MA342R—Covering Spaces and Fundamental Groups
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1.7. Neighbourhoods and Closures in Metric Spaces

Lemma 1.8

Let X be a metric space with distance function d, let p be a point of X and let N be a subset of X, where $p \in N$. Then N is a neighbourhood of p in X if and only if there exists some positive real number δ for which

$$\{x \in X : d(x, p) < \delta\} \subset N.$$

Proof

Let $B_X(p, \delta) = \{x \in X : d(x, p) < \delta\}$ for all positive real numbers δ. Then the open ball $B_X(p, \delta)$ in X of radius δ about the point p is an open set in X (see Lemma 1.1). It follows from the definition of neighbourhoods of points in topological spaces that if there exists some positive real number δ for which $B_X(p, \delta) \subset N$ then N is a neighbourhood of p in X.

Conversely suppose that N is a neighbourhood of p in X. Then there exists an open set W in X such that $p \in W$ and $W \subset N$. The definition of open sets in metric spaces then ensures the existence of a positive real number δ for which $B_X(p, \delta) \subset W$. Then $B_X(p, \delta) \subset N$. The result follows. $\blacksquare$

Lemma 1.9

Let X be a metric space with distance function d, let A be a subset of X, and let p be a point of X. Then p belongs to the closure $\overline{A}$ of A in X if and only if, given any positive real number δ, there exists some element x of A that satisfies $d(x, p) < \delta$.

Proof

The complement of the closure $\overline{A}$ of A is the interior of the complement $X \setminus A$ of A (see Proposition 1.7). It follows that $p \in \overline{A}$ if and only if p does not belong to the interior of $X \setminus A$. Now a point of X belongs to the interior of $X \setminus A$ if and only if $X \setminus A$ is a neighbourhood of that point (see Lemma 1.5). It follows that $p \in \overline{A}$ if and only if $X \setminus A$ is not a neighbourhood of p in X. It then follows from Lemma 1.8 that $p \in \overline{A}$ if and only if, for all positive real numbers δ, the open ball in X of radius δ about the point p intersects A. The result follows. ∎

1.8. Subspace Topologies

Lemma 1.10

Let X be a topological space with topology τ, and let A be a subset of X. Let τ_A be the collection of all subsets of A that are of the form $V \cap A$ for $V \in \tau$. Then τ_A is a topology on the set A.

Proof

The empty set $\emptyset$ belongs to τ_A, because $\emptyset$ is open in X and $\emptyset = A \cap \emptyset$. Also $A \in \tau_A$, because X is open in itself and $A = X \cap A$.

Let $\mathcal{C}$ be a collection of subsets of A, where $W \in \tau_A$ for all $W \in \mathcal{C}$, and let Y be the union of the subsets of A belonging to the collection $\mathcal{C}$. Then for each $W \in \mathcal{C}$ there exists an open set V_W in X for which $W = A \cap V_W$. Let Z be the union of the open sets V_W as W ranges over the collection $\mathcal{C}$. Then

$$Y = \bigcup_{W \in \mathcal{C}} W = \bigcup_{W \in \mathcal{C}} (A \cap V_W) = A \cap \bigcup_{W \in \mathcal{C}} V_W = A \cap Z.$$

Moreover Z is open in X. It follows that $Y \in \tau_A$. Thus any union of subsets of A belonging to τ_A must itself belong to τ_A.

Now let $W_1, W_2, \ldots, W_m$ be subsets of A that each belong to the collection τ_A. Then there exist open sets $V_1, V_2, \ldots, V_m$ in X such that $W_i = A \cap V_i$ for $i = 1, 2, \ldots, m$. Then

$$W_1 \cap W_2 \cap \cdots \cap W_r = A \cap V,$$

where

$$V = V_1 \cap V_2 \cap \cdots \cap V_r.$$

Now V is a finite intersection of subsets of X that are open in X. It follows that V is itself open in X, and therefore

$$W_1 \cap W_2 \cap \cdots \cap W_r \in \tau_A.$$

We have thus shown that τ_A is a topology on A, as required. ∎

Definition

Let X be a topological space and let A be a subset of X. The *subspace topology* on A is the topology on A whose open sets are characterized by the following criterion:

> *A subset W of A is open with respect to the subspace topology on A if and only if there exists some open set V in X for which $W = A \cap V$.*

Proposition 1.11

Let X be a metric space with distance function d, let A be a subset of X, let p be a point of A and let N be a subset of A for which $p \in N$. Then N is a neighbourhood of p with respect to the subspace topology on A if and only if there exists some positive real number δ such that

$$\{x \in A : d(x, p) < \delta\} \subset N.$$

Proof

Let

$$B_A(p, \delta) = \{x \in A : d(x, p) < \delta\}$$

and

$$B_X(p, \delta) = \{x \in X : d(x, p) < \delta\}$$

for all positive real numbers δ.

Suppose that there exists some positive real number δ for which $B_A(p,\delta) \subset N$. We must show that N is a neighbourhood of p with respect to the subspace topology on A. Now $B_A(p,\delta) = A \cap B_X(p,\delta)$, where $B_X(p,\delta)$ is the open ball in X of radius δ about the point p. Moreover $B_X(p,\delta)$ is open in X (Lemma 1.1) and $A \cap B_X(p,\delta) \subset N$. It follows that N is a neighbourhood of p in A with respect to the subspace topology on A.

Conversely suppose that N is a neighbourhood of p with respect to the subspace topology on A. We must show that there exists some positive real number δ for which $B_A(p,\delta) \subset N$. Now the definitions of neighbourhoods and the subspace topology together ensure the existence of an open set V in X for which $p \in V$ and $A \cap V \subset N$. It then follows from the definition of open sets in metric spaces that there exists some positive real number δ for which $B_X(p,\delta) \subset V$. Then $B_A(p,\delta) \subset A \cap V \subset N$. This completes the proof. ∎

Corollary 1.12

Let X be a metric space with distance function d, and let A be a subset of X. A subset W of A is open with respect to the subspace topology on A if and only if, given any point w of W, there exists some positive real number δ for which

$$\{a \in A : d(a, w) < \delta\} \subset W.$$

Thus the subspace topology on A coincides with the topology on A obtained on regarding A as a metric space whose distance function is the restriction to A of the distance function d on X.

Proof

The subset W is open in A with respect to a given topology on A if and only if it is a neighbourhood of all of its points with respect to that given topology (see Lemma 1.4). The required result therefore follows from Proposition 1.11. ∎

Example

Let X be any subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then the subspace topology on X coincides with the topology on X generated by the Euclidean distance function on X. We refer to this topology as the *usual topology* on X.

Lemma 1.13

Let X be a topological space, let A be a subset of X, and let B be a subset of A. Then B is closed in A (relative to the subspace topology on A) if and only if $B = A \cap F$ for some closed subset F of X.

Proof

Suppose that $B = A \cap F$ for some closed subset F of X. Let $V = X \setminus F$. Then V is an open set in X, and

$$A \setminus B = A \setminus (A \cap F) = A \cap (X \setminus F) = A \cap V.$$

Moreover the definition of the subpace topology on A ensures that $A \cap V$ is open in A. Thus the complement $A \setminus B$ of B in A is open in A, and therefore the subset B of A is itself closed in A.

Conversely suppose that B is closed in A. Then $A \setminus B$ is open in the subspace topology on A, and therefore there exists some open set V in X such that $A \setminus B = A \cap V$. Let $F = X \setminus V$. Then F is closed in X, and

$$A \cap F = A \cap (X \setminus V) = A \setminus (A \cap V) = A \setminus (A \setminus B) = B.$$

The result follows. $\blacksquare$

Lemma 1.14

Let X be a topological space, let V be an open set in X, and let W be a subset of V. Then W is open in V if and only if W is open in X.

Proof

If W is open in X then $W = V \cap W$ and therefore W is open in V.

Conversely suppose that the set W is open in V. It then follows from the definition of subspace topologies that $W = V \cap E$ for some open set E in X. But then W is an intersection of two open sets, and is thus itself open in X. $\blacksquare$

Lemma 1.15

Let X be a topological space, let F be a closed set in X, and let G be a subset of F. Then G is closed in F if and only if G is closed in X.

Proof

If G is closed in X then $G = F \cap G$ and therefore G is closed in F.

Conversely suppose that the set G is closed in F. It then follows from Lemma 1.13 that $G = F \cap H$ for some closed set H in X. But then G is an intersection of two closed sets, and is thus itself closed in X (see Proposition 1.3). ∎

1.9. Hausdorff Spaces

Definition

A topological space X is said to be a *Hausdorff space* if and only if it satisfies the following *Hausdorff Axiom*:

- if x and y are distinct points of X then there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Lemma 1.16

Any subset of a Hausdorff space is itself a Hausdorff space (with respect to the subspace topology).

Proof

Let A be a subset of a Hausdorff space X and let x and y be distinct points of A. Then there exist open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Let $U_A = A \cap U$ and $V_A = A \cap V$. Then U_A and V_A are subsets of A that are open in the subspace topology on A. Moreover $x \in U_A$, $y \in V_A$ and $U_A \cap V_A = \emptyset$. The result follows. ∎

Lemma 1.17

All metric spaces are Hausdorff spaces.

Proof

Let X be a metric space with distance function d, and let x and y be points of X, where $x \neq y$. Let $\varepsilon = \frac{1}{2}d(x, y)$. Then the open balls $B_X(x, \varepsilon)$ and $B_X(y, \varepsilon)$ of radius ε centred on the points x and y are open sets (see Lemma 1.1). If $B_X(x, \varepsilon) \cap B_X(y, \varepsilon)$ were non-empty then there would exist $z \in X$ satisfying $d(x, z) < \varepsilon$ and $d(z, y) < \varepsilon$. But this is impossible, since it would then follow from the Triangle Inequality that $d(x, y) < 2\varepsilon$, contrary to the choice of ε. Thus $x \in B_X(x, \varepsilon)$, $y \in B_X(y, \varepsilon)$, $B_X(x, \varepsilon) \cap B_X(y, \varepsilon) = \emptyset$. This shows that the metric space X is a Hausdorff space. ∎

We now give an example of a topological space which is not a Hausdorff space.

Example

Let X be an infinite set. The *cofinite topology* on X is defined as follows: a subset U of X is open (with respect to the cofinite topology) if and only if either $U = \emptyset$ or else $X \setminus U$ is finite. It is a straightforward exercise to verify that the topological space axioms are satisfied, so that the set X is a topological space with respect to this cofinite topology. Now the intersection of any two non-empty open sets in this topology is always non-empty. (Indeed if U and V are non-empty open sets then $U = X \setminus F_1$ and $V = X \setminus F_2$, where F_1 and F_2 are finite subsets of X. But then $U \cap V = X \setminus (F_1 \cup F_2)$, which is non-empty, since $F_1 \cup F_2$ is finite and X is infinite.) It follows immediately from this that an infinite set X is not a Hausdorff space with respect to the the cofinite topology on X.