MA342R—Covering Spaces and Fundamental Groups
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1.14. Bases for Topologies

Proposition 1.26

Let X be a set, let β be a collection of subsets of X, and let τ be the collection consisting of the empty set, together with all subsets of X that are unions of sets belonging to the collection β. Then τ is a topology on X if and only if the following conditions are satisfied:—

(i) *the set X is the union of the subsets belonging to the collection β;*

(ii) *given subsets $B_1, B_2 \in \beta$, and given any point p of $B_1 \cap B_2$, there exists some $B \in \beta$ such that $p \in B$ and $B \subset B_1 \cap B_2$.*

Proof

First suppose that τ is a topology on X. Then $X \in \tau$. But any subset of X that belongs to τ is a union of sets belonging to β. Therefore X is a union of subsets belonging to the collection β, and thus condition (i) is satisfied.

Moreover the intersection of any two open subsets of a topological space is required to be open. Thus if τ is a topology on X, and if $B_1, B_2 \in \beta$, then $B_1, B_2 \in \tau$ and therefore $B_1 \cap B_2 \in \tau$. It follows that $B_1 \cap B_2$ is a union of subsets of X that belong to β, and therefore, given any $p \in B_1 \cap B_2$, there exists $B \in \beta$ such that $p \in B$ and $B \subset B_1 \cap B_2$. Thus condition (ii) is satisfied.

Conversely we must prove that if the collection β of subsets of a set X satisfies conditions (i) and (ii) then the collection τ of unions of sets belonging to β is a topology on X.

The empty set belongs to τ. Condition (i) ensures that the whole set X belongs to τ. It follows directly from the definition of τ that any union of sets belonging to τ is a union of sets belonging to β, and therefore itself belongs to τ.

It therefore only remains to show that the intersection of any finite collection of sets belonging to τ belongs to τ. It suffices to prove that the intersection of two sets belonging to τ belongs to τ. Let $V_1, V_2 \in \tau$, and let $p \in V_1 \cap V_2$. Then V_1 and V_2 are union of sets belonging to β, and therefore there exist $B_1, B_2 \in \beta$ such that $p \in B_1$, $p \in B_2$, $B_1 \subset V_1$, and $B_2 \subset V_2$. Now condition (ii) ensures the existence of $B_p \in \beta$ such that $p \in B_p$ and $B_p \subset B_1 \cap B_2$. Then $B_p \subset V_1 \cap V_2$. It follows that the set $V_1 \cap V_2$ is the union of all subsets B of $V_1 \cap V_2$ that belong to β, and therefore $V_1 \cap V_2$ itself belongs to τ. It then follows by induction on the number of sets involved that the intersection of any finite number of subsets of X belonging to τ must itself belong to τ. Thus τ is a topology on the set X, as required. ∎

Definition

Let X be a set. A collection β of subsets of X is said to be a *base* for a topology on X if the following conditions are satisfied:—

(i) the set X is the union of the subsets belonging to the collection β;

(ii) given subsets $B_1, B_2 \in \beta$, and given any point p of $B_1 \cap B_2$, there exists some $B \in \beta$ such that $p \in B$ and $B \subset B_1 \cap B_2$.

If β is a base for a topology on X then the topology generated by β is the topology whose open sets are those subsets of X that are unions of sets belonging to the base β.

Lemma 1.27

Let X be a set, and let β be a base for a topology on X. A non-empty subset V is open in X with respect to the topology generated by β if and only if, given any point v of V, there exists $B \in \beta$ such that $v \in B$ and $B \subset V$.

Proof

This result follows directly from the fact that the non-empty open sets in X are those subsets of X that are unions of sets belonging to the base β. $\blacksquare$

Example

Let X be a metric space. Then the collection of all open balls of positive radius centred on points of X is a base for the topology on X generated by the distance function on X.

1.15. Product Topologies

The *Cartesian product* $X_1 \times X_2 \times \cdots \times X_n$ of sets $X_1, X_2, \ldots, X_n$ is defined to be the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for $i = 1, 2, \ldots, n$.

The sets $\mathbb{R}^2$ and $\mathbb{R}^3$ are the Cartesian products $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ respectively.

Let $X_1, X_2, X_3, \ldots, X_n$ be topological spaces, and let V_i and W_i be open sets in X_i for $i = 1, 2, \ldots, n$. Then

$$(V_1 \times V_2 \times \cdots \times V_n) \cap (W_1 \times W_2 \times \cdots \times W_n) = E_1 \times E_2 \times \cdots \times E_n,$$

where $E_i = V_i \cap W_i$ for $i = 1, 2, \ldots, n$. The intersection of two open sets in a topological space is always itself open. Therefore E_i is an open set in X_i for $i = 1, 2, \ldots, n$. It follows from this that if β is the collection of subsets of $X_1 \times X_2 \times \cdots \times X_n$ that are of the form $V_1 \times V_2 \times \cdots \times V_n$, where V_i is open in X_i for $i = 1, 2, \ldots, n$, then β is the base for a topology on $X_1 \times X_2 \times \cdots \times X_n$. This topology is the *product topology* on this Cartesian product of topological spaces. Lemma 1.27 ensures that a non-empty subset W of $X_1 \times X_2 \times \cdots \times X_n$ is open in $X_1 \times X_2 \times \cdots \times X_n$ with respect to this product topology if and only if, given any point $(x_1, x_2, \ldots, x_n)$ of W, there exist open sets $V_1, V_2, \ldots, V_n$ such that $x_i \in V_i$ for $i = 1, 2, \ldots, n$ and

$$V_1 \times V_2 \times \cdots \times V_n \subset W.$$

The definition of the product topology is then encapsulated in the following formal definition.

Definition

Let $X_1, X_2, \ldots, X_n$ be topological spaces. The *product topology* on the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is the unique topology on this Cartesian product of sets that satisfies the following criterion:

> *a non-empty subset W of the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is open with respect to the product topology if and only if, given any point $(x_1, x_2, \ldots, x_n)$ of W, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $x_i \in V_i$ for $i = 1, 2, \ldots, n$ and*
>
> $$V_1 \times V_2 \times \cdots \times V_n \subset W.$$

The following result follows directly from the definition of the product topology.

Lemma 1.28

Let $X_1, X_2, \ldots, X_n$ be topological spaces, let p be a point of $X_1 \times X_2 \times \cdots \times X_n$, and let N be a subset of $X_1 \times X_2 \times \cdots \times X_n$ for which $p \in N$. Then N is a neighbourhood of p in X if and only if there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ for which $p \in V_1 \times V_2 \cdots \times V_n$ and $V_1 \times V_2 \times \cdots \times V_n \subset N$.

Lemma 1.29

Let $X_1, X_2, \ldots, X_n$ and Z be topological spaces. Then a function $f\colon X_1 \times X_2 \times \cdots \times X_n \to Z$ is continuous at a point p of $X_1 \times X_2 \times \cdots \times X_n$ if and only if, and given any open set W in Z containing $f(p)$, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ for which $p \in V_1 \times V_2 \cdots \times V_n$ and $f(V_1 \times V_2 \times \cdots \times V_n) \subset W$.

Proof

Given any neighbourhood N of $f(p)$, there exists an open set W in Y such that $f(p) \in W$ and $W \subset N$. It follows from this that the function f is continuous at p if and only if $f^{-1}(W)$ is a neighbourhood of p in X for all open sets W in Y for which $f(p) \in W$. The result therefore follows on applying Lemma 1.28. ∎

Let $X_1, X_2, \ldots, X_n$ be topological spaces, and let V_i be an open set in X_i for $i = 1, 2, \ldots, n$. It follows directly from the definition of the product topology that $V_1 \times V_2 \times \cdots \times V_n$ is open in $X_1 \times X_2 \times \cdots \times X_n$.

Proposition 1.30

Let $X = X_1 \times X_2 \times \cdots \times X_n$, where $X_1, X_2, \ldots, X_n$ are topological spaces and X is given the product topology, and for each i, let $p_i \colon X \to X_i$ denote the projection function which sends $(x_1, x_2, \ldots, x_n) \in X$ to x_i. Let $f \colon Z \to X$ mapping a topological space Z into X and let z be a point of Z. Then $f \colon Z \to X$ is continuous at z if and only if $p_i \circ f \colon Z \to X_i$ is continuous at z for $i = 1, 2, \ldots, n$.

Proof

Let V be an open set in X_i. Then

$$p_i^{-1}(V) = X_1 \times \cdots \times X_{i-1} \times V \times X_{i+1} \times \cdots \times X_n,$$

and therefore $p_i^{-1}(V)$ is open in X. Thus $p_i \colon X \to X_i$ is continuous for all i. It follows that if the function $f \colon Z \to X$ is continuous at a point z of Z then the composition functions $p_i \circ f$ are also continuous at z for $i = 1, 2, \ldots, n$ (see Lemma 1.22).

Conversely suppose that $f\colon Z \to X$ is a function with the property that $p_i \circ f$ is continuous at z for $i = 1, 2, \ldots, n$, where $z \in Z$. Let N be a neighbourhood of $f(z)$ in X. Then there exist $V_1, V_2, \ldots, V_n$, where V_i is open in X_i for $i = 1, 2, \ldots, n$, such that $f(z) \in V_1 \times V_2 \times \cdots \times V_n$ and $V_1 \times V_2 \times \cdots \times V_n \subset N$ (see Lemma 1.28). Let

$$W_z = f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \cap \cdots \cap f_n^{-1}(V_n),$$

where $f_i = p_i \circ f$ for $i = 1, 2, \ldots, n$. Then $z \in W_z$, and the continuity of $f_1, f_2, \ldots, f_n$ ensures that W_z is an open set in Z. Moreover $f(z') \in V_1 \times V_2 \times \cdots \times V_n$ for all $z' \in W_z$, and therefore $W_z \subset f^{-1}(N)$. We have thus shown that $f^{-1}(N)$ is a neighbourhood of z for all neighbourhoods N of $f(z)$. It follows that $f\colon Z \to X$ is continuous at z, as required. ∎

Proposition 1.31

The usual topology on $\mathbb{R}^n$ coincides with the product topology on $\mathbb{R}^n$ obtained on regarding $\mathbb{R}^n$ as the Cartesian product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ of n copies of the real line $\mathbb{R}$.

Proof

We must show that a subset W of $\mathbb{R}^n$ is open with respect to the usual topology if and only if it is open with respect to the product topology.

Let W be a subset of $\mathbb{R}^n$ that is open with respect to the usual topology, and let $\mathbf{q} \in W$. Then there exists some positive real number δ such that $B(\mathbf{q}, \delta) \subset W$, where

$$B(\mathbf{q}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{q}| < \delta\}.$$

Let $J_1, J_2, \ldots, J_n$ be the open intervals in $\mathbb{R}$ defined by

$$J_i = \left\{ t \in \mathbb{R} : q_i - \frac{\delta}{\sqrt{n}} < t < q_i + \frac{\delta}{\sqrt{n}} \right\} \qquad (i = 1, 2, \ldots, n),$$

Then $J_1, J_2, \ldots, J_n$ are open sets in $\mathbb{R}$. Moreover

$$\{\mathbf{q}\} \subset J_1 \times J_2 \times \cdots \times J_n \subset B(\mathbf{q}, \delta) \subset W,$$

since

$$|\mathbf{x} - \mathbf{q}|^2 = \sum_{i=1}^{n} (x_i - q_i)^2 < n \left(\frac{\delta}{\sqrt{n}} \right)^2 = \delta^2$$

for all $\mathbf{x} \in J_1 \times J_2 \times \cdots \times J_n$. This shows that any subset W of $\mathbb{R}^n$ that is open with respect to the usual topology on $\mathbb{R}^n$ is also open with respect to the product topology on $\mathbb{R}^n$.

Conversely suppose that W is a subset of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$, and let $\mathbf{q} \in W$. Then there exist open sets $V_1, V_2, \ldots, V_n$ in $\mathbb{R}$ containing $q_1, q_2, \ldots, q_n$ respectively such that $V_1 \times V_2 \times \cdots \times V_n \subset W$. Now we can find $\delta_1, \delta_2, \ldots, \delta_n$ such that $\delta_i > 0$ and $(q_i - \delta_i, q_i + \delta_i) \subset V_i$ for all i. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \ldots, \delta_n$. Then $\delta > 0$, and

$$B(\mathbf{q}, \delta) \subset V_1 \times V_2 \times \cdots \times V_n \subset W,$$

for if $\mathbf{x} \in B(\mathbf{q}, \delta)$ then $|x_i - q_i| < \delta_i$ for $i = 1, 2, \ldots, n$. This shows that any subset W of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$ is also open with respect to the usual topology on $\mathbb{R}^n$. ∎

The following result is now an immediate corollary of Proposition 1.31 and Proposition 1.30.

Corollary 1.32

Let X be a topological space and let $f: X \to \mathbb{R}^n$ be a function from X to $\mathbb{R}^n$. Let us write

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

for all $x \in X$, where the components $f_1, f_2, \ldots, f_n$ of f are functions from X to $\mathbb{R}$. The function f is continuous if and only if its components $f_1, f_2, \ldots, f_n$ are all continuous.

Let $f \colon X \to \mathbb{R}$ and $g \colon X \to \mathbb{R}$ be continuous real-valued functions on some topological space X. We claim that $f + g$, $f - g$ and $f.g$ are continuous. Now it is a straightforward exercise to verify that the sum and product functions $s \colon \mathbb{R}^2 \to \mathbb{R}$ and $p \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $p(x, y) = xy$ are continuous, and $f + g = s \circ h$ and $f.g = p \circ h$, where $h \colon X \to \mathbb{R}^2$ is defined by $h(x) = (f(x), g(x))$. Moreover it follows from Corollary 1.32 that the function h is continuous, and compositions of continuous functions are continuous. Therefore $f + g$ and $f.g$ are continuous, as claimed. Also $-g$ is continuous, and $f - g = f + (-g)$, and therefore $f - g$ is continuous. If in addition the continuous function g is non-zero everywhere on X then $1/g$ is continuous (since $1/g$ is the composition of g with the reciprocal function $t \mapsto 1/t$), and therefore f/g is continuous.

Lemma 1.33

The Cartesian product $X_1 \times X_2 \times \ldots X_n$ of Hausdorff spaces $X_1, X_2, \ldots, X_n$ is Hausdorff.

Proof

Let $X = X_1 \times X_2 \times \ldots, X_n$, and let u and v be distinct points of X, where $u = (x_1, x_2, \ldots, x_n)$ and $v = (y_1, y_2, \ldots, y_n)$. Then $x_i \neq y_i$ for some integer i between 1 and n. But then there exist open sets U and V in X_i such that $x_i \in U$, $y_i \in V$ and $U \cap V = \emptyset$ (since X_i is a Hausdorff space). Let $p_i \colon X \to X_i$ denote the projection function. Then $p_i^{-1}(U)$ and $p_i^{-1}(V)$ are open sets in X, since p_i is continuous. Moreover $u \in p_i^{-1}(U)$, $v \in p_i^{-1}(V)$, and $p_i^{-1}(U) \cap p_i^{-1}(V) = \emptyset$. Thus X is Hausdorff, as required. ∎

1.16. Identification Maps and Quotient Topologies

Definition

Let X and Y be topological spaces and let $q\colon X \to Y$ be a function from X to Y. The function q is said to be an *identification map* if and only if the following conditions are satisfied:

- the function $q\colon X \to Y$ is surjective,
- a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

It follows directly from the definition that any identification map is continuous. Moreover, in order to show that a continuous surjection $q\colon X \to Y$ is an identification map, it suffices to prove that if V is a subset of Y with the property that $q^{-1}(V)$ is open in X then V is open in Y.

Lemma 1.34

Let X be a topological space, let Y be a set, and let $q\colon X \to Y$ be a surjection. Then there is a unique topology on Y for which the function $q\colon X \to Y$ is an identification map.

Proof

Let τ be the collection consisting of all subsets U of Y for which $q^{-1}(U)$ is open in X. Now $q^{-1}(\emptyset) = \emptyset$, and $q^{-1}(Y) = X$, so that $\emptyset \in \tau$ and $Y \in \tau$.

Let $\{V_\alpha : \alpha \in A\}$ be a collection of subsets of Y indexed by a set A. Then it is a straightforward exercise to verify that

$$\bigcup_{\alpha\in A} q^{-1}(V_\alpha) = q^{-1}\left(\bigcup_{\alpha\in A} V_\alpha\right),$$

and

$$\bigcap_{\alpha\in A} q^{-1}(V_\alpha) = q^{-1}\left(\bigcap_{\alpha\in A} V_\alpha\right)$$

(i.e., given any collection of subsets of Y, the union of the preimages of the sets is the preimage of the union of those sets, and the intersection of the preimages of the sets is the preimage of the intersection of those sets). It follows easily from this that unions and finite intersections of sets belonging to τ must themselves belong to τ. Thus τ is a topology on Y, and the function $q\colon X \to Y$ is an identification map with respect to the topology τ. Clearly τ is the unique topology on Y for which the function $q\colon X \to Y$ is an identification map. ∎

Definition

Let X be a topological space, let Y be a set, and let $q\colon X \to Y$ be a surjection. The unique topology on Y for which the function q is an identification map is referred to as the *quotient topology* (or *identification topology*) on Y.

Lemma 1.35

Let X and Y be topological spaces and let $q\colon X \to Y$ be an identification map. Let Z be a topological space, and let $f\colon Y \to Z$ be a function from Y to Z. Then the function f is continuous if and only if the composition function $f \circ q\colon X \to Z$ is continuous.

Proof

Suppose that f is continuous. Then the composition function $f \circ q$ is a composition of continuous functions and hence is itself continuous.

Conversely suppose that $f \circ q$ is continuous. Let U be an open set in Z. Then $q^{-1}(f^{-1}(U))$ is open in X (since $f \circ q$ is continuous), and hence $f^{-1}(U)$ is open in Y (since the function q is an identification map). Therefore the function f is continuous, as required. ∎

Example

Let S^1 be the unit circle in $\mathbb{R}^2$, and let $q\colon [0,1] \to S^1$ be the map that sends $t \in [0,1]$ to $(\cos 2\pi t, \sin 2\pi t)$. Then $q\colon [0,1] \to S^1$ is an identification map, and therefore a function $f\colon S^1 \to Z$ from S^1 to some topological space Z is continuous if and only if $f \circ q\colon [0,1] \to Z$ is continuous.

Example

Let S^n be the n-sphere, consisting of all points $\mathbf{x}$ in $\mathbb{R}^{n+1}$ satisfying $|\mathbf{x}| = 1$. Let $\mathbb{R}P^n$ be the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin (i.e., $\mathbb{R}P^n$ is the set of all one-dimensional vector subspaces of $\mathbb{R}^{n+1}$). Let $q\colon S^n \to \mathbb{R}P^n$ denote the function which sends a point $\mathbf{x}$ of S^n to the element of $\mathbb{R}P^n$ represented by the line in $\mathbb{R}^{n+1}$ that passes through both $\mathbf{x}$ and the origin. Note that each element of $\mathbb{R}P^n$ is the image (under q) of exactly two antipodal points $\mathbf{x}$ and $-\mathbf{x}$ of S^n. The function q induces a corresponding quotient topology on $\mathbb{R}P^n$ such that $q\colon S^n \to \mathbb{R}P^n$ is an identification map. The set $\mathbb{R}P^n$, with this topology, is referred to as *real projective n-dimensional space*. In particular $\mathbb{R}P^2$ is referred to as the *real projective plane*. It follows from Lemma 1.35 that a function $f\colon \mathbb{R}P^n \to Z$ from $\mathbb{R}P^n$ to any topological space Z is continuous if and only if the composition function $f \circ q\colon S^n \to Z$ is continuous.