X \setminus \{(0, v)\}$ of X is then connected but not path-connected. Indeed let $Y = X \setminus \{(0, v)\}$. Then the point $(n^{-1}, 1)$ of Y can be joined to $(0, 0)$ by a path in Y, and therefore belongs to the same connected component of Y as the point $(0, 0)$. Also every open set in Y containing the point $(0, 1)$ contains points $(n^{-1}, 0)$ for sufficiently large positive integers n, and thus the point $(0, 1)$ belongs to the closure of the connected component of Y that contains the point $(0, 0)$. But connected components of topological spaces are closed (Proposition 1.63). Therefore the point $(0, 1)$ belongs to the same connected component of Y as the point $(0, 0)$. Moreover every point of Y can be joined by a path to at least one of the points $(0, 0)$ and $(0, 1)$. It follows that Y is connected. But there is no path in Y from $(0, 0)$ to $(0, 1)$, and therefore Y is not path-connected.

Definition

A topological space X is said to be *locally contractible* if, given any point p of X, and given any open set N for which $p \in N$, there exists a contractible open set W for which $p \in W$ and $W \subset N$.

Definition

A topological space is said to be *locally Euclidean* of dimension n if, if, given any point p of X, and given any open set N for which $p \in N$, there exists an open set W satisfying $p \in W$ and $W \subset N$ that is homeomorphic to an open set in $\mathbb{R}^n$.

Lemma 1.70

Locally Euclidean topological spaces are locally contractible, and are therefore locally path-connected and locally connected.

Proof

The result follows directly on combining the relevant definitions with the result stated in Lemma 1.69. ∎

Definition

A *topological manifold* of dimension n is a Hausdorff space with a countable base of open sets that is locally Euclidean of dimension n.

It follows from Lemma 1.70 that topological manifolds are locally contractible, and are therefore locally path-connected and locally connected.