Module MA342R: Covering Spaces and Fundamental Groups
Hilary Term 2017
Section 5: Discontinuous Group Actions and Orbit Spaces

D. R. Wilkins

Contents

5 Discontinuous Group Actions and Orbit Spaces

5.1 Discontinuous Group Actions

Definition Let G be a group, and let X be a set. The group G is said to *act* on the set X (on the left) if each element g of G determines a corresponding function $\theta_g\colon X \to X$ from the set X to itself, where

(i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

(ii) the function θ_e determined by the identity element e of G is the identity function of X.

Let G be a group acting on a set X. Given any element x of X, the *orbit* $[x]_G$ of x (under the group action) is defined to be the subset $\{\theta_g(x) : g \in G\}$ of X, and the *stabilizer* of x is defined to the the subgroup $\{g \in G : \theta_g(x) = x\}$ of the group G. Thus the orbit of an element x of X is the set consisting of all points of X to which x gets mapped under the action of elements of the group G. The stabilizer of x is the subgroup of G consisting of all elements of this group that fix the point x. The group G is said to act *freely* on X if $\theta_g(x) \neq x$ for all $x \in X$ and $g \in G$ satisfying $g \neq e$. Thus the group G acts freely on X if and only if the stabilizer of every element of X is the trivial subgroup of G.

Let e be the identity element of G. Then $x = \theta_e(x)$ for all $x \in X$, and therefore $x \in [x]_G$ for all $x \in X$, where $[x]_G = \{\theta_g(x) : g \in G\}$.

Let x and y be elements of G for which $[x]_G \cap [y]_G$ is non-empty, and let $z \in [x]_G \cap [y]_G$. Then there exist elements h and k of G such that $z = \theta_h(x) = \theta_k(y)$. Then $\theta_g(z) = \theta_{gh}(x) = \theta_{gk}(y)$, $\theta_g(x) = \theta_{gh^{-1}}(z)$ and $\theta_g(y) = \theta_{gk^{-1}}(z)$ for all $g \in G$, and therefore $[x]_G = [z]_G = [y]_G$. It follows from this that the group action partitions the set X into orbits, so that each element of X determines an orbit which is the unique orbit for the action of G on X to which it belongs. We denote by X/G the set of orbits for the action of G on X.

Now suppose that the group G acts on a topological space X. Then there is a surjective function $q\colon X \to X/G$, where $q(x) = [x]_G$ for all $x \in X$. This surjective function induces a quotient topology on the set of orbits: a subset U of X/G is open in this quotient topology if and only if $q^{-1}(U)$ is an open set in X (see Lemma 1.34). We define the *orbit space* X/G for the action of G on X to be the topological space whose underlying set is the set of orbits for the action of G on X, the topology on X/G being the quotient topology induced by the function $q\colon X \to X/G$. This function $q\colon X \to X/G$

is then an identification map: we shall refer to it as the *quotient map* from X to X/G.

We shall be concerned here with situations in which a group action on a topological space gives rise to a covering map. The relevant group actions are those where the group acts *freely and properly discontinuously* on the topological space.

Definition Let G be a group with identity element e, and let X be a topological space. The group G is said to act *freely and properly discontinuously* on X if each element g of G determines a corresponding continuous map $\theta_g: X \to X$, where the following conditions are satisfied:

(i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

(ii) the continuous map θ_e determined by the identity element e of G is the identity map of X;

(iii) given any point x of X, there exists an open set U in X such that $x \in U$ and $\theta_g(U) \cap U = \emptyset$ for all $g \in G$ satisfying $g \neq e$.

Let G be a group which acts freely and properly discontinuously on a topological space X. Given any element g of G, the corresponding continuous function $\theta_g: X \to X$ determined by X is a homeomorphism. Indeed it follows from conditions (i) and (ii) in the above definition that $\theta_{g^{-1}} \circ \theta_g$ and $\theta_g \circ \theta_{g^{-1}}$ are both equal to the identity map of X, and therefore $\theta_g: X \to X$ is a homeomorphism with inverse $\theta_{g^{-1}}: X \to X$.

Remark The terminology 'freely and properly discontinuously' is traditional, but is hardly ideal. The adverb 'freely' refers to the requirement that $\theta_g(x) \neq x$ for all $x \in X$ and for all $g \in G$ satisfying $g \neq e$. The adverb 'discontinuously' refers to the fact that, given any point x of G, the elements of the orbit $\{\theta_g(x) : g \in G\}$ of x are separated; it does not signify that the functions defining the action are in any way discontinuous or badly-behaved. The adverb 'properly' refers to the fact that, given any compact subset K of X, the number of elements of g for which $K \cap \theta_g(K) \neq \emptyset$ is finite. Moreover the definitions of *properly discontinuous actions* in textbooks and in sources of reference are not always in agreement: some say that an action of a group G on a topological space X (where each group element determines a corresponding homeomorphism of the topological space) is *properly discontinuous* if, given any $x \in X$, there exists an open set U in X such that the number of elements g of the group for which $g(U) \cap U \neq \emptyset$ is finite; others say that the action is *properly discontinuous* if it satisfies the conditions given in the definition above for a group acting freely and properly discontinuously on the set. William Fulton, in his textbook *Algebraic topology: a first course* (Springer, 1995), introduced the term 'evenly' in place of 'freely and properly discontinuously', but this change in terminology does not appear to have been generally adopted.

5.2 Orbit Spaces

Example The cyclic group C_2 of order 2 consists of a set $\{e, a\}$ with two elements e and a, together with a group multiplication operation defined so that $e^2 = a^2 = e$ and $ea = ae = a$. The identity element of C_2 is thus e.

Let us represent the n-dimensional sphere S^n as the unit sphere in $\mathbb{R}^{n+1}$ centred on the origin. Let $\theta_e\colon S^n \to S^n$ be the identity map of S^n and let $\theta_a\colon S^n \to S^n$ be the antipodal map of S^n, defined such that $\theta_a(\mathbf{x}) = -\mathbf{x}$ for all $\mathbf{x} \in S^n$. Then the group C_2 acts on S^n (on the left) so that elements a and e of S^n correspond under this action to the homeomorphisms θ_e and θ_a respectively. Points $\mathbf{x}$ and $\mathbf{y}$ are said to be *antipodal* to one another if and only if $\mathbf{y} = -\mathbf{x}$. Each orbit for the action of C_2 on S^n thus consists of a pair of antipodal points on S^n.

Let $\mathbf{p}$ be an element of S^n, and let

$$U = \{\mathbf{x} \in S^n : \mathbf{x} \,.\, \mathbf{p} > 0\}.$$

Then U is open in S^n and $\mathbf{p} \in U$. Also

$$\theta_a(U) = \{\mathbf{x} \in S^n : \mathbf{x} \,.\, \mathbf{p} < 0\},$$

and therefore $U \cap \theta_a(U) = \emptyset$. It follows that the group C_2 acts freely and properly discontinuously on S^n.

Distinct points of S^n belong to the same orbit under the action of C_2 on S^n if and only if the line in $\mathbb{R}^{n+1}$ passing through those points also passes through the origin. It follows that lines in $\mathbb{R}^{n+1}$ that pass through the origin are in one-to-one correspondence with orbits for the action of C_2 on S^n. The orbit space S^n/C_2 thus represents the set of lines through the origin in $\mathbb{R}^{n+1}$. We define n-dimensional *real projective space* $\mathbb{R}P^n$ to be the topological space whose elements are the lines in $\mathbb{R}^{n+1}$ passing through the origin, with the topology obtained on identifying $\mathbb{R}P^n$ with the orbit space S^n/C_2. The quotient map $q\colon S^n \to \mathbb{R}P^n$ then sends each point $\mathbf{x}$ of S^n to the orbit consisting of the two points $\mathbf{x}$ and $-\mathbf{x}$. Thus each pair of antipodal points on the n-dimenionsional sphere S^n determines a single point of n-dimensional real projective space $\mathbb{R}P^n$.

Proposition 5.1 *Let G be a group acting freely and properly discontinuously on a topological space X, let X/G denote the resulting orbit space, and let $q: X \to X/G$ be the quotient map that sends each element of X to its orbit under the action of the group G. Let $p: X \to Y$ be a continuous surjective map from X to a topological space Y. Suppose that elements x and x' of X satisfy $p(x) = p(x')$ if and only if $q(x) = q(x')$. Suppose also $p(V)$ is open in Y for every open set V in X. Then the surjective continuous map $p: X \to Y$ induces a homeomorphism $h: X/G \to Y$ between the topological spaces X/G and Y, where $h(q(x)) = p(x)$ for all $x \in X$.*

Proof The function $h: X/G \to Y$ is continuous because $p: X \to Y$ is continuous and $q: X \to Y$ is a quotient map (see Lemma 1.35). Moreover it is surjective because $p: X \to Y$ is a surjection, and it is injective because elements x and x' satisfy $p(x) = p(x')$ if and only if $q(x) = q(x')$. It follows that $h: X/G \to Y$ is a bijection.

Let W be an open set in X/G. It follows from the definition of the quotient topology that $q^{-1}(W)$ is open in X. The map p maps open sets to open sets. Therefore $p(q^{-1}(W))$ is open in Y. But $p(q^{-1}(W)) = h(W)$. Thus $h(W)$ is open in Y for every open set W in X, and therefore $h^{-1}: Y \to X/W$ is continuous. Thus the continuous bijection $h: X/G \to Y$ is a homeomorphism, as required. ∎

Corollary 5.2 *Let the group $\mathbb{Z}$ act on the real line $\mathbb{R}$ by translation, where the action sends each integer n to the translation $\theta_n: \mathbb{R} \to \mathbb{R}$ defined such that $\theta_n(t) = t + n$ for all real numbers t. Let $\mathbb{R}/\mathbb{Z}$ denote the orbit space for this action, and let $q: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map that sends each real number to its orbit under the action of the group $\mathbb{Z}$. Let S^1 denote the unit circle centred on the origin in $\mathbb{R}^2$, let $p: \mathbb{R} \to S^1$ be defined such that*

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

for all real numbers t, and let $h: \mathbb{R}/\mathbb{Z} \to S^1$ be the map defined such that $h(q(t)) = p(t)$ for all real numbers t. Then $h: \mathbb{R}/\mathbb{Z} \to S^1$ is a homeomorphism.

Proof The map $p: \mathbb{R} \to S^1$ maps open sets to open sets. The result therefore follows directly on applying Proposition 5.1. ∎

Corollary 5.3 *Let the group $\mathbb{Z}$ act by translation on the complex plane $\mathbb{C}$, where the action sends each integer n to the translation $\theta_n: \mathbb{C} \to \mathbb{C}$ defined such that $\theta_n(z) = z + n$ for all complex numbers z, where $i^2 = -1$. Let $\mathbb{C}/\mathbb{Z}$ denote the orbit space for this action, and let $q: \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ be the quotient* *map that sends each complex number to its orbit under the action of the group $\mathbb{Z}$. Let $p: \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be defined such that $p(z) = \exp(2\pi i z)$ for all complex numbers z, and let $h: \mathbb{C}/\mathbb{Z} \to \mathbb{C} \setminus \{0\}$ be the map defined such that $h(q(z)) = p(z)$ for all complex numbers z. Then $h: \mathbb{C}/\mathbb{Z} \to \mathbb{C} \setminus \{0\}$ is a homeomorphism.*

Proof We show that the map $p: \mathbb{C} \to \mathbb{C} \setminus \{0\}$ maps open sets to open sets. Let V be an open set in $\mathbb{C}$, and let u and v be real numbers for which $u + iv \in V$. Then there exist real numbers θ_1, θ_2 and positive real numbers r_1 and r_2 satisfying the inequalities

$$\theta_1 < 2\pi u < \theta_2 \text{ and } \log r_1 < -2\pi v < \log r_2$$

where θ_1 and θ_2 are close enough to $2\pi u$ and $\log r_1$ and $\log r_2$ are close enough to $-2\pi v$ to ensure that $s + it \in V$ for all real numbers s and t that satisfy the inequalities

$$\theta_1 < 2\pi s < \theta_2 \text{ and } \log r_1 < -2\pi t < \log r_2.$$

It then follows that $u + iv \in N$ and $N \subset p(V)$, where

$$N = \{re^{i\theta} : r_1 < r < r_2 \text{ and } \theta_1 < \theta < \theta_2\}.$$

Now N is an open set in $\mathbb{C} \setminus \{0\}$. It follows that $p(V)$ is a neighbourhood of $p(u+iv)$. We have now shown that the set $p(V)$ is a neighbourhood of each of its points. It follows that $p(V)$ is open in $\mathbb{C} \setminus \{0\}$. We conclude therefore that the map $p: \mathbb{C} \setminus \mathbb{C} \setminus \{0\}$ maps open sets to open sets. It then follows directly from Proposition 5.1 that $h: \mathbb{C}/\mathbb{Z} \to \mathbb{C} \setminus \{0\}$ is a homeomorphism. ■

Proposition 5.4 *Let G be a group acting freely and properly discontinuously on a topological space X, let X/G denote the resulting orbit space, and let $q: X \to X/G$ be the quotient map that sends each element of X to its orbit under the action of the group G. Let $p: X \to Y$ be a continuous surjective map from X to a Hausdorff topological space Y. Suppose that elements x and x' of X satisfy $p(x) = p(x')$ if and only if $q(x) = q(x')$. Suppose also that there exists a compact subset K of X that intersects every orbit for the action of G on X. Then the surjective continuous map $p: X \to Y$ induces a homeomorphism $h: X/G \to Y$ between the topological spaces X/G and Y, where $h(q(x)) = p(x)$ for all $x \in X$.*

Proof The function $h: X/G \to Y$ is continuous because X is continuous and $q: X \to Y$ is a quotient map (see Lemma 1.35). Moreover it is surjective because $p: X \to Y$ is a surjection, and it is injective because elements x and

x' satisfy $p(x) = p(x')$ if and only if $q(x) = q(x')$. It follows that $h: X/G \to Y$ is a bijection.

The orbit space X/G is compact, because it is the image $q(K)$ of the compact set K under the continuous map $q: X \to X/G$. (see Lemma 1.39). Thus $h: X/G \to Y$ is a continuous bijection from a compact topological space to a Hausdorff space. This map is therefore a homeomorphism (see Theorem 1.45). ∎

Example Let the group $\mathbb{Z}$ of integers under addition act by translation on the real line $\mathbb{R}$ by translation so that, under this action, an integer n corresponds to the homeomorphism $\theta_n: \mathbb{R} \to \mathbb{R}$ defined such that $\theta_n(t) = t+n$ for all real numbers t. Let $q: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map onto the orbit space, and let $p: \mathbb{R} \to S^1$ be defined such that

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

for all real numbers t, and let $h: \mathbb{R}/\mathbb{Z} \to S^1$ be the map defined such that $h(q(t)) = p(t)$ for all real numbers t.

Now S^1 is a Hausdorff space, as it is a subset of the metric space $\mathbb{R}^2$. Also the map $p: \mathbb{R} \to S^1$ is surjective. Real numbers t_1 and t_2 satisfy $p(t_1) = p(t_2)$ if and only if $t_1 = t_2 + n$ for some integer n. It follows that $p(t_1) = p(t_2)$ if and only if $q(t_1) = q(t_2)$. The compact subset $[0, 1]$ of $\mathbb{R}$ intersects every orbit for the action of $\mathbb{Z}$ on $\mathbb{R}$. It therefore follows from Proposition 5.4 that $h: \mathbb{R}/\mathbb{Z} \to S^1$ is a homeomorphism. (This result was also shown to follow from the fact that $p: \mathbb{R} \to S^1$ maps open sets to open sets: see Corollary 5.2.)

Example Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be defined so that

$$f(s,t) = ((2 + \cos 2\pi t)\ \cos 2\pi s,\ \ (2 + \cos 2\pi t)\ \sin 2\pi s,\ \ \sin 2\pi t)$$

for all $(s,t) \in \mathbb{R}^2$, and let $Y = f(\mathbb{R}^2)$. Then Y is a torus in $\mathbb{R}^3$ that bounds the 'solid doughnut' consisting of those points of $\mathbb{R}^3$ whose distance from the circle in the plane $z = 0$ of radius 2 centred on the origin is less than one. Points (s_1, t_1) and (s_2, t_2) of $\mathbb{R}^2$ satisfy $f(s_1, t_1) = f(s_2, t_2)$ if and only if $s_1 - t_1$ and $s_2 - t_2$ are integers. Let the group $\mathbb{Z}^2$ act on $\mathbb{R}^2$ by translation, so that, under this action, an element (m, n) of $\mathbb{Z}^2$ corresponds to the homeomorphism $\theta_{(m,n)}: \mathbb{R}^2 \to \mathbb{R}^2$ from $\mathbb{R}^2$ to itself defined so that $\theta_{(m,n)}(s,t) = (s + m, t + n)$ for all $(s,t) \in \mathbb{R}^2$.

Let δ be a real number satisfying $0 < \delta \le \frac{1}{2}$, and, for all $(s,t) \in \mathbb{R}^2$, let $B((s,t), \delta)$ denote the open disk in $\mathbb{R}^2$ of radius δ centred on the point (s,t). Then

$$B((s + m, t + n), \delta) \cap B((s,t), \delta) = \emptyset$$

for all integers m and n for which $(m, n) \neq (0, 0)$. It follows that the group $\mathbb{Z}^2$ acts freely and properly discontinuously on $\mathbb{R}^2$ by translation. Let $\mathbb{R}^2/\mathbb{Z}^2$ be the orbit space determined by this action, let $q: \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ be the quotient map sending each point of $\mathbb{R}^2$ to its orbit under the action of $\mathbb{Z}^2$, and let $h: \mathbb{R}^2/\mathbb{Z}^2 \to Y$ be the function from $\mathbb{R}^2/\mathbb{Z}^2$ to the surface Y defined so that $h(q(s, t)) = f(s, t))$ for all $(s, t) \in \mathbb{R}^2$.

Now the unit square $[0, 1] \times [0, 1]$ is a compact subset of $\mathbb{R}^2$ that intersects every orbit for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. It follows directly from Proposition 5.4 that $h: \mathbb{R}^2/\mathbb{Z}^2 \to Y$ is a homeomorphism. Thus the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ represents a 2-dimensional torus.

Proposition 5.5 *Let G be a group acting freely and properly discontinuously on a topological space X. Then the quotient map $q: X \to X/G$ from X to the corresponding orbit space X/G is a covering map.*

Proof The quotient map $q: X \to X/G$ is surjective. Let V be an open set in X. Then $q^{-1}(q(V))$ is the union $\bigcup_{g \in G} \theta_g(V)$ of the open sets $\theta_g(V)$ as g ranges over the group G, since $q^{-1}(q(V))$ is the subset of X consisting of all elements of X that belong to the orbit of some element of V. But any union of open sets in a topological space is an open set. We conclude therefore that if V is an open set in X then $q(V)$ is an open set in X/G.

Let x be a point of X. Then there exists an open set U in X such that $x \in U$ and $\theta_g(U) \cap U = \emptyset$ for all $g \in G$ satisfying $g \neq e$. Now $q^{-1}(q(U)) = \bigcup_{g \in G} \theta_g(U)$. We claim that the sets $\theta_g(U)$ are disjoint. Let g and h be elements of G. Suppose that $\theta_g(U) \cap \theta_h(U) \neq \emptyset$. Then $\theta_{h^{-1}}(\theta_g(U) \cap \theta_h(U)) \neq \emptyset$. But $\theta_{h^{-1}}: X \to X$ is a bijection, and therefore

$$\theta_{h^{-1}}(\theta_g(U) \cap \theta_h(U)) = \theta_{h^{-1}}(\theta_g(U)) \cap \theta_{h^{-1}}(\theta_h(U)) = \theta_{h^{-1}g}(U) \cap U,$$

and therefore $\theta_{h^{-1}g}(U) \cap U \neq \emptyset$. It follows that $h^{-1}g = e$, where e denotes the identity element of G, and therefore $g = h$. Thus if g and h are elements of g, and if $g \neq h$, then $\theta_g(U) \cap \theta_h(U) = \emptyset$. We conclude therefore that the preimage $q^{-1}(q(U))$ of $q(U)$ is the disjoint union of the sets $\theta_g(U)$ as g ranges over the group G. Moreover each these sets $\theta_g(U)$ is an open set in X.

Now $U \cap [u]_G = \{u\}$ for all $u \in U$, since $[u]_G = \{\theta_g(u) : g \in G\}$ and $U \cap \theta_g(U) = \emptyset$ when $g \neq e$. Thus if u and v are elements of U, and if $q(u) = q(v)$ then $[u]_G = [v]_G$ and therefore $u = v$. It follows that the restriction $q|U: U \to X/G$ of the quotient map q to U is injective, and therefore q maps U bijectively onto $q(U)$. But q maps open sets onto open sets, and any continuous bijection that maps open sets onto open sets is a homeomorphism. We conclude therefore that the restriction of $q: X \to X/G$ to the open set U maps U homeomorphically onto $q(U)$. Moreover, given any element g of G, the quotient map q satisfies $q = q \circ \theta_{g^{-1}}$, and the homeomorphism $\theta_{g^{-1}}$ maps $\theta_g(U)$ homeomorphically onto U. It follows that the quotient map q maps $\theta_g(U)$ homeomorphically onto $q(U)$ for all $g \in U$. We conclude therefore that $q(U)$ is an evenly covered open set in X/G whose preimage $q^{-1}(q(U))$ is the disjoint union of the open sets $\theta_g(U)$ as g ranges over the group G. It follows that the quotient map $q: X \to X/G$ is a covering map, as required. ∎

5.3 Fundamental Groups of Orbit Spaces

Theorem 5.6 *Let G be a group acting freely and properly discontinuously on a path-connected topological space X, let $q: X \to X/G$ be the quotient map from X to the orbit space X/G, and let x_0 be a point of X. Then there exists a surjective homomorphism $\lambda: \pi_1(X/G, q(x_0)) \to G$ with the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(x_0)$ for any loop γ in X/G based at $q(x_0)$, where $\tilde{\gamma}$ denotes the unique path in X for which $\tilde{\gamma}(0) = x_0$ and $q \circ \tilde{\gamma} = \gamma$. The kernel of this homomorphism is the subgroup $q_\#(\pi_1(X, x_0))$ of $\pi_1(X/G, q(x_0))$.*

Proof Let $\gamma: [0,1] \to X/G$ be a loop in the orbit space with $\gamma(0) = \gamma(1) = q(x_0)$. It follows from the Path Lifting Theorem for covering maps (Theorem 4.5) that there exists a unique path $\tilde{\gamma}: [0,1] \to X$ for which $\tilde{\gamma}(0) = x_0$ and $q \circ \tilde{\gamma} = \gamma$. Now $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ must belong to the same orbit, since $q(\tilde{\gamma}(0)) = \gamma(0) = \gamma(1) = q(\tilde{\gamma}(1))$. Therefore there exists some element g of G such that $\tilde{\gamma}(1) = \theta_g(x_0)$. This element g is uniquely determined, since the group G acts freely on X. Moreover the value of g is determined by the based homotopy class $[\gamma]$ of γ in $\pi_1(X/G, q(x_0))$. Indeed it follows from Proposition 4.7 that if σ is a loop in X/G based at $q(x_0)$, if $\tilde{\sigma}$ is the lift of σ starting at x_0 (so that $q \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = x_0$), and if $[\gamma] = [\sigma]$ in $\pi_1(X/G, q(x_0))$ (so that $\gamma \simeq \sigma$ rel $\{0,1\}$), then $\tilde{\gamma}(1) = \tilde{\sigma}(1)$. We conclude therefore that there exists a well-defined function

$$\lambda: \pi_1(X/G, q(x_0)) \to G,$$

which is characterized by the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(x_0)$ for any loop γ in X/G based at $q(x_0)$, where $\tilde{\gamma}$ denotes the unique path in X for which $\tilde{\gamma}(0) = x_0$ and $q \circ \tilde{\gamma} = \gamma$.

Now let $\alpha: [0,1] \to X/G$ and $\beta: [0,1] \to X/G$ be loops in X/G based at $q(x_0)$, and let $\tilde{\alpha}: [0,1] \to X$ and $\tilde{\beta}: [0,1] \to X$ be the lifts of α and β respectively starting at x_0, so that $q \circ \tilde{\alpha} = \alpha$, $q \circ \tilde{\beta} = \beta$ and $\tilde{\alpha}(0) = \tilde{\beta}(0) = x_0$. Then $\tilde{\alpha}(1) = \theta_{\lambda([\alpha])}(x_0)$ and $\tilde{\beta}(1) = \theta_{\lambda([\beta])}(x_0)$. Then the path $\theta_{\lambda([\alpha])} \circ \tilde{\beta}$ is also a lift of the loop β, and is the unique lift of β starting at $\tilde{\alpha}(1)$. Let $\alpha.\beta$

be the concatenation of the loops α and β, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then the unique lift of $\alpha.\beta$ to X starting at x_0 is the path $\sigma\colon [0,1] \to X$, where

$$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \theta_{\lambda([\alpha])}(\tilde{\beta}(2t-1)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

It follows that

$$\begin{aligned} \theta_{\lambda([\alpha][\beta])}(x_0) &= \theta_{\lambda([\alpha.\beta])}(x_0) = \sigma(1) = \theta_{\lambda([\alpha])}(\tilde{\beta}(1)) \\ &= \theta_{\lambda([\alpha])}(\theta_{\lambda([\beta])}(x_0)) = \theta_{\lambda([\alpha])\lambda([\beta])}(x_0) \end{aligned}$$

and therefore $\lambda([\alpha][\beta]) = \lambda([\alpha])\lambda([\beta])$. Therefore the function

$$\lambda\colon \pi_1(X/G, q(x_0)) \to G$$

is a homomorphism.

Let $g \in G$. Then there exists a path α in X from x_0 to $\theta_g(x_0)$, since the space X is path-connected. Then $q \circ \alpha$ is a loop in X/G based at $q(x_0)$, and $g = \lambda([q \circ \alpha])$. This shows that the homomorphism λ is surjective.

Let $\gamma\colon [0,1] \to X/G$ be a loop in X/G based at $q(x_0)$. Suppose that $[\gamma] \in \ker \lambda$. Then $\tilde{\gamma}(1) = \theta_e(x_0) = x_0$, and therefore $\tilde{\gamma}$ is a loop in X based at x_0. Moreover $[\gamma] = q_\#[\tilde{\gamma}]$, and therefore $[\gamma] \in q_\#(\pi_1(X, x_0))$. On the other hand, if $[\gamma] \in q_\#(\pi_1(X, x_0))$ then $\gamma = q \circ \tilde{\gamma}$ for some loop $\tilde{\gamma}$ in X based at x_0 (see Proposition 4.9). But then $x_0 = \tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(x_0)$, and therefore $\lambda([\gamma]) = e$, where e is the identity element of G. Thus $\ker \lambda = q_\#(\pi_1(X, x_0))$, as required. ∎

Corollary 5.7 *Let G be a group acting freely and properly discontinuously on a path-connected topological space X, let $q\colon X \to X/G$ be the quotient map from X to the orbit space X/G, and let x_0 be a point of X. Then $q_\#(\pi_1(X, x_0))$ is a normal subgroup of the fundamental group $\pi_1(X/G, q(x_0))$ of the orbit space, and*

$$\frac{\pi_1(X/G, q(x_0))}{q_\#(\pi_1(X, x_0))} \cong G.$$

Proof The subgroup $q_\#(\pi_1(X, x_0))$ is the kernel of the homomorphism

$$\lambda\colon \pi_1(X/G, q(x_0)) \to G$$

described in the statement of Theorem 5.6. It is therefore a normal subgroup of $\pi_1(X/G, q(x_0))$, since the kernel of any homomorphism is a normal subgroup. The homomorphism λ is surjective, and the image of any group homomorphism is isomorphism of the quotient of its domain by its kernel. The result follows. ∎

Corollary 5.8 *Let G be a group acting freely and properly discontinuously on a simply-connected topological space X, let $q: X \to X/G$ be the quotient map from X to the orbit space X/G, and let x_0 be a point of X. Then $\pi_1(X/G, q(x_0)) \cong G$.*

Proof This is a special case of Corollary 5.7. ∎

Example The group $\mathbb{Z}$ of integers under addition acts freely and properly discontinuously on the real line $\mathbb{R}$. Indeed each integer n determines a corresponding homeomorphism $\theta_n: \mathbb{R} \to \mathbb{R}$, where $\theta_n(x) = x + n$ for all $x \in \mathbb{R}$. Moreover $\theta_m \circ \theta_n = \theta_{m+n}$ for all $m, n \in \mathbb{Z}$, and θ_0 is the identity map of $\mathbb{R}$. If $U = (-\frac{1}{2}, \frac{1}{2})$ then $\theta_n(U) \cap U = \emptyset$ for all non-zero integers n.

The real line $\mathbb{R}$ is simply-connected. It therefore follows from Corollary 5.8 that $\pi_1(\mathbb{R}/\mathbb{Z}, b) \cong \mathbb{Z}$ for any point b of $\mathbb{R}/\mathbb{Z}$.

Let $q: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map from the real line $\mathbb{R}$ to the orbit space $\mathbb{R}/\mathbb{Z}$ that sends each real number to its orbit under the action of the group of integers, let $p: \mathbb{R} \to S^1$ be defined such that

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

for all $t \in \mathbb{R}$. Then $p(t_1) = p(t_2)$ for all real numbers t_1 and t_2 satisfying $q(t_1) = q(t_2)$. Thus there is a well-defined function $h: \mathbb{R}/\mathbb{Z} \to S^1$ characterized by the property that $h(q(t)) = p(t)$ for all real numbers t.

The continuous map $h: \mathbb{R}/\mathbb{Z} \to S^1$ is a homeomorphism (see Corollary 5.2). It follows that

$$\pi_1(S^1, h(b)) \cong \pi_1(\mathbb{R}/\mathbb{Z}, b) \cong \mathbb{Z}$$

for all $b \in \mathbb{R}/\mathbb{Z}$. This shows that Theorem 3.9 concerning the fundamental group of the circle can be obtained as a special case of the more general result Corollary 5.8 concerning fundamental groups of orbit spaces obtained via discontinuous group actions on simply-connected topological spaces.

Example The group $\mathbb{Z}^n$ of ordered n-tuples of integers under addition acts freely and properly discontinuously on $\mathbb{R}^n$, where

$$\theta_{(m_1,m_2,\ldots,m_n)}(x_1, x_2, \ldots, x_n) = (x_1 + m_1, x_2 + m_2, \ldots, x_n + m_n)$$

for all $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is an n-dimensional torus, homeomorphic to the product of n circles. It follows from Corollary 5.8 that the fundamental group of this n-dimensional torus is isomorphic to the group $\mathbb{Z}^n$.

Example Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ centred on the origin, and let C_2 denote the cyclic group of order 2. Then $C_2 = \{e, a\}$, where $e^2 = a^2 = e$ and $ea = ae = a$. The group C_2 acts freely and discontinuously on S^n, where e acts as the identity map of S^n and a acts as the antipodal map sending $\mathbf{x}$ to $-\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The orbit space S^n/C^2 is homeomorphic to real projective n-dimensional space $\mathbb{R}P^n$. Now the n-dimensional sphere is simply-connected if $n > 1$. It follows from Corollary 5.8 that the fundamental group of $\mathbb{R}P^n$ is isomorphic to the cyclic group C_2 when $n > 1$.

Note that S^0 is a pair of points, and $\mathbb{R}P^0$ is a single point. Also S^1 is a circle (which is not simply-connected) and $\mathbb{R}P^1$ is homeomorphic to a circle. Moreover, for any $b \in S^1$, the homomorphism $q_\#: \pi_1(S^1, b) \to \pi_1(\mathbb{R}P^1, q(b))$ corresponds to the homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ that sends each integer n to $2n$. This is consistent with the conclusions of Corollary 5.7 in this example.

Example Given a pair (m, n) of integers, let $\theta_{m,n}: \mathbb{R}^2 \to \mathbb{R}^2$ be the homeomorphism of the plane $\mathbb{R}^2$ defined such that

$$\theta_{m,n}(x, y) = (x + m, (-1)^m y + n)$$

for all $(x, y) \in \mathbb{R}^2$. Let (m_1, n_1) and (m_2, n_2) be ordered pairs of integers. Then

$$\theta_{m_1,n_1} \circ \theta_{m_2,n_2} = \theta_{m_1+m_2,n_1+(-1)^{m_1}n_2}.$$

Let Γ be the group whose elements are represented as ordered pairs of integers, where the group operation # on Γ is defined such that

$$(m_1, n_1)\#(m_2, n_2) = (m_1 + m_2, n_1 + (-1)^{m_1} n_2)$$

for all $(m_1, n_1), (m_2, n_2) \in \Gamma$. The group Γ is non-Abelian, and its identity element is $(0, 0)$. This group acts on the plane $\mathbb{R}^2$: given $(m, n) \in \Gamma$ the corresponding symmetry $\theta_{m,n}$ is a translation if m is even, and is a glide reflection if m is odd.

Given a pair (m, n) of integers, the corresponding homeomorphism $\theta_{m,n}$ maps an open disk about the point (x, y) onto an open disk of the same radius about the point $\theta_{(m,n)}(x, y)$. It follows that if D is the open disk of radius $\frac{1}{2}$ about the point (x, y), and if $D \cap \theta_{m,n}(D)$ is non-empty, then $(m, n) = (0, 0)$. Thus the group Γ maps freely and properly discontinuously on the plane $\mathbb{R}^2$.

Now each orbit intersects the closed unit square S, where $S = [0, 1] \times [0, 1]$. If $0 < x < 1$ and $0 < y < 1$ then the orbit of (x, y) intersects the square S in one point, namely the point (x, y). If $0 < x < 1$, then the orbit of $(x, 0)$ intersects the square in two points $(x, 0)$ and $(x, 1)$. If $0 < y < 1$ then the orbit of $(0, y)$ intersects the square S in the two points $(0, y)$ and $(1, 1 - y)$. (Note that $(1, 1 - y) = \theta_{1,1}(0, y)$.) And the orbit of any corner of the square S intersects the square in the four corners of the square. The restriction $q|S$ of the quotient map $q: \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ to the square S is a continuous surjection defined on the square: one can readily verify that it is an identification map. It follows that the orbit space $\mathbb{R}^2/\Gamma$ is homeomorphic to the identification space obtained from the closed square S by identifying together the points $(x, 0)$ and $(x, 1)$ where the real number x satisfies $0 < x < 1$, identifying together the points $(0, y)$ and $(1, 1 - y)$ where the real number y satisfies $0 < y < 1$, and identifying together the four corners of the square. The identification space obtained in this fashion is a closed non-orientable surface, first described by Felix Klein in 1882, and now known as the *Klein bottle*. Apparently the surface was initially referred to as the *Kleinsche Fläche* (Klein's Surface), but this name was incorrectly translated into English, and, as a result the surface is now referred to as the Klein Bottle (*Kleinsche Flasche*).

The plane $\mathbb{R}^2$ is simply-connected. It follows from Corollary 5.8 that the fundamental group of the Klein bottle is isomorphic to the group Γ defined above.