MA342R—Covering Spaces and Fundamental Groups School of Mathematics, Trinity College Hilary Term 2017 Lecture 7 (January 30, 2017)

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1.20. Finite Cartesian Products of Compact Spaces

Theorem 1.49

A Cartesian product of a finite number of compact spaces is itself compact.

Proof

It suffices to prove that the product of two compact topological spaces X and Y is compact, since the general result then follows easily by induction on the number of compact spaces in the product.

Let $\mathcal{V}$ be an open cover of $X \times Y$. Then for each $x \in X$ and $y \in Y$ there exists an open set $V_{x,y}$ in $X \times Y$ belonging to the open cover $\mathcal{V}$ for which $(x, y) \in V_{x,y}$. It then follows from the definition of the product topology on $X \times Y$ that there exist an open set $D_{x,y}$ in X and an open set $E_{x,y}$ in Y such that $x \in D_{x,y}$, $y \in E_{x,y}$ and $D_{x,y} \times E_{x,y} \subset V_{x,y}$.

Now the compactness of the topological space Y ensures that for each $x \in X$ there exists a finite subset $B(x)$ of Y for which $\bigcup_{y \in B(x)} E_{x,y} = Y$. Let $W_x = \bigcap_{y \in B(x)} D_{x,y}$. Then W_x is the intersection of a finite number of open sets in X, and is therefore itself an open set in X. Moreover

$$\begin{aligned} W_x \times Y &\subset \bigcup_{y \in B(x)} (W_x \times E_{x,y}) \subset \bigcup_{y \in B(x)} (D_{x,y} \times E_{x,y}) \\ &\subset \bigcup_{y \in B(x)} V_{x,y}. \end{aligned}$$

It then follows from the compactness of the topological space X that there exists a finite subset A of X for which $\bigcup_{x \in A} W_x = X$. Let

$$C = \{(x, y) : x \in A \text{ and } y \in B(x)\},$$

and, for each $(x, y) \in C$, let $V_{x,y}$ be an open set in $X \times Y$ belonging to the open cover $\mathcal{V}$ for which $D_{x,y} \times E_{x,y} \subset V_{x,y}$. Now C is a finite set, and

$$\begin{aligned} X \times Y &= \bigcup_{x \in A} (W_x \times Y) \subset \bigcup_{x \in A} \bigcup_{y \in B(x)} V_{x,y} \\ &= \bigcup_{(x,y) \in C} V_{x,y}. \end{aligned}$$

Thus $(V_{x,y} : (x, y) \in C)$ is an open cover of $X \times Y$. Moreover it is a finite subcover of the open cover $\mathcal{V}$. We have thus shown that $X \times Y$ is compact, as required. $\blacksquare$

Theorem 1.50

Let K be a subset of $\mathbb{R}^n$. Then K is compact if and only if K is both closed and bounded.

Proof

Suppose that K is compact. Then K is closed, since $\mathbb{R}^n$ is Hausdorff, and a compact subset of a Hausdorff space is closed (by Corollary 1.43). For each natural number m, let B_m be the open ball of radius m about the origin, given by $B_m = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < m\}$. Then $\{B_m : m \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^n$. It follows from the compactness of K that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that $K \subset B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_k}$. But then $K \subset B_M$, where M is the maximum of $m_1, m_2, \ldots, m_k$, and thus K is bounded.

Conversely suppose that K is both closed and bounded. Then there exists some real number L such that K is contained within the closed cube C given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem (Theorem 1.37), and C is the Cartesian product of n copies of the compact set $[-L, L]$. It follows from Theorem 1.49 that C is compact. But K is a closed subset of C, and a closed subset of a compact topological space is itself compact, by Lemma 1.38. Thus K is compact, as required. $\blacksquare$

Let K be a closed bounded subset of $\mathbb{R}^n$. It follows from Theorem 1.48 and Theorem 1.50 that any continuous function $f\colon K \to \mathbb{R}^k$ is uniformly continuous.