MA342R—Covering Spaces and Fundamental Groups
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1.21. Connected Topological Spaces

Definition

A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed.

Lemma 1.51

A topological space X is connected if and only if there do not exist non-empty open subsets V and W of X for which both $V \cup W = X$ and $V \cap W = \emptyset$.

Proof

Suppose that X is connected. Let V and W be open subsets of X. If $V \cup W = X$ and $V \cap W = \emptyset$ then $V = X \setminus W$, and thus the subset V of X is both open and closed. It follows from the connectedness of X that either $V = \emptyset$ or else $V = X$. Moreover $W = X$ in the case when $V = \emptyset$, and $W = \emptyset$ in the case when $V = X$. Thus the sets V and W cannot both be non-empty. We conclude that if the topological space X is connected then there cannot exist non-empty open sets V and W for which both $V \cap W = X$ and $V \cap W = \emptyset$.

Conversely let X be a topological space that does not contain non-empty open sets V and W with the property that both $V \cup W = X$ and $V \cap W = \emptyset$. Let V be a subset of X that is both open and closed, and let $W = X \setminus V$. Then the sets V and W are both open in X, $V \cup W = X$ and $V \cap W = \emptyset$. It follows that the open sets V and W cannot both be non-empty, and therefore either $V = \emptyset$ or else $W = \emptyset$, in which case $V = X$. This shows that X is connected, as required. ∎

The following two lemmas are immediate consequences of Lemma 1.51

Lemma 1.52

A topological space X is connected if and only if it has the following property: if U and V are non-empty open sets in X such that $U \cup V = X$, then $U \cap V$ is non-empty.

Lemma 1.53

A topological space X is connected if and only if it has the following property: if U and V are non-empty open sets in X such that $U \cap V = \emptyset$, then $U \cup V$ is a proper subset of X.

Definition

A topological space D is *discrete* if every subset of D is open in D.

Example

The set $\mathbb{Z}$ integers with the usual topology is an example of a discrete topological space. Indeed, given any integer n, the set $\{n\}$ is open in $\mathbb{Z}$, because it is the intersection of $\mathbb{Z}$ with the open ball in $\mathbb{R}$ of radius $\frac{1}{2}$ about n. Any non-empty subset S of $\mathbb{Z}$ is the union of the sets $\{n\}$ as n ranges over the elements of S. Therefore every subset of $\mathbb{Z}$ is open in $\mathbb{Z}$, and thus $\mathbb{Z}$, with the usual topology, is a discrete topological space.

Proposition 1.54

Let X be a topological space, and let D be a discrete topological space with at least two elements. Then X is connected if and only if every continuous function from X to D is constant.

Proof

Suppose that X is connected. Let $f \colon X \to D$ be a continuous function from X to D, let $d \in f(X)$, and let $Z = f^{-1}(\{d\})$. Now $\{d\}$ is both open and closed in D. It follows from the continuity of $f \colon X \to D$ that Z is both open and closed in X. Moreover Z is non-empty. It follows from the connectedness of X that $Z = X$, and thus $f \colon X \to D$ is constant.

Now suppose that X is not connected. Then there exists a non-empty proper subset Z of X that is both open and closed in X. Let d_1 and d_2 be elements of D, where $d_1 \neq d_2$, and let $f \colon X \to D$ be defined so that

$$f(x) = \begin{cases} d_1 & \text{if } x \in Z; \\ d_2 & \text{if } x \in X \setminus Z. \end{cases}$$

If V is a subset of D then $f^{-1}(V)$ is one of the following four sets: $\emptyset$, Z, $X \setminus Z$, X. It follows that $f^{-1}(V)$ is open in X for all subsets V of D. Therefore $f \colon X \to D$ is continuous. But the function $f \colon X \to D$ is not constant, because Z is a non-empty proper subset of X. The result follows. ∎

The following results follow immediately from Proposition 1.54.

Corollary 1.55

A topological space X is connected if and only if every continuous function $f\colon X \to \{0,1\}$ from X to the discrete topological space $\{0,1\}$ is constant.

Corollary 1.56

A topological space X is connected if and only if every continuous function $f\colon X \to \mathbb{Z}$ from X to the set $\mathbb{Z}$ of integers is constant.

Example

Let $X = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. The topological space X is not connected. Indeed if $f\colon X \to \mathbb{Z}$ is defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

then f is continuous on X but is not constant.

Definition

A subset I of the set $\mathbb{R}$ of real numbers is said to be an *interval* if $(1-t)c + td \in I$ for all $c \in I$, $d \in I$ and $t \in [0, 1]$.

Using the Least Upper Bound Property of the real number system one can show that a non-empty subset of the set $\mathbb{R}$ of real numbers is an interval if and only if it can be expressed in one of the following forms: $[a, b]$, $[a, b)$, $(a, b]$, (a, b), $[a, +\infty)$, $(a, +\infty)$, $(-\infty, b]$, $(-\infty, b)$, $(-\infty, \infty)$.

Theorem 1.57

Every interval in the set $\mathbb{R}$ of real numbers is connected.

Proof

An interval consisting of a single real number is clearly connected.

Throughout the remainder of the proof let I be an interval with more than one element, and let V and W be disjoint non-empty subsets of I that are both open in I. We shall show that $V \cup W$ must then be a proper subset of I.

Now there must exist real numbers c and d belonging to the interval I and satisfying $c < d$ for which c belongs to one of the sets V and W and d belongs to the other. We may suppose, without loss of generality, that $c \in V$ and $d \in W$.

Let $z = \sup([c,d] \cap V)$. If $t \in [c,d] \cap V$ then there exists some positive real number δ such that $(t-\delta, t+\delta) \cap [c,d] \subset V$, and therefore $t \neq z$. It follows that $z \notin V$. Similarly if $t \in [c,d] \cap W$ then there exists some positive real number δ such that $(t-\delta, t+\delta) \cap [c,d] \subset W$, But then $(t-\delta, t+\delta) \cap [c,d] \cap V = \emptyset$, because $V \cap W = \emptyset$, and therefore $t \neq z$. It follows that $z \notin W$.

We have now shown that $z \notin V \cup W$. But $z \in I$. It follows that $V \cup W$ is a proper subset of I. We conclude that the interval I is connected (see Lemma 1.51, see also Lemma 1.53). ∎

Corollary 1.58

Let $f\colon I \to \mathbb{Z}$ be a continuous integer-valued function defined on an interval I in the real line. Then $f\colon I \to \mathbb{Z}$ is a constant function.

Proof

The result follows directly on combining the results of Corollary 1.56 and Theorem 1.57. ∎

Lemma 1.59

Let X be a topological space, and let A be a subset of X. Then A is connected (with respect to the subspace topology on A) if and only if, given open sets V and W in X for which $A \cap V \neq \emptyset$, $A \cap W \neq \emptyset$ and $A \subset V \cup W$, the set $A \cap V \cap W$ is non-empty.

Proof

A subset of A is open in the subspace topology if and only if it is of the form $A \cap V$ for some open set V in X. It follows from Lemma 1.52 that A is connected if and only if, given any open sets V and W in X for which $A \cap V \neq \emptyset$, $A \cap W \neq \emptyset$ and $(A \cap V) \cup (A \cap W) = A$, the set $(A \cap V) \cap (A \cap W)$ is the emptyset. Now $(A \cap V) \cup (A \cap W) = A$ if and only if $A \subset V \cup W$, and $(A \cap V) \cap (A \cap W) = \emptyset$ if and only if $A \cap V \cap W = \emptyset$. The result therefore follows directly on applying Lemma 1.52. ∎

Lemma 1.60

Let X be a topological space and let A be a connected subset of X. Then the closure $\overline{A}$ of A is connected.

Proof

Let V and W be open sets in X for which $V \cap \overline{A} \neq \emptyset$, $W \cap \overline{A} \neq \emptyset$, and $\overline{A} \subset V \cup W$. The definition of the closure of A in X ensures that if A is a subset of a closed subset F of X then $\overline{A}$ is also a subset of F. Now $X \setminus V$ and $X \setminus W$ are closed subsets of X and $\overline{A}$ is not a subset of either $X \setminus V$ or $X \setminus W$. It follows that A is not a subset of either $X \setminus V$ or $X \setminus W$ and therefore $V \cap A \neq \emptyset$ and $W \cap A \neq \emptyset$ (see Lemma 1.6). Also $A \subset V \cup W$. It follows from the connectedness of A that $A \cap V \cap W \neq \emptyset$ (see Lemma 1.59). Therefore $\overline{A} \cap V \cap W \neq \emptyset$. We conclude from this that $\overline{A}$ is connected, as required. ∎

Alternative Proof

Let $f\colon \overline{A} \to \mathbb{Z}$ be a continuous function mapping the closure $\overline{A}$ of A into the set $\mathbb{Z}$ of integers. It follows from Corollary 1.56 that the restriction of the function f to the connected set A is constant. Therefore there exists some integer n such that $f(x) = n$ for all $x \in A$.

Let $B = \{x \in \overline{A} : f(x) = n\}$. Now $\{n\}$ is closed in $\mathbb{Z}$. It follows from the continuity of f that the set B is closed in the subspace topology on $\overline{A}$. Therefore $B = \overline{A} \cap F$ for some closed subset F of X. But $\overline{A}$ is itself closed in X. It follows that B is closed in X, and therefore $\overline{A} \subset B$. Thus $B = \overline{A}$, and therefore the continuous function $f\colon \overline{A} \to \mathbb{Z}$ is constant on $\overline{A}$. The required result therefore follows from Corollary 1.56. ∎

Lemma 1.61

Let $f\colon X \to Y$ be a continuous function between topological spaces X and Y, and let A be a connected subset of X. Then $f(A)$ is connected.

Proof

Let V and W be open sets in Y for which $V \cap f(X) \neq \emptyset$, $W \cap f(X) \neq \emptyset$ and $f(X) \subset V \cup W$. Then $f^{-1}(V) \neq \emptyset$, $f^{-1}(W) \neq \emptyset$ and $X \subset f^{-1}(V) \cup f^{-1}(W)$. It follows from the connectedness of X that $f^{-1}(V) \cap f^{-1}(W) \neq \emptyset$. Let $x \in f^{-1}(V) \cap f^{-1}(W)$. Then $f(x) \in V \cap W$, and therefore $f(X) \cap V \cap W \neq \emptyset$. It follows from Lemma 1.59 that the subset $f(X)$ of Y is connected, as required. ∎

Alternative Proof

Let $g\colon f(A) \to \mathbb{Z}$ be any continuous integer-valued function on $f(A)$. Then $g \circ f\colon A \to \mathbb{Z}$ is a continuous integer-valued function on A. It follows from Corollary 1.56 that $g \circ f$ is constant on A. Therefore g is constant on $f(A)$. We deduce from Corollary 1.56 that $f(A)$ is connected. $\blacksquare$

1.22. Products of Connected Topological Spaces

Lemma 1.62

The Cartesian product $X \times Y$ of connected topological spaces X and Y is itself connected.

Proof

Let $f\colon X \times Y \to \mathbb{Z}$ be a continuous integer-valued function from $X \times Y$ to Z. Choose $x_0 \in X$ and $y_0 \in Y$. The function $x \mapsto f(x, y_0)$ is continuous on X, and is thus constant. Therefore $f(x, y_0) = f(x_0, y_0)$ for all $x \in X$. Now fix x. The function $y \mapsto f(x, y)$ is continuous on Y, and is thus constant. Therefore

$$f(x, y) = f(x, y_0) = f(x_0, y_0)$$

for all $x \in X$ and $y \in Y$. We deduce from Corollary 1.56 that $X \times Y$ is connected. ∎

We deduce immediately that a finite Cartesian product of connected topological spaces is connected.

1.23. Connected Components of Topological Spaces

Proposition 1.63

Let X be a topological space. For each $x \in X$, let S_x be the union of all connected subsets of X that contain x. Then

(i) *S_x is connected,*

(ii) *S_x is closed,*

(iii) *if $x, y \in X$, then either $S_x = S_y$, or else $S_x \cap S_y = \emptyset$.*

Proof

Let $f\colon S_x \to \mathbb{Z}$ be a continuous integer-valued function on S_x, for some $x \in X$. Let y be any point of S_x. Then, by definition of S_x, there exists some connected set A containing both x and y. But then f is constant on A, and thus $f(x) = f(y)$. This shows that the function f is constant on S_x. We deduce that S_x is connected. This proves (i). Moreover the closure $\overline{S_x}$ is connected, by Lemma 1.60. Therefore $\overline{S_x} \subset S_x$. This shows that S_x is closed, proving (ii).

Finally, suppose that x and y are points of X for which $S_x \cap S_y \neq \emptyset$. Let $f\colon S_x \cup S_y \to \mathbb{Z}$ be any continuous integer-valued function on $S_x \cup S_y$. Then f is constant on both S_x and S_y. Moreover the value of f on S_x must agree with that on S_y, since $S_x \cap S_y$ is non-empty. We deduce that f is constant on $S_x \cup S_y$. Thus $S_x \cup S_y$ is a connected set containing both x and y, and thus $S_x \cup S_y \subset S_x$ and $S_x \cup S_y \subset S_y$, by definition of S_x and S_y. We conclude that $S_x = S_y$. This proves (iii). ∎

Given any topological space X, the connected subsets S_x of X defined as in the statement of Proposition 1.63 are referred to as the *connected components* of X. We see from Proposition 1.63, part (iii) that the topological space X is the disjoint union of its connected components.

Example

The connected components of $\{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ are

$$\{(x, y) \in \mathbb{R}^2 : x > 0\} \quad \text{and} \quad \{(x, y) \in \mathbb{R}^2 : x < 0\}.$$

Example

The connected components of

$$\{t \in \mathbb{R} : |t - n| < \tfrac{1}{2} \text{ for some integer } n\}.$$

are the sets J_n for all $n \in \mathbb{Z}$, where $J_n = (n - \frac{1}{2}, n + \frac{1}{2})$.