MA342R—Covering Spaces and Fundamental Groups
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David R. Wilkins

1.17. Compact Topological Spaces

Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to *cover* A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{V}$ and $\mathcal{W}$ are open covers of some topological space X then $\mathcal{W}$ is said to be a *subcover* of $\mathcal{V}$ if and only if every open set belonging to $\mathcal{W}$ also belongs to $\mathcal{V}$.

Definition

A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 1.36

Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{V}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{V}$ such that $A \subset V_1 \cup V_2 \cup \cdots \cup V_r$.

Proof

A subset B of A is open in A (with respect to the subspace topology on A) if and only if $B = A \cap V$ for some open set V in X. The desired result therefore follows directly from the definition of compactness. ∎

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem*. The proof of this theorem uses the *Least Upper Bound Principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 1.37 (Heine-Borel Theorem in One Dimension)

Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.

Proof

Let $\mathcal{V}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let S be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by some finite collection of open sets belonging to $\mathcal{V}$, and let $s = \sup S$. Now $s \in W$ for some open set W belonging to $\mathcal{V}$. Moreover W is open in $\mathbb{R}$, and therefore there exists some positive real number δ such that $(s - \delta, s + \delta) \subset W$. Moreover $s - \delta$ is not an upper bound for the set S, hence there exists some $\tau \in S$ satisfying $\tau > s - \delta$. It follows from the definition of S that $[a, \tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{V}$.

Let $t \in [a, b]$ satisfy $\tau \leq t < s + \delta$. Then

$$[a, t] \subset [a, \tau] \cup (s - \delta, s + \delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $s \in S$, and moreover $s = b$, since otherwise s would not be an upper bound of the set S. Thus $b \in S$, and therefore $[a, b]$ is covered by a finite collection of open sets belonging to $\mathcal{V}$, as required. ∎

Lemma 1.38

Let A be a closed subset of some compact topological space X. Then A is compact.

Proof

Let $\mathcal{V}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{V}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{V}$ that belong to this finite subcover. It follows from Lemma 1.36 that A is compact, as required. ∎

Lemma 1.39

Let $f\colon X \to Y$ be a continuous function between topological spaces X and Y, and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.

Proof

Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$. Then A is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact. ∎

Lemma 1.40

Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.

Proof

Let $V_j = \{x \in X : -j < f(x) < j\}$ for all positive integers j. For each integer j the subset V_j of X is the preimage under the continuous map f of the open interval $(-j, j)$, and moreover $(-j, j)$ is open in $\mathbb{R}$. It follows from the continuity of f that V_j is an open set in X for all positive integers j. Moreover the compact topological space X is covered by these open sets. It follows from the compactness of X that there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X = V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let N be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $-N < f(x) < N$ for all $x \in X$. The result follows. ∎

Proposition 1.41

Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.

Proof

The function $f\colon X \to \mathbb{R}$ is bounded on X (Lemma 1.40). Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. For each positive integer j let $V_j = \{x \in X : f(x) < M - 1/j\}$. Then the set V_j is an open set in X, being the preimage of an open interval in $\mathbb{R}$ under the continuous map f. If $j_1, j_2, \ldots, j_k$ are positive integers then

$$V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k} = V_N$$

where N is the largest of the positive integers $j_1, j_2, \ldots, j_k$.

Moreover V_N is a proper subset of X, because $M - 1/N$ is not an upper bound on the values of the function f on X. It follows that X cannot covered by any finite collection of sets from the collection $(V_j : j \in \mathbb{N})$. It then follows from the compactness of X that $(V_j : j \in \mathbb{N})$ is not an open cover of X, and therefore there exists $v \in X$ for which $f(v) = M$. Applying this argument with f replaced by $-f$, we conclude that there also exists $u \in X$ for which $f(u) = m$. Then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required. ∎

1.18. Compact Subsets of Hausdorff Spaces

Proposition 1.42

Let X be a Hausdorff topological space, and let K be a compact subset of X. Let x be a point of $X \setminus K$. Then there exist open sets V and W in X such that $x \in V$, $K \subset W$ and $V \cap W = \emptyset$.

Proof

For each point $y \in K$ there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since X is a Hausdorff space). But then there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of K such that K is contained in $W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}$, since K is compact. Define

$$V = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \qquad W = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Then V and W are open sets, $x \in V$, $K \subset W$ and $V \cap W = \emptyset$, as required. ∎

Corollary 1.43

A compact subset of a Hausdorff topological space is closed.

Proof

Let K be a compact subset of a Hausdorff topological space X. It follows immediately from Proposition 1.42 that, for each $x \in X \setminus K$, there exists an open set V_x such that $x \in V_x$ and $V_x \cap K = \emptyset$. But then $X \setminus K$ is equal to the union of the open sets V_x as x ranges over all points of $X \setminus K$, and any set that is a union of open sets is itself an open set. We conclude that $X \setminus K$ is open, and thus K is closed. ∎

Lemma 1.44

Let $f: X \to Y$ be a continuous function from a compact topological space X to a Hausdorff space Y. Then $f(K)$ is closed in Y for every closed set K in X.

Proof

If K is a closed set in X, then K is compact (Lemma 1.38), and therefore $f(K)$ is compact (Lemma 1.39). But any compact subset of a Hausdorff space is closed (Corollary 1.43). Thus $f(K)$ is closed in Y, as required. ∎

Theorem 1.45

A continuous bijection $f\colon X \to Y$ from a compact topological space X to a Hausdorff space Y is a homeomorphism.

Proof

Let $g\colon Y \to X$ be the inverse of the bijection $f\colon X \to Y$. If U is open in X then $X \setminus U$ is closed in X, and hence $f(X \setminus U)$ is closed in Y (see Lemma 1.44). But $f(X \setminus U) = g^{-1}(X \setminus U) = Y \setminus g^{-1}(U)$. It follows that $g^{-1}(U)$ is open in Y for every open set U in X. Therefore $g\colon Y \to X$ is continuous, and thus $f\colon X \to Y$ is a homeomorphism. ∎

Proposition 1.46

A continuous surjection $f\colon X \to Y$ from a compact topological space X to a Hausdorff space Y is an identification map.

Proof

Let U be a subset of Y. We claim that $Y \setminus U = f(K)$, where $K = X \setminus f^{-1}(U)$. Clearly $f(K) \subset Y \setminus U$. Also, given any $y \in Y \setminus U$, there exists $x \in X$ satisfying $y = f(x)$, since $f\colon X \to Y$ is surjective. Moreover $x \in K$, since $f(x) \notin U$. Thus $Y \setminus U \subset f(K)$, and hence $Y \setminus U = f(K)$, as claimed.

We must show that the set U is open in Y if and only if $f^{-1}(U)$ is open in X. First suppose that $f^{-1}(U)$ is open in X. Then K is closed in X, and hence $f(K)$ is closed in Y, by Lemma 1.44. It follows that U is open in Y. Conversely if U is open in Y then $f^{-1}(U)$ is open in X, since $f\colon X \to Y$ is continuous. Thus the surjection $f\colon X \to Y$ is an identification map. ∎

Example

Let S^1 be the unit circle in $\mathbb{R}^2$, defined by $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and let $q\colon [0, 1] \to S^1$ be defined by $q(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in [0, 1]$. It has been shown that the map q is an identification map. This also follows directly from the fact that $q\colon [0, 1] \to S^1$ is a continuous surjection from the compact space $[0, 1]$ to the Hausdorff space S^1.

1.19. The Lebesgue Lemma and Uniform Continuity

Definition

Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists a non-negative real number K such that $d(x, y) \leq K$ for all $x, y \in A$. The smallest real number K with this property is referred to as the *diameter* of A, and is denoted by $\mathrm{diam}\, A$. (Note that $\mathrm{diam}\, A$ is the supremum of the values of $d(x, y)$ as x and y range over all points of A.)

Lemma 1.47 (Lebesgue Lemma)

Let (X, d) be a compact metric space and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{V}$.

Proof

Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{V}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{V}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_r$. Then $\delta > 0$.

Suppose that A is a subset of X whose diameter is less than δ. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{V}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{V}$, as required. ∎

Let $\mathcal{V}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{V}$ is a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{V}$. The Lebesgue Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

Definition

Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f\colon X \to Y$ be a function from X to Y. The function f is said to be *uniformly continuous* on X if and only if, given $\varepsilon > 0$, there exists some positive real number δ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x and x' of X satisfying $d_X(x, x') < \delta$. (The value of δ should be independent of both x and x'.)

Theorem 1.48

Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.

Proof

Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $f\colon X \to Y$ be a continuous function from X to Y. We must show that f is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}\left(B_Y(y, \frac{1}{2}\varepsilon)\right)$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about y in Y. Now the open ball $B_Y(y, \frac{1}{2}\varepsilon)$ is an open set in Y, and f is continuous. Therefore V_y is open in X for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 1.47) that there exists some positive real number δ such that every subset of X whose diameter is less than δ is a subset of some set V_y. Let x and x' be points of X satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$, which is less than δ. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f : X \to Y$ is uniformly continuous, as required. ∎