MA342R—Covering Spaces and Fundamental Groups
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1.10. Continuous Maps between Topological Spaces

Definition

A function $f\colon X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

$$f^{-1}(V) = \{x \in X : f(x) \in V\}.$$

A continuous function from X to Y is often referred to as a *map* from X to Y.

Lemma 1.18

Let X, Y and Z be topological spaces, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions. Then the composition $g \circ f\colon X \to Z$ of the functions f and g is continuous.

Proof

Let V be an open set in Z. Then $g^{-1}(V)$ is open in Y (because g is continuous), and then $f^{-1}(g^{-1}(V))$ is open in X (because f is continuous). But $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Thus the composition function $g \circ f$ is continuous. ∎

Lemma 1.19

Let X and Y be topological spaces, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(G)$ is closed in X for every closed subset G of Y.

Proof

If G is any subset of Y then $X \setminus f^{-1}(G) = f^{-1}(Y \setminus G)$ (i.e., the complement of the preimage of G is the preimage of the complement of G). The result therefore follows immediately from the definitions of continuity and closed sets. ∎

Definition

Let X and Y be topological spaces, let $f\colon X \to Y$ be a function from X to Y and let p be a point of X. The function f is said to be *continuous at* p if $f^{-1}(N)$ is a neighbourhood of p in X for all neighbourhoods N of $f(p)$ in Y.

Proposition 1.20

Let X and Y be topological spaces and let $f\colon X \to Y$ be a function from X to Y. Then the function f is continuous on X if and only if it is continuous at each point of X.

Proof

Suppose that $f\colon X \to Y$ be continuous on X. Let p be a point of X and let N be a neighbourhood of $f(p)$. Then there exists an open set V in Y for which $f(p) \in V$ and $V \subset N$. The continuity of f ensures that $f^{-1}(V)$ is open in X. Moreover $p \in f^{-1}(V)$ and $f^{-1}(V) \subset f^{-1}(N)$. It follows that $f^{-1}(N)$ is a neighbourhood of p in X. This shows that $f\colon X \to Y$ is continuous at each point p of X.

Conversely suppose that $f\colon X \to Y$ is continuous at each point of X. Let V be an open set in Y. Then, given any point p of $f^{-1}(V)$, there exists an open set W_p for which $p \in W_p$ and $W_p \subset f^{-1}(V)$, because the function f is continuous at p. Then $f^{-1}(V) = \bigcup_{p \in f^{-1}(V)} W_p$. Thus $f^{-1}(V)$ is a union of open subsets of X, and is therefore itself open in X. We conclude that $f\colon X \to Y$ is continuous on X. $\blacksquare$

Lemma 1.21

Let X and Y be topological spaces, let $f \colon X \to Y$ be a function from X to Y and let p be a point of X. Then $f \colon X \to Y$ is continuous at p if and only if, given any neighbourhood N of $f(p)$, there exists a neighbourhood M of p for which $f(M) \subset N$.

Proof

Let N be a neighbourhood of $f(p)$ in Y. Suppose that there exists a neighbourhood M of p in X for which $f(M) \subset N$. The definition of neighbourhoods of points in topological spaces then ensures that there exists an open set W in X for which $p \in W$ and $W \subset M$. Then $f(W) \subset N$ and therefore $W \subset f^{-1}(N)$. It follows that $f^{-1}(N)$ is a neighbourhood of p in X, and thus the function f is continuous at p.

Conversely suppose that the function f is continuous at p. Let N be a neighbourhood of $f(p)$ in Y, and let $M = f^{-1}(N)$. Then M is a neighbourhood of p in X, because the function f is continuous at p, and $f(M) \subset N$. The result follows. ∎

Lemma 1.22

Let X, Y and Z be topological spaces, let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions, and let p be a point of X. Suppose that $f\colon X \to Y$ is continuous at p and that $g\colon Y \to Z$ is continuous at $f(p)$. Then the composition $g \circ f\colon X \to Z$ of the functions f and g is continuous at p.

Proof

Let N be a neighbourhood of $g(f(p))$ in Z. Then $g^{-1}(N)$ is a neighbourhood of $f(p)$ in Y (because g is continuous), and then $f^{-1}(g^{-1}(N))$ is a neighbourhood of p in X (because f is continuous). But $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$. Thus the composition function $g \circ f$ is continuous at p. ∎

Proposition 1.23

Let X and Y be topological spaces and let $f: X \to Y$ be a function from X to Y. Then $f: X \to Y$ is continuous if and only if, given any point p of X, there exists some open set W in X such that $p \in W$ and the restriction $f|W: W \to Y$ of the function f to W is continuous on W.

Proof

Suppose that $f: X \to Y$ is continuous. Let W be an open set in X, and let V be an open set in Y. Then the preimage $f^{-1}(V)$ of V is open in X. Now $(f|W)^{-1}(V) = f^{-1}(V) \cap W$. It follows that $(f|W)^{-1}(V)$ is open with respect to the subspace topology on W.

Conversely suppose that, given any point p of X, there exists an open set W in X such that $p \in W$ and $f|W \colon W \to Y$ is continuous. Let p be a point of X and let W be an open set in X for which $p \in W$ and $f|W \colon \colon W \to Y$ is continuous. Let N be a neighbourhood of $f(p)$ in Y. Then $(f|W)^{-1}(N)$ is a neighbourhood of p in W. It follows from the definition of the subspace topology on W that there exists an open set E in X for which $p \in E$ and $f(E \cap W) \subset N$. But then $E \cap W$ is an open set in X, because both E and W are open sets in X. It follows that $f^{-1}(N)$ is an open neighbourhood of p in X. We have thus shown that the function f is continuous at p. It then follows from Proposition 1.20 that $f \colon X \to Y$ is continuous, as required. ∎

1.11. The Pasting Lemma

We now show that, if a topological space X is the union of a finite collection of closed sets, and if a function from X to some topological space is continuous on each of these closed sets, then that function is continuous on X. The names *Pasting Lemma* and *Gluing Lemma* are both used to refer to this result.

Lemma 1.24 (Pasting Lemma)

Let X and Y be topological spaces, let $f\colon X \to Y$ be a function from X to Y, and let $X = A_1 \cup A_2 \cup \cdots \cup A_k$, where $A_1, A_2, \ldots, A_k$ are closed sets in X. Suppose that the restriction of f to the closed set A_i is continuous for $i = 1, 2, \ldots, k$. Then $f\colon X \to Y$ is continuous.

Proof

Let p be a point of X, and let N be a neighbourhood of $f(p)$. The continuity of the restriction of f to each closed set A_i ensures the existence of open sets W_i for $i = 1, 2, \ldots, k$ such that $W_i \cap A_i = \emptyset$ whenever $p \notin A_i$ and $f(W_i \cap A_i) \subset N$ whenever $p \in A_i$. Let

$$W = W_1 \cap W_2 \cap \cdots \cap W_k$$

Then W is an open set in X, and $p \in W$. Moreover if $x \in W$ then there exists some integer i between 1 and k for which $x \in A_i$ and $p \in A_i$. Then $x \in W_i \cap A_i$, and therefore $f(x) \in N$. We conclude from this that the function f is continuous at each point p of X. It follows that the function f is continuous on X (see Proposition 1.20). ∎

Alternative Proof

A function $f\colon X \to Y$ is continuous if and only if $f^{-1}(G)$ is closed in X for every closed set G in Y (Lemma 1.19). Let G be an closed set in Y. Then $f^{-1}(G) \cap A_i$ is closed in the subspace topology on A_i for $i = 1, 2, \ldots, k$, because the restriction of f to A_i is continuous for each i. But A_i is closed in X, and therefore a subset of A_i is closed in A_i if and only if it is closed in X (see Lemma 1.15). Therefore $f^{-1}(G) \cap A_i$ is closed in X for $i = 1, 2, \ldots, k$. Now $f^{-1}(G)$ is the union of the sets $f^{-1}(G) \cap A_i$ for $i = 1, 2, \ldots, k$. It follows that $f^{-1}(G)$, being a finite union of closed sets, is itself closed in X. It now follows from Lemma 1.19 that $f\colon X \to Y$ is continuous. ∎

Example

Let Y be a topological space, and let $\alpha\colon [0,1] \to Y$ and $\beta\colon [0,1] \to Y$ be continuous functions defined on the interval $[0,1]$, where $\alpha(1) = \beta(0)$. Let $\gamma\colon [0,1] \to Y$ be defined by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now $\gamma|[0,\frac{1}{2}] = \alpha \circ \rho$ where $\rho\colon [0,\frac{1}{2}] \to [0,1]$ is the continuous function defined by $\rho(t) = 2t$ for all $t \in [0,\frac{1}{2}]$. Thus $\gamma|[0,\frac{1}{2}]$ is continuous, being a composition of two continuous functions. Similarly $\gamma|[\frac{1}{2},1]$ is continuous. The subintervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ are closed in $[0,1]$, and $[0,1]$ is the union of these two subintervals. It follows from Lemma 1.24 that $\gamma\colon [0,1] \to Y$ is continuous.

Example

Let X be the surface of a closed cube in $\mathbb{R}^3$ and let $f: X \to Y$ be a function mapping X into a topological space Y. The topological space X is the union of the six square faces of the cube, and each of these faces is a closed subset of X. The Pasting Lemma Lemma 1.24 ensures that the function f is continuous if and only if its restrictions to each of the six faces of the cube is continuous on that face.

We now present a couple of examples to show that the conclusions of the Pasting Lemma (Lemma 1.24) do not follow when the conditions stated in that lemma are relaxed.

Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined so that

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases}$$

and let $A_1 = \{x \in \mathbb{R} : x \leq 0\}$ and $A_2 = \{x \in \mathbb{R} : x > 0\}$. The restriction of the function f to each of the subsets A_1 and A_2 of $\mathbb{R}$ is continuous on that subset, but the function f itself is not continuous on $\mathbb{R}$. This does not contradict the Pasting Lemma because the subset A_2 of $\mathbb{R}$ is not closed in $\mathbb{R}$.

Example

Let

$$X = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{Z} \text{ and } n > 0 \right\},$$

and let $f : X \to \mathbb{R}$ be defined so that $f(0) = 0$ and $f(1/n) = n$ for all positive integers n. For each $x \in X$, the set $\{x\}$ is a closed subset of X, and the restriction of f to each of these one-point subsets is continuous on that subset. But the function f itself is not continuous on X. This does not contradict the Pasting Lemma because the number of these one-point closed subsets of X is infinite.

1.12. Continuous Functions between Metric Spaces

The following proposition shows that the definition of continuity for functions between topological spaces is consistent with the standard definition of continuity for functions between metric spaces that is expressed directly in terms of distance functions on those metric spaces.

Proposition 1.25

Let X and Y be metric spaces with distance functions d_X and d_Y respectively, let $f\colon X \to Y$ be a function from X to Y, and let p be a point of X. Then the following two conditions are equivalent:

(i) *given any neighbourhood N of $f(p)$ in Y, there exists a neighbourhood M of p in X for which $f(M) \subset N$;*

(ii) *given any positive real number ε, there exists some positive real number δ such that $d_Y(f(x), f(p)) < \varepsilon$ for all points x of X for which $d(x, p) < \delta$.*

(iii) *the function $f\colon X \to Y$ is continuous at p.*

Proof

Suppose that, given any neighbourhood N of $f(p)$ in Y, there exists a neighbourhood M of p for which $f(M) \subset N$. Let some positive real number ε be given. Then the open ball $B_Y(f(p), \varepsilon)$ of radius ε about the point $f(p)$ is a neighbourhood of $f(p)$ in Y. It follows that there exists a neighbourhood M of p for which $f(M) \subset B_Y(f(p), \varepsilon)$. There then exists some positive real number δ such that $B_X(p, \delta) \subset M$ (see Lemma 1.8). If $x \in X$ satisfies $d_X(x, p) < \delta$ then $x \in M$ and therefore $f(x) \in B_Y(f(p), \varepsilon)$. But then $d_Y(f(x), f(p)) < \varepsilon$. Thus (i) implies (ii).

Conversely suppose that, given any positive real number ε, there exists some positive real number δ such that $d_Y(f(x), f(p)) < \varepsilon$ for all points x of X for which $d(x, p) < \delta$. Let N be a neighbourhood of $f(p)$. Then there exists some positive real number ε for which $B_Y(f(p), \varepsilon) \subset N$, where $B_Y(f(p), \varepsilon)$ denotes the open ball of radius ε about the point $f(p)$. There then exists some positive real number δ for which $f(B_X(p, \delta)) \subset B_Y(f(p), \varepsilon)$, where $B_X(p, \delta)$ denotes the open ball of radius δ about the point p. Let $M = B_X(p, \delta)$. Then M is a neighbourhood of p in X and $f(M) \subset N$. Thus (ii) implies (i).

The equivalence of (i) and (iii), for functions between general topological spaces, was proved in Lemma 1.21. This completes the proof. ∎

1.13. Homeomorphisms

Definition

Let X and Y be topological spaces. A function $h\colon X \to Y$ is said to be a *homeomorphism* if and only if the following conditions are satisfied:

- the function $h\colon X \to Y$ is both injective and surjective (so that the function $h\colon X \to Y$ has a well-defined inverse $h^{-1}\colon Y \to X$),
- the function $h\colon X \to Y$ and its inverse $h^{-1}\colon Y \to X$ are both continuous.

Two topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism $h\colon X \to Y$ from X to Y.

If $h\colon X \to Y$ is a homeomorphism between topological spaces X and Y then h induces a one-to-one correspondence between the open sets of X and the open sets of Y. Thus the topological spaces X and Y can be regarded as being identical as topological spaces.