Module MA342R: Covering Spaces and Fundamental Groups
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Section 6: The Topology of Closed Surfaces

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6 The Topology of Closed Surfaces

6.1 Affine Independence

Definition Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *affinely independent* (or *geometrically independent*) if the only solution of the linear system

$$\begin{cases} \sum_{j=0}^{q} s_j \mathbf{v}_j &= \mathbf{0}, \\ \sum_{j=0}^{q} s_j &= 0 \end{cases}$$

is the trivial solution $s_0 = s_1 = \cdots = s_q = 0$.

Lemma 6.1 *Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be points of Euclidean space $\mathbb{R}^k$ of dimension k. Then the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent if and only if the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.*

Proof Suppose that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent. Let $s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equation

$$\sum_{j=1}^{q} s_j(\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

Then $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$, where $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$s_0 = s_1 = \cdots = s_q = 0.$$

It follows that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Conversely, suppose that these displacement vectors are linearly independent. Let $s_0, s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equations $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$. Then $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$\mathbf{0} = \sum_{j=0}^{q} s_j \mathbf{v}_j = s_0 \mathbf{v}_0 + \sum_{j=1}^{q} s_j \mathbf{v}_j = \sum_{j=1}^{q} s_j(\mathbf{v}_j - \mathbf{v}_0).$$

It follows from the linear independence of the displacement vectors $\mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$ that

$$s_1 = s_2 = \cdots = s_q = 0.$$

But then $s_0 = 0$ also, because $s_0 = -\sum_{j=1}^{q} s_j$. It follows that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent, as required. ∎

It follows from Lemma 6.1 that any set of affinely independent points in $\mathbb{R}^k$ has at most $k+1$ elements. Moreover if a set consists of affinely independent points in $\mathbb{R}^k$, then so does every subset of that set.

6.2 Simplices in Euclidean Spaces

Definition A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent points of $\mathbb{R}^k$. The points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are referred to as the *vertices* of the simplex. The non-negative integer q is referred to as the *dimension* of the simplex.

Example A 0-simplex in a Euclidean space $\mathbb{R}^k$ is a single point of that space.

Example A 1-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least one is a line segment in that space. Indeed let λ be a 1-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}$ and $\mathbf{w}$. Then

$$\begin{aligned} \lambda &= \{s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq s \leq 1,\ 0 \leq t \leq 1 \text{ and } s + t = 1\} \\ &= \{(1-t)\mathbf{v} + t\,\mathbf{w} : 0 \leq t \leq 1\}, \end{aligned}$$

and thus λ is a line segment in $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$.

Example A 2-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least two is a triangle in that space. Indeed let τ be a 2-simplex in $\mathbb{R}^k$ with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Then

$$\tau = \{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq r, s, t \leq 1 \text{ and } r + s + t = 1\}.$$

Let $\mathbf{x} \in \tau$. Then there exist $r, s, t \in [0, 1]$ such that $\mathbf{x} = r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w}$ and $r + s + t = 1$. If $r = 1$ then $\mathbf{x} = \mathbf{u}$. Suppose that $r < 1$. Then

$$\mathbf{x} = r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big)$$

where $p = \dfrac{t}{1-r}$. Moreover $0 < r \leq 1$ and $0 \leq p \leq 1$. Moreover the above formula determines a point of the 2-simplex τ for each pair of real numbers r and p satisfying $0 \leq r \leq 1$ and $0 \leq p \leq 1$. Thus

$$\tau = \left\{ r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big) : 0 \leq p, r \leq 1. \right\}.$$

Now the point $(1-p)\mathbf{v} + p\mathbf{w}$ traverses the line segment $\mathbf{v}\,\mathbf{w}$ from $\mathbf{v}$ to $\mathbf{w}$ as p increases from 0 to 1. It follows that τ is the set of points that lie on line segments with one endpoint at $\mathbf{u}$ and the other at some point of the line segment $\mathbf{v}\,\mathbf{w}$. This set of points is thus a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

A 3-dimensional simplex is a tetrahedron. Higher-dimensional simplices are the higher-dimensional analogues of points, line segments, triangles and tetrahedra.

6.3 Faces of Simplices

Definition Let σ and τ be simplices in $\mathbb{R}^k$. We say that τ is a *face* of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a *proper face* if it is not equal to σ itself. An r-dimensional face of σ is referred to as an *r-face* of σ. A 1-dimensional face of σ is referred to as an *edge* of σ.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

6.4 Simplicial Complexes in Euclidean Spaces

Definition A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The *dimension* of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex. The union of all the simplices of K is a compact subset $|K|$ of $\mathbb{R}^k$ referred to as the *polyhedron* of K. (The polyhedron is compact since it is both closed and bounded in $\mathbb{R}^k$.)

Example Let K_σ consist of some n-simplex σ together with all of its faces. Then K_σ is a simplicial complex of dimension n, and $|K_\sigma| = \sigma$.

Lemma 6.2 *Let K be a simplicial complex, and let X be a subset of some Euclidean space. A function $f\colon |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.*

Proof Each simplex of the simplicial complex K is a closed subset of the polyhedron $|K|$ of the simplicial complex K. The numbers of simplices belonging to the simplicial complex is finite. The result therefore follows from a straightforward application of the Pasting Lemma (Lemma 1.24). ∎

Lemma 6.3 *Let K be a finite collection of triangles, edges and points in some Euclidean space. Then K is a two-dimensional simplicial complex if and only if the following conditions are all satisfied:—*

(i) the edges and vertices of any triangle belonging to K themselves belong to K;

(ii) the endpoints of any edge belonging to K are vertices belonging to K;

(iii) if two distinct triangles belonging to K have a non-empty intersection, then that intersection is either a single common edge or a single common vertex of both triangles;

(iv) if a triangle belonging to K intersects an edge belonging to K then either the edge is an edge of the triangle or else the intersection of the triangle and edge is a vertex of the triangle that is an endpoint of the edge;

(v) if two distinct edges belonging to K have a non-empty intersection then that intersection is a common vertex (or endpoint) of both edges;

(vi) if a vertex belongs to a triangle then it is a vertex of that triangle, and if a vertex belongs to an edge then it is an endpoint of that edge.

Proof Consider a finite collection K of simplices of dimension two in a Euclidean space. The simplices belonging to K are points, line segments or triangles. Conditions (i) and (ii) in the statement of the lemma are equivalent to the condition that every face of a simplex belonging to the collection K must itself belong to that collection. Similarly conditions (iii), (iv), (v) and (vi) in the statement of the lemma are equivalent to the condition that any two simplices of K whose intersection is non-empty intersect in a common face. The result therefore follows from the definition of a simplicial complex, applied in the special case where the simplices of the complex are of dimension at most two. ∎

Let K be a two-dimensional simplicial complex in some Euclidean space. The *polyhedron* $|K|$ of K is the union of all the triangles, edges and vertices belonging to the collection K.

Lemma 6.4 *The polyhedron of a two-dimensional simplicial complex is a compact Hausdorff space.*

Proof The simplicial complex K is a finite collection of triangles, edges and vertices in some ambient Euclidean space, and each triangle, edge and vertex in the collection is a closed bounded subset of this ambient Euclidean space. Now a subset of a Euclidean space is compact if and only if it is both closed and bounded. It follows that each of the triangles, edges and vertices belonging to K is a compact subset of the ambient Euclidean space. Moreover it follows directly from the definition of compactness that any finite union of compact topological spaces is itself compact. Therefore the polyhedron $|K|$ of K is a compact subset of the ambient Euclidean space. This ambient Euclidean space is a Hausdorff space (as it is a metric space, and all metric spaces are Hausdorff spaces), and any subset of a Hausdorff space is itself a Hausdorff space (with the subspace topology). Therefore the polyhedron $|K|$ of K is a compact Hausdorff space, as required. ∎

Definition Let $\mathbf{p}$ be a point of the polyhedron $|K|$ of the two-dimensional simplicial complex K. The *star neighbourhood* $\mathrm{st}_K(\mathbf{p})$ of the point $\mathbf{p}$ in $|K|$ is defined to be the subset of $|K|$ whose complement is the union of all triangles, edges and vertices belonging to K that do not contain the point $\mathbf{p}$.

Lemma 6.5 *Let K be a two-dimensional simplicial complex, and let $\mathbf{p}$ be a point of K. Then the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ of the point $\mathbf{p}$ of $|K|$ is an open subset of $|K|$, and moreover $\mathbf{p} \in \mathrm{st}_K(\mathbf{p})$.*

Proof A two-dimensional simplicial complex is a finite collection of triangles, edges and vertices in some ambient Euclidean space. Each of those triangles, edges and vertices is a closed subset of the ambient Euclidean space, and therefore the union of any finite collection of such triangles, edges and vertices is a closed subset of the ambient Euclidean space.

Now, given any point $\mathbf{p}$ of $|K|$, the complement $|K| \setminus \mathrm{st}_K(\mathbf{p})$ of the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ of $\mathbf{p}$ in $|K|$ is by definition the union of all triangles, edges and vertices belonging to K that do not contain the point $\mathbf{p}$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{p})$ is closed in $|K|$, and $\mathbf{p} \notin |K| \setminus \mathrm{st}_K(\mathbf{p})$. Therefore $\mathrm{st}_K(\mathbf{p})$ is open in $|K|$, and $\mathbf{p} \in \mathrm{st}_K(\mathbf{p})$, as required. ∎

6.5 Triangulated Closed Surfaces

Definition A *topological closed surface* is a compact Hausdorff space that may be covered by open sets, where each of these open sets is homeomorphic to a open set in the Euclidean plane.

An open set in the Euclidean plane is a union of open disks in that plane. It follows that a compact Hausdorff space is a topological closed surface if and only if it can be covered by open sets, where each of these open sets is homeomorphic to a open disk in the Euclidean plane.

Proposition 6.6 *Let K be a two-dimensional simplicial complex which satisfies the following two conditions:—*

(i) every edge belonging to K is an edge of exactly two triangles belonging to K;

(ii) given any vertex $\mathbf{v}$ belonging to K, the triangles that have $\mathbf{v}$ as vertex can be listed as a finite sequence $T_1, T_2, \ldots, T_m$, where $m > 1$, where T_i and T_{i-1} intersect along a common edge when $1 < i \leq m$, and where T_m and T_1 also intersect along a common edge.

Then the polyhedron $|K|$ of K is a topological closed surface.

Proof The polyhedron $|K|$ of the two-dimensional simplicial complex K is a compact Hausdorff space. We shall prove that the star neighbourhood of each point of $|K|$ is homeomorphic to an open disk.

Now suppose that the point $\mathbf{p}$ belongs to a triangle T of K with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ but does not lie on any edge of that triangle. Then the triangle T is the only member of the collection K of triangles, edges and vertices that contains the point $\mathbf{p}$. It follows that the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ consists of all points of the triangle T that do not lie on any edge of T. Thus $\mathrm{st}_K(\mathbf{p})$ is homeomorphic to the interior of a triangle in the Euclidean plane.

Next suppose that the point $\mathbf{p}$ belongs to an edge of K with vertices $\mathbf{v}$ and $\mathbf{w}$ but is not an endpoint of that edge. The edge is an edge of exactly two triangles belonging to K, because K represents a triangulated closed surface. Let these two triangles be $\mathbf{v}\,\mathbf{w}\,\mathbf{x}$ and $\mathbf{v}\,\mathbf{w}\,\mathbf{y}$. The conditions in the definition of two-dimensional complex ensure that the only members of the collection K

that contain the point $\mathbf{p}$ are the edge $\mathbf{v}\,\mathbf{w}$ and the two triangles $\mathbf{v}\,\mathbf{w}\,\mathbf{x}$ and $\mathbf{v}\,\mathbf{w}\,\mathbf{y}$. It follows that the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ of the point $\mathbf{p}$ in $|K|$ consists of all points of the union of these two triangles that do not lie on any of the edges $\mathbf{v}\,\mathbf{x}$, $\mathbf{x}\,\mathbf{w}$, $\mathbf{w}\,\mathbf{y}$ and $\mathbf{y}\,\mathbf{v}$. It follows from this that $\mathrm{st}_K(\mathbf{p})$ is homeomorphic to the interior of a quadrilateral in the Euclidean plane.

Finally suppose that $\mathbf{v}$ is a vertex belonging to K. Then the triangles that have $\mathbf{v}$ as vertex can be listed as a finite sequence $T_1, T_2, \ldots, T_m$, where $m > 1$, where T_i and T_{i-1} intersect along a common edge when $1 < i \leq m$, and where T_m and T_1 also intersect along a common edge. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be the vertices of these triangles distinct from $\mathbf{v}$, ordered so that the triangles T_m and T_1 intersect along the edge $\mathbf{v}\,\mathbf{w}_1$ and the triangles T_i and T_{i-1} intersect along the edge $\mathbf{v}\,\mathbf{w}_i$ for $i < i \leq m$. Then T_i is the triangle $\mathbf{v}\,\mathbf{w}_i\,\mathbf{w}_{i+1}$ for $i = 1, 2, \ldots, m-1$, and T_m is the triangle $\mathbf{v}\,\mathbf{w}_m\,\mathbf{w}_1$. The triangles of K that have $\mathbf{v}$ as a vertex are thus in the configuration depicted in Figure 1. The union of these triangles $T_1, T_2, \ldots, T_m$ is then homeomorphic to a convex

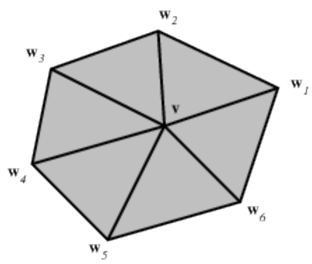

Figure 1

polygon in the Euclidean plane. The union of those edges

$$\mathbf{w}_m\,\mathbf{w}_1, \quad \mathbf{w}_1\,\mathbf{w}_2, \quad \cdots \quad \mathbf{w}_{m-1}\,\mathbf{w}_m$$

of these triangles that do not have $\mathbf{v}$ as one endpoint corresponds under this homeomorphism to the boundary of the convex polygon, and therefore the star neighbourhood $\mathrm{st}_K(\mathbf{v})$ of $\mathbf{v}$ in $|K|$ is homeomorphic to the interior of a convex polygon in the Euclidean plane.

We have thus shown that, given any point $\mathbf{p}$ of the polyhedron of K, the star neighbourhood of the point $\mathbf{p}$ is an open set in $|K|$ which is homeomorphic to the interior of a convex polygon in the Euclidean plane. The interior of such a polygon is homeomorphic to a disk. The result follows. ∎

Lemma 6.7 *Let K be a two-dimensional simplicial complex which satisfies the two conditions listed in the statement of Proposition 6.6 that ensure that the polyhedron $|K|$ of K is a topological closed surface. Then this polyhedron* *is a connected topological space if and only if, given any two triangles σ and τ of K, we can find a sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$ of triangles of K with $\sigma = \sigma_1$ and $\tau = \sigma_k$, where σ_{i-1} and σ_i intersect in a common edge for $i = 2, 3, \ldots, k$.*

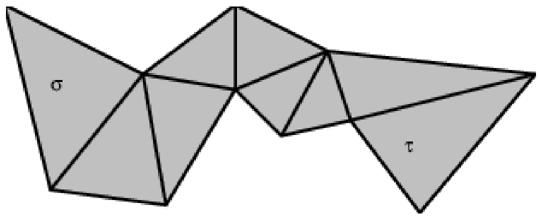

Figure 2

Proof Let σ_0 be a triangle in K, and let F be the subset of the polyhedron $|K|$ of K which is the union of all triangles that can be joined to σ_0 by a finite sequence of triangles belonging to K, where successive triangles in this sequence intersect along a common edge. Then F is a finite union of triangles, and those trianges are closed subsets of $|K|$, and therefore F is itself a closed subset of $|K|$.

Let $\mathbf{p}$ be a point of F. If $\mathbf{p}$ does not lie on any edge belonging to K then the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ belongs to just one triangle belonging to K, and moreover this triangle must then be a subset of F (or else the point $\mathbf{p}$ would not belong to F). Thus if $\mathbf{p} \in F$ does not like on any edge belonging to K then $\mathrm{st}_K(\mathbf{p}) \subset F$.

Next suppose that the point $\mathbf{p}$ of F lies on some edge belonging to K but is not an endpoint of that edge. Then the point $\mathbf{p}$ belongs to exactly two triangles of K that intersect along a common edge (because the two-dimensional simplicial complex represents a closed surface). At least one of these triangles must be contained in the set F (since $\mathbf{p} \in F$) and therefore both triangles are contained in F. But the star neighbourhood of the point $\mathbf{p}$ is contained in the union of those two triangles. Therefore $\mathrm{st}_K(\mathbf{p}) \subset F$ in this case also.

Finally suppose that the point $\mathbf{p}$ is a vertex of K. Then the requirement that the two-dimensional simplicial complex K represent a triangulated closed surface ensures that if at least one of the triangles belonging to K with a vertex at $\mathbf{p}$ is contained in F then every triangle belonging to K with a vertex at $\mathbf{v}$ must be contained in F. It follows that $\mathrm{st}_K(\mathbf{p}) \subset F$.

We have now shown that, given any point $\mathbf{p}$ of F, the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ of $\mathbf{p}$ in $|K|$ is a subset of F. But this star neighbourhood is an open subset of $|K|$ (see Lemma 6.5). Therefore the subset F of $|K|$ is both open and closed in $|K|$. Thus if the topological space $|K|$ is connected then $F = |K|$.

Every point of a topological space belongs to unique connected component which is the union of all connected subsets of the topological space that contain the given point. It follows that every triangle belonging to K is contained in a some connected component of $|K|$, and if two triangles belonging to K intersect along a common edge, or at a common vertex, then both belong to the same connected component of $|K|$. It follows that the set F is contained in some connected component of $|K|$. Thus if the topological space $|K|$ is not connected then F is a proper subset of $|K|$. We deduce that $F = |K|$ if and only if $|K|$ is a connected topological space. The result follows. ∎

Lemma 6.8 *Let K be a triangulated closed surface whose polyhedron $|K|$ is a connected topological space. Then $|K|$ is homeomorphic to the topological space obtained from a filled polygon with an even number of edges by identifying edges in pairs (i.e., given any edge with endpoints* $\mathbf{a}$ *and* $\mathbf{b}$*, there exists exactly one other edge with endpoints* $\mathbf{c}$ *and* $\mathbf{d}$ *such that* $(1-t)\mathbf{a} + t\mathbf{b}$ *is identified with* $(1-t)\mathbf{c} + t\mathbf{d}$ *for all* $t \in [0,1]$*).*

Proof Suppose that we have constructed some subcomplex L of K whose polyhedron is homeomorphic to the identification space obtained from a filled polygon P_L by identifying some of the edges of that polygon in pairs. Let $q_L \colon P_L \to |L|$ denote the identification map.

Suppose that e is an edge of P_L that is not identified to any other edge of L. Then e corresponds under the identification map to some edge e' of L. Moreover only one of the two triangles in K adjoining the edge e' belongs to L. Thus there is some triangle σ of $K \setminus L$ which has e' as one of its edges. Let M be the subcomplex of K obtained on adjoining to L the triangle σ, together with all its edges and vertices. We now extend the polygon P_L by attaching a triangle T along the free edge e to obtain a filled polygon P_M, where $P_M = P_L \cup T$ and $P_L \cap T = e$. We also extend the identification map $q_L \colon P_L \to |L|$ over this attached triangle to obtain an identification map $q_M \colon P_M \to |M|$, where $q_M|P_L = q_L$ and $q_M|T$ is a simplicial homeomorphism mapping the triangle T onto σ. Then the new identification map $q_M \colon P_M \to |M|$ also identifies some of the edges of the polygon P_M in pairs.

If we successively add triangles to build up a polygon in this fashion, we eventually obtain a subcomplex L of K whose polyhedron is homeomorphic to the identification space obtained from a filled polygon on identifying all of the edges of that polygon in pairs. But then, given any two triangles of K that intersect along a common edge, either both triangles belong to L, or else neither triangle belongs to L. It now follows from Lemma 6.7 that $L = K$, and thus the polyhedron of K is an identification space of the prescribed type. ∎

6.6 The Topological Classification of Closed Surfaces

We wish to classify up to homeomorphism the identification spaces obtained from polygons by identifying edges in pairs.

Suppose that we are given a polygon with its edges identified in pairs. Choose an orientation (i.e., a 'direction') on each edge of the polygon in such a way that, for each pair of identified edges, the orientations on those edges correspond under the identification of these edges. Denote each pair of identified edges by some letter $a, b, c, \ldots$. Suppose that we travel round the boundary of the polygon in the anticlockwise direction, starting at some chosen vertex. We obtain a *surface symbol* consisting of a sequence of symbols taken from $a, a^{-1}, b, b^{-1}, c, c^{-1}$, ordered so as to represent the order in which the corresponding edges of the polygon are traversed (on travelling round the polygon in the anticlockwise direction), and where an edge represented by some letter x occurs in the surface symbol as 'x' if the chosen orientation on the edge agrees with the anticlockise orientation, or as 'x^{-1}' if the chosen orientation on the edge is opposite to the anticlockwise orientation. For example, the surface symbol $xyx^{-1}y^{-1}$ represents the torus (see Figure 3), and the surface symbol $xy^{-1}xy^{-1}$ represents the real projective plane (see Figure 4).

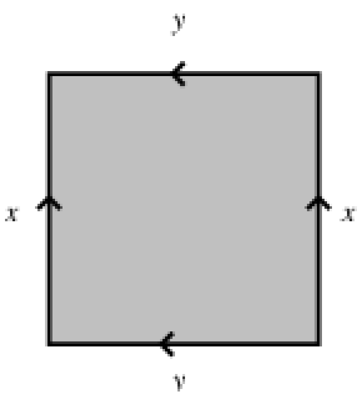

Figure 3: Torus

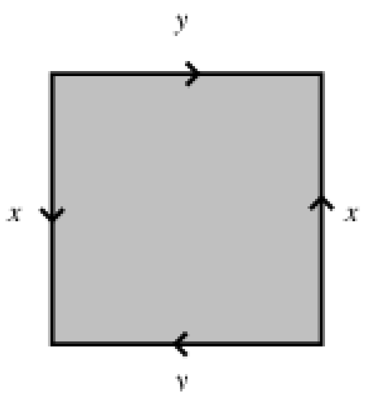

Figure 4: Real projective plane

Lemma 6.8 shows that any connected triangulated closed surface can be described by such a *surface symbol*. This is a finite sequence of symbols of the form $a, a^{-1}, b, b^{-1}, c, c^{-1}, \ldots,\ a, b, c, \ldots$ representing some suitable list of 'letters' that label the pairs of identified edges of the polygon representing the surface. Each 'letter' x present occurs exactly twice in the surface symbol, either as 'x' or as 'x^{-1}'. Conversely any surface symbol of this form determines a scheme for identifying in pairs the edges of a suitable polygon to obtain a closed surface. We wish to determine necessary and sufficient conditions for determining whether or not the surfaces obtained in this way from two such surface symbols are homeomorphic.

Let us use capital letters $A, B, C, \ldots$ to denote (possibly empty) sequences of symbols taken from the list $a, a^{-1}, b, b^{-1}, c, c^{-1}, \ldots$. Also if A is such a sequence of symbols, given by $A = a_1 a_2 \cdots a_n$, then we write A^{-1} for the sequence $a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$ (where $(x^{-1})^{-1} \equiv x$ for any letter x.) Using these conventions, we now state three rules which enable one to transform one surface symbol into another in such a way that the two surface symbols represent surfaces that are homeomorphic.

- **Rule 1.** cyclically permute the symbols occurring in the surface symbol,
- **Rule 2.** Replace $ABxCDxE$ by $AyDB^{-1}yC^{-1}E$ (where y represents some letter not occurring in the first surface symbol).
- **Rule 3.** Replace $ABxCDx^{-1}E$ by $AyDCy^{-1}BE$.
- **Rule 4.** Replace $Axx^{-1}B$ or $Ax^{-1}xB$ by AB, provided that AB contains at least two letters (each occurring twice).

Lemma 6.9 *The application of Rules 1–4 to a surface symbol gives a new surface symbol such that the surfaces determined by the two surface symbols are homeomorphic.*

Proof Rule 1 corresponds to traversing the boundary of the polygon starting from a different vertex. Rules 2–4 are justified by the simple 'cut and paste' operations depicted in Figures 5, 6 and 7. ∎

Rules 1–4 allow the reduction of surface symbols to certain standard forms. Now each letter x in a surface symbol occurs exactly twice; if the letter x either occurs both times as 'x' or else occurs both times as 'x^{-1}', then we call the occurrence of the letter x in the surface symbol a *similar pair*; otherwise we call the occurrence of this letter a *reversed pair*. Two reversed pairs in some given surface symbol are said to *interlock* if they occur in the order $\cdots y \cdots z \cdots y^{-1} \cdots z^{-1} \cdots$ (after interchanging y and y^{-1}, or z and z^{-1}, if necessary).

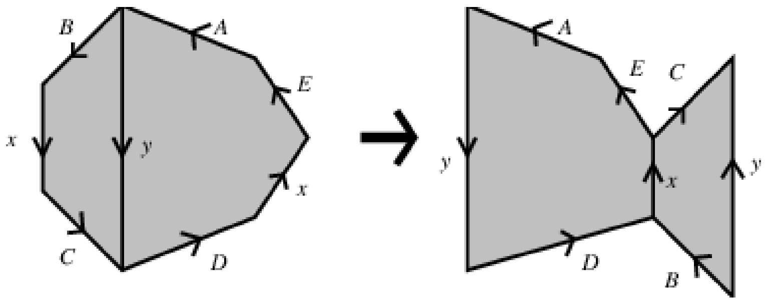

Figure 5: Rule 2

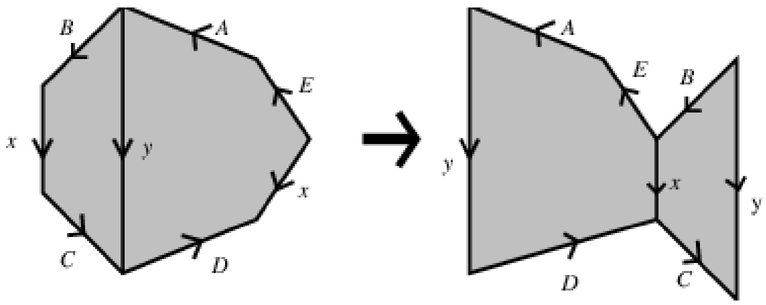

Figure 6: Rule 3

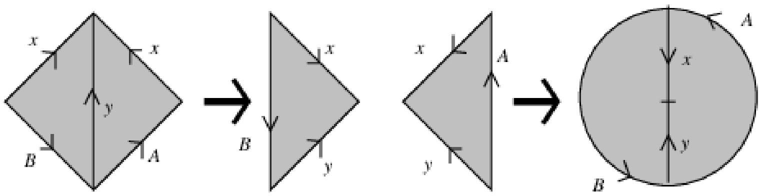

Figure 7: Rule 4

Proposition 6.10 *Any surface symbol can be reduced, by suitable applications of the transformations described in Rules 1–4 above and their inverses, to one of the following canonical forms:—*

$$x_1y_1x_1^{-1}y_1^{-1}\cdots x_gy_gx_g^{-1}y_g^{-1} \qquad (g \geq 1),$$

$$x_1x_1x_2x_2\cdots x_hx_h \qquad (h \geq 2),$$

$$xx^{-1}yy^{-1}, \qquad xxyy^{-1}.$$

Proof First we note that if C is any sequence of the form $x_1x_1x_2x_2\ldots$, then a sequence of transformations

$$CDxExF \to CyD^{-1}yE^{-1}F \to CzzDE^{-1}F$$

of the type specified by Rule 2 will transform any surface symbol $CDxExF$ with a similar pair to one of the form $CzzDE^{-1}$. Repeated applications of this procedure reduce any surface symbol to one of the form AB, where A is of the form form $x_1x_1x_2x_2\cdots x_rx_r$ and B contains only reversed pairs (where either A or B may be empty).

One can now use Rule 3 in order to reduce a surface symbol of the form AB to one of the form ACD, where C is of the form

$$y_1z_1y_1^{-1}z_1^{-1}\cdots y_sz_sy_s^{-1}z_s^{-1},$$

and D contains only non-interlocking reversed pairs. Indeed if E is any surface symbol of the required form, then successive applications of Rule 3 show that any surface symbol of the form $EFaGbHa^{-1}Ib^{-1}J$ can be transformed to $Eefe^{-1}f^{-1}FIHGJ$ by the following sequence of transformations:

$$\begin{aligned} EFaGbHa^{-1}Ib^{-1}J &\to EcGbHc^{-1}FIb^{-1}J \\ &\to EcGdFIHc^{-1}d^{-1}J \\ &\to EeFIHGde^{-1}d^{-1}J \\ &\to Eefe^{-1}f^{-1}FIHGJ. \end{aligned}$$

The stated reduction of AB to ACD now follows by induction on the number of interlocking reversed pairs.

If A is non-empty then one can reduce a surface symbol of the form ACD (where A, C and D are as above) to one of the form ED, where E is of the form $x_1x_1x_2x_2\cdots$ and D contains only non-interlocking reversed pairs. This follows from successive applications of the following sequence of transformations (which are inverses of transformations of the type specified by Rule 2):

$$Fxxaba^{-1}b^{-1}G \to Fyb^{-1}a^{-1}ya^{-1}b^{-1}G \to Fyay^{-1}accG \leftarrow FyyddccG.$$

Now consider D, which consists only of non-interlocking reversed pairs. Let $\cdots x \cdots x^{-1} \cdots$ be the closest reversed pair occurring in D. Then x and x^{-1} must be adjacent (since otherwise D would contain two interlocking reversed pairs). We can therefore 'cancel' xx^{-1}, by Rule 4, provided that the resultant symbol always contains at least two letters. It follows that any surface symbol with more than two letters can be reduced to one or other of the first two canonical forms specified (with $g \geq 1$ or $h \geq 2$).

Finally consider surface symbols with two letters occurring. The procedures described above reduce such a surface symbol to one of the following forms: $xyx^{-1}y^{-1}$, $xxyy$, $xxyy^{-1}$, $xx^{-1}yy^{-1}$, $xyy^{-1}x^{-1}$. The first four of these are included in the list of canonical forms. The symbol $xyy^{-1}x^{-1}$ reduces to $zz^{-1}yy^{-1}$ on cyclically permuting the symbol (according to Rule 1) and replacing x^{-1} and x by z and z^{-1} respectively, as required. $\blacksquare$

Let M_g $(g \geq 1)$ be the space obtained from a regular $4g$-sided polygon by identifying the edges according to the sequence $x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_g y_g x_g^{-1} y_g^{-1}$ (see Figure 8), and let N_h $(h \geq 2)$ be defined similarly using $x_1 x_1 \cdots x_h x_h$ (see Figure 9). Also let M_0 and N_1 be surfaces whose surface symbols are $xx^{-1}yy^{-1}$ and $xxyy^{-1}$ respectively. We have so far proved that the polyhedron of any connected triangulated closed surface is homeomorphic to one of the spaces M_g $(g \geq 0)$ or N_h $(h \geq 1)$.

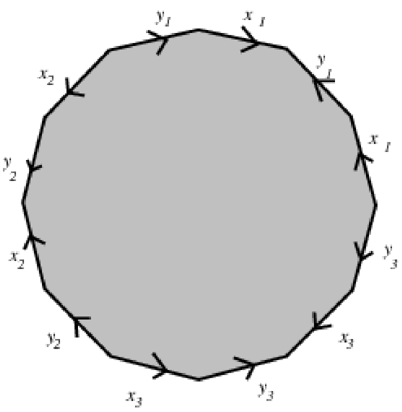

Figure 8: The surface M_g $(g = 3)$

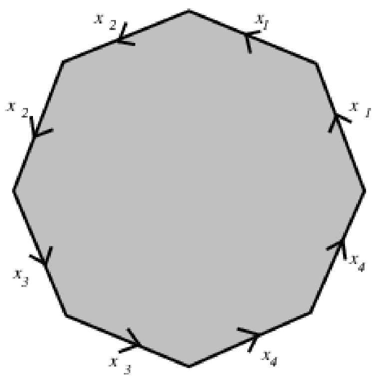

Figure 9: The surface N_h $(h = 4)$