MA342R—Covering Spaces and Fundamental Groups
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1. Results concerning Topological Spaces

1.1. Topological Spaces

Definition

A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

- **(i)** the empty set $\emptyset$ and the whole set X are open sets,
- **(ii)** the union of any collection of open sets is itself an open set,
- **(iii)** the intersection of any *finite* collection of open sets is itself an open set.

The collection consisting of all the open sets in a topological space X is referred to as a *topology* on the set X.

Remark

If it is necessary to specify explicitly the topology on a topological space then one denotes by (X, τ) the topological space whose underlying set is X and whose topology is τ. However if no confusion will arise then it is customary to denote this topological space simply by X.

1.2. Subsets of Euclidean Space

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. The *Euclidean distance* $|\mathbf{x} - \mathbf{y}|$ between two points $\mathbf{x}$ and $\mathbf{y}$ of X is defined as follows:

$$|\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. The Euclidean distances between any three points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of X satisfy the *Triangle Inequality*:

$$|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|.$$

A subset V of X is said to be *open* in X if, given any point $\mathbf{v}$ of V, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{v}| < \delta\} \subset V.$$

The empty set is also considered to be open in X.

Both $\emptyset$ and X are open sets in X. Also it is not difficult to show that any union of open sets in X is open in X, and that any finite intersection of open sets in X is open in X. (This will be proved in more generality for open sets in metric spaces.) Thus the collection of open sets in a subset X of a Euclidean space $\mathbb{R}^n$ satisfies the topological space axioms. Thus every subset X of $\mathbb{R}^n$ is a topological space with these open sets. This topology on a subset X of $\mathbb{R}^n$ is referred to as the *usual topology* on X, generated by the Euclidean distance function.

In particular $\mathbb{R}^n$ is itself a topological space.

1.3. Open Sets in Metric Spaces

Definition

A *metric space* (X, d) consists of a set X together with a *distance function* $d\colon X \times X \to [0, +\infty)$ on X satisfying the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

The quantity $d(x, y)$ should be thought of as measuring the *distance* between the points x and y. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ is referred to as the *Triangle Inequality*. The elements of a metric space are usually referred to as *points* of that metric space.

An n-dimensional Euclidean space $\mathbb{R}^n$ is a metric space with with respect to the *Euclidean distance function d*, defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Any subset X of $\mathbb{R}^n$ may be regarded as a metric space whose distance function is the restriction to X of the Euclidean distance function on $\mathbb{R}^n$ defined above.

Definition

Let (X, d) be a metric space. Given a point x of X and $r \geq 0$, the *open ball* $B_X(x, r)$ of *radius* r about x in X is defined by

$$B_X(x, r) = \{x' \in X : d(x', x) < r\}.$$

Definition

Let (X, d) be a metric space. A subset V of X is said to be an *open set* if and only if the following condition is satisfied:

- given any point v of V there exists some positive real number δ such that $B_X(v, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in this case.)

Lemma 1.1

Let X be a metric space with distance function d, and let x_0 be a point of X. Then, for any $r > 0$, the open ball $B_X(x_0, r)$ of radius r about x_0 is an open set in X.

Proof

Let $x \in B_X(x_0, r)$. We must show that there exists some positive real number δ such that $B_X(x, \delta) \subset B_X(x_0, r)$. Now $d(x, x_0) < r$, and hence $\delta > 0$, where $\delta = r - d(x, x_0)$. Moreover if $x' \in B_X(x, \delta)$ then

$$d(x', x_0) \leq d(x', x) + d(x, x_0) < \delta + d(x, x_0) = r,$$

by the Triangle Inequality, hence $x' \in B_X(x_0, r)$. Thus $B_X(x, \delta) \subset B_X(x_0, r)$, showing that $B_X(x_0, r)$ is an open set, as required. ∎

Proposition 1.2

Let X be a metric space. The collection of open sets in X has the following properties:—

- **(i)** *the empty set $\emptyset$ and the whole set X are both open sets;*
- **(ii)** *the union of any collection of open sets is itself an open set;*
- **(iii)** *the intersection of any* finite *collection of open sets is itself an open set.*

Proof

The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. Thus (i) is satisfied.

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself an open set. Let $x \in U$. Then $x \in V$ for some open set V belonging to the collection $\mathcal{A}$. Therefore there exists some positive real number δ such that $B_X(x, \delta) \subset V$. But $V \subset U$, and thus $B_X(x, \delta) \subset U$. This shows that U is open. Thus (ii) is satisfied.

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of open sets in X, and let $V = V_1 \cap V_2 \cap \cdots \cap V_k$. Let $x \in V$. Now $x \in V_j$ for all j, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(x, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of open sets.) Moreover $B_X(x, \delta) \subset B_X(x, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(x, \delta) \subset V$. This shows that the intersection V of the open sets $V_1, V_2, \ldots, V_k$ is itself open. Thus (iii) is satisfied. ∎

Any metric space may be regarded as a topological space. Indeed let X be a metric space with distance function d. We recall that a subset V of X is an *open set* if and only if, given any point v of V, there exists some positive real number δ such that

$$\{x \in X : d(x, v) < \delta\} \subset V.$$

Proposition 1.2 shows that the topological space axioms are satisfied by the collection of open sets in any metric space. We refer to this collection of open sets as the *topology* generated by the distance function d on X.

1.4. Further Examples of Topological Spaces

Example

Given any set X, one can define a topology on X where every subset of X is an open set. This topology is referred to as the *discrete topology* on X.

Example

Given any set X, one can define a topology on X in which the only open sets are the empty set $\emptyset$ and the whole set X.

1.5. Closed Sets

Definition

Let X be a topological space. A subset F of X is said to be a *closed set* if and only if its complement $X \setminus F$ is an open set.

We recall that the complement of the union of some collection of subsets of some set X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets. The following result therefore follows directly from the definition of a topological space.

Proposition 1.3

Let X be a topological space. Then the collection of closed sets of X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are closed sets,*

(ii) *the intersection of any collection of closed sets is itself a closed set,*

(iii) *the union of any finite collection of closed sets is itself a closed set.*

1.6. Neighbourhoods, Closures and Interiors

Definition

Let X be a topological space, and let x be a point of X. Let N be a subset of X which contains the point x. Then N is said to be a *neighbourhood* of the point x if and only if there exists an open set W for which $x \in W$ and $W \subset N$.

Lemma 1.4

Let X be a topological space. A subset V of X is open in X if and only if V is a neighbourhood of each point belonging to V.

Proof

It follows directly from the definition of neighbourhoods that an open set V is a neighbourhood of any point belonging to V. Conversely, suppose that V is a subset of X which is a neighbourhood of each $v \in V$. Then, given any point v of V, there exists an open set W_v such that $v \in W_v$ and $W_v \subset V$. Thus V is an open set, since it is the union of the open sets W_v as v ranges over all points of V. ∎

Definition

Let X be a topological space and let A be a subset of X. The *interior* A° of A in X is defined to be the union of all of the open subsets of X that are subsets of A.

Let X be a topological space and let A be a subset of X. It follows from the definition of a topological space that the union of open subsets of X is itself a open subset of X. It follows directly from this that the interior A° of A in X is the subset of X uniquely characterized by the following two properties:—

(i) the interior A° of A is an open set contained in A,

(ii) if W is any open set contained in A then W is contained in A°.

Lemma 1.5

Let X be a topological space, let A be a subset of X, and let p be a point of A. Then p belongs to the interior A° if and only if A is a neighbourhood of the point p.

Proof

It follows from the definition of interiors that the point p belongs to the interior of A if and only if there exists an open set W such that $p \in W$ and $W \subset A$. It then follows from the definition of neighbourhoods that this is the case if and only if the set A is a neighbourhood of the point p. ∎

Definition

Let X be a topological space and let A be a subset of X. The *closure* $\overline{A}$ of A in X is defined to be the intersection of all of the closed subsets of X that contain A.

Let X be a topological space and let A be a subset of X. Any intersection of closed subsets of X is itself a closed subset of X (see Proposition 1.3). It follows directly from this that the closure $\overline{A}$ of A in X is the subset of X uniquely characterized by the following two properties:—

- **(i)** the closure $\overline{A}$ of A is a closed set containing A,
- **(ii)** if F is any closed set containing A then F contains $\overline{A}$.

Lemma 1.6

Let X be a topological space, let A be a subset of X, let $\overline{A}$ be the closure of A in X, and let V be an open set. Then $V \cap A = \emptyset$ if and only if $V \cap \overline{A} = \emptyset$.

Proof

Suppose that $V \cap A = \emptyset$. Then $A \subset X \setminus V$. Now the complement $X \setminus V$ of V is a closed set, and $\overline{A}$ is by definition the intersection of all closed sets that contain the subset A. It follows that $\overline{A} \subset X \setminus V$, and therefore $V \cap \overline{A} = \emptyset$.

Conversely suppose that $V \cap \overline{A} = \emptyset$. Then $V \cap A = \emptyset$, because A is a subset of $\overline{A}$. The result follows. ∎

Proposition 1.7

Let X be a topological space, and let A be a subset of X. Let A° and $\overline{A}$ denote the interior and closure respectively of A, and let $(X \setminus A)^\circ$ and $\overline{X \setminus A}$ denote the interior and closure respectively of the complement $X \setminus A$ of A in X. Then

$$X \setminus \overline{A} = (X \setminus A)^\circ \quad \text{and} \quad X \setminus A^\circ = \overline{X \setminus A}$$

(i.e., the complement of the closure of A is the interior of the complement of A, and the complement of the interior of A is the closure of the complement of A).

Proof

The interior $(X \setminus A)^\circ$ of $X \setminus A$ is by definition the union of all open subsets of X that are contained in $X \setminus A$. But an open subset V is contained in $X \setminus A$ if and only if $V \cap A = \emptyset$. It follows from Lemma 1.6 that $V \subset X \setminus A$ if and only if $V \subset X \setminus \overline{A}$. We conclude from this that $(X \setminus A)^\circ \subset X \setminus \overline{A}$. But $X \setminus \overline{A}$ is itself an open set contained in $X \setminus A$, and therefore $X \setminus \overline{A} \subset (X \setminus A)^\circ$. It follows that

$$(X \setminus A)^\circ = X \setminus \overline{A}.$$

Similarly $(X \setminus B)^\circ = X \setminus \overline{B}$, where $B = X \setminus A$, and thus $A^\circ = X \setminus \overline{B}$. Taking complements, we find that

$$X \setminus A^\circ = \overline{B} = \overline{X \setminus A}.$$

This completes the proof. ∎