Module MA342R: Covering Spaces and Fundamental Groups
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Part I (Section 1)

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Contents

1 Results concerning Topological Spaces

1.1 Topological Spaces

Definition A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

(i) the empty set $\emptyset$ and the whole set X are open sets,

(ii) the union of any collection of open sets is itself an open set,

(iii) the intersection of any *finite* collection of open sets is itself an open set.

The collection consisting of all the open sets in a topological space X is referred to as a *topology* on the set X.

Remark If it is necessary to specify explicitly the topology on a topological space then one denotes by (X, τ) the topological space whose underlying set is X and whose topology is τ. However if no confusion will arise then it is customary to denote this topological space simply by X.

1.2 Subsets of Euclidean Space

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. The *Euclidean distance* $|\mathbf{x} - \mathbf{y}|$ between two points $\mathbf{x}$ and $\mathbf{y}$ of X is defined as follows:

$$|\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. The Euclidean distances between any three points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of X satisfy the *Triangle Inequality*:

$$|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|.$$

A subset V of X is said to be *open* in X if, given any point $\mathbf{v}$ of V, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{v}| < \delta\} \subset V.$$

The empty set is also considered to be open in X.

Both $\emptyset$ and X are open sets in X. Also it is not difficult to show that any union of open sets in X is open in X, and that any finite intersection of open sets in X is open in X. (This will be proved in more generality for open sets in metric spaces.) Thus the collection of open sets in a subset X of a Euclidean space $\mathbb{R}^n$ satisfies the topological space axioms. Thus every subset X of $\mathbb{R}^n$ is a topological space with these open sets. This topology on a subset X of $\mathbb{R}^n$ is referred to as the *usual topology* on X, generated by the Euclidean distance function.

In particular $\mathbb{R}^n$ is itself a topological space.

1.3 Open Sets in Metric Spaces

Definition A *metric space* (X, d) consists of a set X together with a *distance function* $d\colon X \times X \to [0, +\infty)$ on X satisfying the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

The quantity $d(x, y)$ should be thought of as measuring the *distance* between the points x and y. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ is referred to as the *Triangle Inequality.* The elements of a metric space are usually referred to as *points* of that metric space.

An n-dimensional Euclidean space $\mathbb{R}^n$ is a metric space with with respect to the *Euclidean distance function* d, defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Any subset X of $\mathbb{R}^n$ may be regarded as a metric space whose distance function is the restriction to X of the Euclidean distance function on $\mathbb{R}^n$ defined above.

Definition Let (X, d) be a metric space. Given a point x of X and $r \geq 0$, the *open ball* $B_X(x, r)$ of *radius* r about x in X is defined by

$$B_X(x, r) = \{x' \in X : d(x', x) < r\}.$$

Definition Let (X, d) be a metric space. A subset V of X is said to be an *open set* if and only if the following condition is satisfied:

- given any point v of V there exists some positive real number δ such that $B_X(v, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in this case.)

Lemma 1.1 *Let X be a metric space with distance function d, and let x_0 be a point of X. Then, for any $r > 0$, the open ball $B_X(x_0, r)$ of radius r about x_0 is an open set in X.*

Proof Let $x \in B_X(x_0, r)$. We must show that there exists some positive real number δ such that $B_X(x, \delta) \subset B_X(x_0, r)$. Now $d(x, x_0) < r$, and hence $\delta > 0$, where $\delta = r - d(x, x_0)$. Moreover if $x' \in B_X(x, \delta)$ then

$$d(x', x_0) \leq d(x', x) + d(x, x_0) < \delta + d(x, x_0) = r,$$

by the Triangle Inequality, hence $x' \in B_X(x_0, r)$. Thus $B_X(x, \delta) \subset B_X(x_0, r)$, showing that $B_X(x_0, r)$ is an open set, as required. ∎

Proposition 1.2 *Let X be a metric space. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open sets;

(ii) the union of any collection of open sets is itself an open set;

(iii) the intersection of any finite *collection of open sets is itself an open set.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. Thus (i) is satisfied.

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself an open set. Let $x \in U$. Then $x \in V$ for some open set V belonging to the collection $\mathcal{A}$. Therefore there exists some positive real number δ such that $B_X(x, \delta) \subset V$. But $V \subset U$, and thus $B_X(x, \delta) \subset U$. This shows that U is open. Thus (ii) is satisfied.

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of open sets in X, and let $V = V_1 \cap V_2 \cap \cdots \cap V_k$. Let $x \in V$. Now $x \in V_j$ for all j, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(x, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of open sets.) Moreover $B_X(x, \delta) \subset B_X(x, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(x, \delta) \subset V$. This shows that the intersection V of the open sets $V_1, V_2, \ldots, V_k$ is itself open. Thus (iii) is satisfied. ∎

Any metric space may be regarded as a topological space. Indeed let X be a metric space with distance function d. We recall that a subset V of X is an *open set* if and only if, given any point v of V, there exists some positive real number δ such that

$$\{x \in X : d(x, v) < \delta\} \subset V.$$

Proposition 1.2 shows that the topological space axioms are satisfied by the collection of open sets in any metric space. We refer to this collection of open sets as the *topology* generated by the distance function d on X.

1.4 Further Examples of Topological Spaces

Example Given any set X, one can define a topology on X where every subset of X is an open set. This topology is referred to as the *discrete topology* on X.

Example Given any set X, one can define a topology on X in which the only open sets are the empty set $\emptyset$ and the whole set X.

1.5 Closed Sets

Definition Let X be a topological space. A subset F of X is said to be a *closed set* if and only if its complement $X \setminus F$ is an open set.

We recall that the complement of the union of some collection of subsets of some set X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets. The following result therefore follows directly from the definition of a topological space.

Proposition 1.3 *Let X be a topological space. Then the collection of closed sets of X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are closed sets,

(ii) the intersection of any collection of closed sets is itself a closed set,

(iii) the union of any finite collection of closed sets is itself a closed set.

1.6 Neighbourhoods, Closures and Interiors

Definition Let X be a topological space, and let x be a point of X. Let N be a subset of X which contains the point x. Then N is said to be a *neighbourhood* of the point x if and only if there exists an open set W for which $x \in W$ and $W \subset N$.

Lemma 1.4 *Let X be a topological space. A subset V of X is open in X if and only if V is a neighbourhood of each point belonging to V.*

Proof It follows directly from the definition of neighbourhoods that an open set V is a neighbourhood of any point belonging to V. Conversely, suppose that V is a subset of X which is a neighbourhood of each $v \in V$. Then, given any point v of V, there exists an open set W_v such that $v \in W_v$ and $W_v \subset V$. Thus V is an open set, since it is the union of the open sets W_v as v ranges over all points of V. ∎

Definition Let X be a topological space and let A be a subset of X. The *interior* A° of A in X is defined to be the union of all of the open subsets of X that are subsets of A.

Let X be a topological space and let A be a subset of X. It follows from the definition of a topological space that the union of open subsets of X is itself a open subset of X. It follows directly from this that the interior A° of A in X is the subset of X uniquely characterized by the following two properties:—

(i) the interior A° of A is an open set contained in A,

(ii) if W is any open set contained in A then W is contained in A°.

Lemma 1.5 *Let X be a topological space, let A be a subset of X, and let p be a point of A. Then p belongs to the interior A° if and only if A is a neighbourhood of the point p.*

Proof It follows from the definition of interiors that the point p belongs to the interior of A if and only if there exists an open set W such that $p \in W$ and $W \subset A$. It then follows from the definition of neighbourhoods that this is the case if and only if the set A is a neighbourhood of the point p. ∎

Definition Let X be a topological space and let A be a subset of X. The *closure* $\overline{A}$ of A in X is defined to be the intersection of all of the closed subsets of X that contain A.

Let X be a topological space and let A be a subset of X. Any intersection of closed subsets of X is itself a closed subset of X (see Proposition 1.3). It follows directly from this that the closure $\overline{A}$ of A in X is the subset of X uniquely characterized by the following two properties:—

(i) the closure $\overline{A}$ of A is a closed set containing A,

(ii) if F is any closed set containing A then F contains $\overline{A}$.

Lemma 1.6 *Let X be a topological space, let A be a subset of X, let $\overline{A}$ be the closure of A in X, and let V be an open set. Then $V \cap A = \emptyset$ if and only if $V \cap \overline{A} = \emptyset$.*

Proof Suppose that $V \cap A = \emptyset$. Then $A \subset X \setminus V$. Now the complement $X \setminus V$ of V is a closed set, and $\overline{A}$ is by definition the intersection of all closed sets that contain the subset A. It follows that $\overline{A} \subset X \setminus V$, and therefore $V \cap \overline{A} = \emptyset$.

Conversely suppose that $V \cap \overline{A} = \emptyset$. Then $V \cap A = \emptyset$, because A is a subset of $\overline{A}$. The result follows. ∎

Proposition 1.7 *Let X be a topological space, and let A be a subset of X. Let A° and $\overline{A}$ denote the interior and closure respectively of A, and let $(X \setminus A)^\circ$ and $\overline{X \setminus A}$ denote the interior and closure respectively of the complement $X \setminus A$ of A in X. Then*

$$X \setminus \overline{A} = (X \setminus A)^\circ \quad \textit{and} \quad X \setminus A^\circ = \overline{X \setminus A}$$

(i.e., the complement of the closure of A is the interior of the complement of A, and the complement of the interior of A is the closure of the complement of A).

Proof The interior $(X \setminus A)^\circ$ of $X \setminus A$ is by definition the union of all open subsets of X that are contained in $X \setminus A$. But an open subset V is contained in $X \setminus A$ if and only if $V \cap A = \emptyset$. It follows from Lemma 1.6 that $V \subset X \setminus A$ if and only if $V \subset X \setminus \overline{A}$. We conclude from this that $(X \setminus A)^\circ \subset X \setminus \overline{A}$. But $X \setminus \overline{A}$ is itself an open set contained in $X \setminus A$, and therefore $X \setminus \overline{A} \subset (X \setminus A)^\circ$. It follows that

$$(X \setminus A)^\circ = X \setminus \overline{A}.$$

Similarly $(X \setminus B)^\circ = X \setminus \overline{B}$, where $B = X \setminus A$, and thus $A^\circ = X \setminus \overline{B}$. Taking complements, we find that

$$X \setminus A^\circ = \overline{B} = \overline{X \setminus A}.$$

This completes the proof. ∎

1.7 Neighbourhoods and Closures in Metric Spaces

Lemma 1.8 *Let X be a metric space with distance function d, let p be a point of X and let N be a subset of X, where $p \in N$. Then N is a neighbourhood of p in X if and only if there exists some positive real number δ for which*

$$\{x \in X : d(x,p) < \delta\} \subset N.$$

Proof Let $B_X(p,\delta) = \{x \in X : d(x,p) < \delta\}$ for all positive real numbers δ. Then the open ball $B_X(p,\delta)$ in X of radius δ about the point p is an open set in X (see Lemma 1.1). It follows from the definition of neighbourhoods of points in topological spaces that if there exists some positive real number δ for which $B_X(p,\delta) \subset N$ then N is a neighbourhood of p in X.

Conversely suppose that N is a neighbourhood of p in X. Then there exists an open set W in X such that $p \in W$ and $W \subset N$. The definition of open sets in metric spaces then ensures the existence of a positive real number δ for which $B_X(p,\delta) \subset W$. Then $B_X(p,\delta) \subset N$. The result follows. ∎

Lemma 1.9 *Let X be a metric space with distance function d, let A be a subset of X, and let p be a point of X. Then p belongs to the closure $\overline{A}$ of A in X if and only if, given any positive real number δ, there exists some element x of A that satisfies $d(x,p) < \delta$.*

Proof The complement of the closure $\overline{A}$ of A is the interior of the complement $X \setminus A$ of A (see Proposition 1.7). It follows that $p \in \overline{A}$ if and only if p does not belong to the interior of $X \setminus A$. Now a point of X belongs to the interior of $X \setminus A$ if and only if $X \setminus A$ is a neighbourhood of that point (see Lemma 1.5). It follows that $p \in \overline{A}$ if and only if $X \setminus A$ is not a neighbourhood of p in X. It then follows from Lemma 1.8 that $p \in \overline{A}$ if and only if, for all positive real numbers δ, the open ball in X of radius δ about the point p intersects A. The result follows. ∎

1.8 Subspace Topologies

Lemma 1.10 *Let X be a topological space with topology τ, and let A be a subset of X. Let τ_A be the collection of all subsets of A that are of the form $V \cap A$ for $V \in \tau$. Then τ_A is a topology on the set A.*

Proof The empty set $\emptyset$ belongs to τ_A, because $\emptyset$ is open in X and $\emptyset = A \cap \emptyset$. Also $A \in \tau_A$, because X is open in itself and $A = X \cap A$.

Let $\mathcal{C}$ be a collection of subsets of A, where $W \in \tau_A$ for all $W \in \mathcal{C}$, and let Y be the union of the subsets of A belonging to the collection $\mathcal{C}$. Then for each $W \in \mathcal{C}$ there exists an open set V_W in X for which $W = A \cap V_W$. Let Z be the union of the open sets V_W as W ranges over the collection $\mathcal{C}$. Then

$$Y = \bigcup_{W \in \mathcal{C}} W = \bigcup_{W \in \mathcal{C}} (A \cap V_W) = A \cap \bigcup_{W \in \mathcal{C}} V_W = A \cap Z.$$

Moreover Z is open in X. It follows that $Y \in \tau_A$. Thus any union of subsets of A belonging to τ_A must itself belong to τ_A.

Now let $W_1, W_2, \ldots, W_m$ be subsets of A that each belong to the collection τ_A. Then there exist open sets $V_1, V_2, \ldots, V_m$ in X such that $W_i = A \cap V_i$ for $i = 1, 2, \ldots, m$. Then

$$W_1 \cap W_2 \cap \cdots \cap W_r = A \cap V,$$

where

$$V = V_1 \cap V_2 \cap \cdots \cap V_r.$$

Now V is a finite intersection of subsets of X that are open in X. It follows that V is itself open in X, and therefore

$$W_1 \cap W_2 \cap \cdots \cap W_r \in \tau_A.$$

We have thus shown that τ_A is a topology on A, as required. ∎

Definition Let X be a topological space and let A be a subset of X. The *subspace topology* on A is the topology on A whose open sets are characterized by the following criterion:

> A subset W of A is open with respect to the subspace topology on A if and only if there exists some open set V in X for which $W = A \cap V$.

Proposition 1.11 *Let X be a metric space with distance function d, let A be a subset of X, let p be a point of A and let N be a subset of A for which $p \in N$. Then N is a neighbourhood of p with respect to the subspace topology on A if and only if there exists some positive real number δ such that*

$$\{x \in A : d(x, p) < \delta\} \subset N.$$

Proof Let

$$B_A(p, \delta) = \{x \in A : d(x, p) < \delta\}$$

and

$$B_X(p, \delta) = \{x \in X : d(x, p) < \delta\}$$

for all positive real numbers δ. Suppose that there exists some positive real number δ for which $B_A(p,\delta) \subset N$. We must show that N is a neighbourhood of p with respect to the subspace topology on A. Now $B_A(p,\delta) = A \cap B_X(p,\delta)$, where $B_X(p,\delta)$ is the open ball in X of radius δ about the point p. Moreover $B_X(p,\delta)$ is open in X (Lemma 1.1) and $A \cap B_X(p,\delta) \subset N$. It follows that N is a neighbourhood of p in A with respect to the subspace topology on A.

Conversely suppose that N is a neighbourhood of p with respect to the subspace topology on A. We must show that there exists some positive real number δ for which $B_A(p,\delta) \subset N$. Now the definitions of neighbourhoods and the subspace topology together ensure the existence of an open set V in X for which $p \in V$ and $A \cap V \subset N$. It then follows from the definition of open sets in metric spaces that there exists some positive real number δ for which $B_X(p,\delta) \subset V$. Then $B_A(p,\delta) \subset A \cap V \subset N$. This completes the proof. ∎

Corollary 1.12 *Let X be a metric space with distance function d, and let A be a subset of X. A subset W of A is open with respect to the subspace topology on A if and only if, given any point w of W, there exists some positive real number δ for which*

$$\{a \in A : d(a,w) < \delta\} \subset W.$$

Thus the subspace topology on A coincides with the topology on A obtained on regarding A as a metric space whose distance function is the restriction to A of the distance function d on X.

Proof The subset W is open in A with respect to a given topology on A if and only if it is a neighbourhood of all of its points with respect to that given topology (see Lemma 1.4). The required result therefore follows from Proposition 1.11. ∎

Example Let X be any subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then the subspace topology on X coincides with the topology on X generated by the Euclidean distance function on X. We refer to this topology as the *usual topology* on X.

Lemma 1.13 *Let X be a topological space, let A be a subset of X, and let B be a subset of A. Then B is closed in A (relative to the subspace topology on A) if and only if $B = A \cap F$ for some closed subset F of X.*

Proof Suppose that $B = A \cap F$ for some closed subset F of X. Let $V = X \setminus F$. Then V is an open set in X, and

$$A \setminus B = A \setminus (A \cap F) = A \cap (X \setminus F) = A \cap V.$$

Moreover the definition of the subpace topology on A ensures that $A \cap V$ is open in A. Thus the complement $A \setminus B$ of B in A is open in A, and therefore the subset B of A is itself closed in A.

Conversely suppose that B is closed in A. Then $A \setminus B$ is open in the subspace topology on A, and therefore there exists some open set V in X such that $A \setminus B = A \cap V$. Let $F = X \setminus V$. Then F is closed in X, and

$$A \cap F = A \cap (X \setminus V) = A \setminus (A \cap V) = A \setminus (A \setminus B) = B.$$

The result follows. ∎

Lemma 1.14 *Let X be a topological space, let V be an open set in X, and let W be a subset of V. Then W is open in V if and only if W is open in X.*

Proof If W is open in X then $W = V \cap W$ and therefore W is open in V.

Conversely suppose that the set W is open in V. It then follows from the definition of subspace topologies that $W = V \cap E$ for some open set E in X. But then W is an intersection of two open sets, and is thus itself open in X. ∎

Lemma 1.15 *Let X be a topological space, let F be a closed set in X, and let G be a subset of F. Then G is closed in F if and only if G is closed in X.*

Proof If G is closed in X then $G = F \cap G$ and therefore G is closed in F.

Conversely suppose that the set G is closed in F. It then follows from Lemma 1.13 that $G = F \cap H$ for some closed set H in X. But then G is an intersection of two closed sets, and is thus itself closed in X (see Proposition 1.3). ∎

1.9 Hausdorff Spaces

Definition A topological space X is said to be a *Hausdorff space* if and only if it satisfies the following *Hausdorff Axiom*:

- if x and y are distinct points of X then there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Lemma 1.16 *Any subset of a Hausdorff space is itself a Hausdorff space (with respect to the subspace topology).*

Proof Let A be a subset of a Hausdorff space X and let x and y be distinct points of A. Then there exist open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Let $U_A = A \cap U$ and $V_A = A \cap V$. Then U_A and V_A are subsets of A that are open in the subspace topology on A. Moreover $x \in U_A$, $y \in V_A$ and $U_A \cap V_A = \emptyset$. The result follows. ∎

Lemma 1.17 *All metric spaces are Hausdorff spaces.*

Proof Let X be a metric space with distance function d, and let x and y be points of X, where $x \neq y$. Let $\varepsilon = \frac{1}{2}d(x, y)$. Then the open balls $B_X(x, \varepsilon)$ and $B_X(y, \varepsilon)$ of radius ε centred on the points x and y are open sets (see Lemma 1.1). If $B_X(x, \varepsilon) \cap B_X(y, \varepsilon)$ were non-empty then there would exist $z \in X$ satisfying $d(x, z) < \varepsilon$ and $d(z, y) < \varepsilon$. But this is impossible, since it would then follow from the Triangle Inequality that $d(x, y) < 2\varepsilon$, contrary to the choice of ε. Thus $x \in B_X(x, \varepsilon)$, $y \in B_X(y, \varepsilon)$, $B_X(x, \varepsilon) \cap B_X(y, \varepsilon) = \emptyset$. This shows that the metric space X is a Hausdorff space. ∎

We now give an example of a topological space which is not a Hausdorff space.

Example Let X be an infinite set. The *cofinite topology* on X is defined as follows: a subset U of X is open (with respect to the cofinite topology) if and only if either $U = \emptyset$ or else $X \setminus U$ is finite. It is a straightforward exercise to verify that the topological space axioms are satisfied, so that the set X is a topological space with respect to this cofinite topology. Now the intersection of any two non-empty open sets in this topology is always non-empty. (Indeed if U and V are non-empty open sets then $U = X \setminus F_1$ and $V = X \setminus F_2$, where F_1 and F_2 are finite subsets of X. But then $U \cap V = X \setminus (F_1 \cup F_2)$, which is non-empty, since $F_1 \cup F_2$ is finite and X is infinite.) It follows immediately from this that an infinite set X is not a Hausdorff space with respect to the the cofinite topology on X.

1.10 Continuous Maps between Topological Spaces

Definition A function $f \colon X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

$$f^{-1}(V) = \{x \in X : f(x) \in V\}.$$

A continuous function from X to Y is often referred to as a *map* from X to Y.

Lemma 1.18 *Let X, Y and Z be topological spaces, and let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Then the composition $g \circ f: X \to Z$ of the functions f and g is continuous.*

Proof Let V be an open set in Z. Then $g^{-1}(V)$ is open in Y (because g is continuous), and then $f^{-1}(g^{-1}(V))$ is open in X (because f is continuous). But $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Thus the composition function $g \circ f$ is continuous. ∎

Lemma 1.19 *Let X and Y be topological spaces, and let $f: X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(G)$ is closed in X for every closed subset G of Y.*

Proof If G is any subset of Y then $X \setminus f^{-1}(G) = f^{-1}(Y \setminus G)$ (i.e., the complement of the preimage of G is the preimage of the complement of G). The result therefore follows immediately from the definitions of continuity and closed sets. ∎

Definition Let X and Y be topological spaces, let $f: X \to Y$ be a function from X to Y and let p be a point of X. The function f is said to be *continuous at p* if $f^{-1}(N)$ is a neighbourhood of p in X for all neighbourhoods N of $f(p)$ in Y.

Proposition 1.20 *Let X and Y be topological spaces and let $f: X \to Y$ be a function from X to Y. Then the function f is continuous on X if and only if it is continuous at each point of X.*

Proof Suppose that $f: X \to Y$ be continuous on X. Let p be a point of X and let N be a neighbourhood of $f(p)$. Then there exists an open set V in Y for which $f(p) \in V$ and $V \subset N$. The continuity of f ensures that $f^{-1}(V)$ is open in X. Moreover $p \in f^{-1}(V)$ and $f^{-1}(V) \subset f^{-1}(N)$. It follows that $f^{-1}(N)$ is a neighbourhood of p in X. This shows that $f: X \to Y$ is continuous at each point p of X.

Conversely suppose that $f: X \to Y$ is continuous at each point of X. Let V be an open set in Y. Then, given any point p of $f^{-1}(V)$, there exists an open set W_p for which $p \in W_p$ and $W_p \subset f^{-1}(V)$, because the function f is continuous at p. Then $f^{-1}(V) = \bigcup_{p \in f^{-1}(V)} W_p$. Thus $f^{-1}(V)$ is a union of open subsets of X, and is therefore itself open in X. We conclude that $f: X \to Y$ is continuous on X. ∎

Lemma 1.21 *Let X and Y be topological spaces, let $f: X \to Y$ be a function from X to Y and let p be a point of X. Then $f: X \to Y$ is continuous at p if and only if, given any neighbourhood N of $f(p)$, there exists a neighbourhood M of p for which $f(M) \subset N$.*

Proof Let N be a neighbourhood of $f(p)$ in Y. Suppose that there exists a neighbourhood M of p in X for which $f(M) \subset N$. The definition of neighbourhoods of points in topological spaces then ensures that there exists an open set W in X for which $p \in W$ and $W \subset M$. Then $f(W) \subset N$ and therefore $W \subset f^{-1}(N)$. It follows that $f^{-1}(N)$ is a neighbourhood of p in X, and thus the function f is continuous at p.

Conversely suppose that the function f is continuous at p. Let N be a neighbourhood of $f(p)$ in Y, and let $M = f^{-1}(N)$. Then M is a neighbourhood of p in X, because the function f is continuous at p, and $f(M) \subset N$. The result follows. ∎

Lemma 1.22 *Let X, Y and Z be topological spaces, let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions, and let p be a point of X. Suppose that $f\colon X \to Y$ is continuous at p and that $g\colon Y \to Z$ is continuous at $f(p)$. Then the composition $g \circ f\colon X \to Z$ of the functions f and g is continuous at p.*

Proof Let N be a neighbourhood of $g(f(p))$ in Z. Then $g^{-1}(N)$ is a neighbourhood of $f(p)$ in Y (because g is continuous), and then $f^{-1}(g^{-1}(N))$ is a neighbourhood of p in X (because f is continuous). But $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$. Thus the composition function $g \circ f$ is continuous at p. ∎

Proposition 1.23 *Let X and Y be topological spaces and let $f\colon X \to Y$ be a function from X to Y. Then $f\colon X \to Y$ is continuous if and only if, given any point p of X, there exists some open set W in X such that $p \in W$ and the restriction $f|W\colon W \to Y$ of the function f to W is continuous on W.*

Proof Suppose that $f\colon X \to Y$ is continuous. Let W be an open set in X, and let V be an open set in Y. Then the preimage $f^{-1}(V)$ of V is open in X. Now $(f|W)^{-1}(V) = f^{-1}(V) \cap W$. It follows that $(f|W)^{-1}(V)$ is open with respect to the subspace topology on W.

Conversely suppose that, given any point p of X, there exists an open set W in X such that $p \in W$ and $f|W\colon W \to Y$ is continuous. Let p be a point of X and let W be an open set in X for which $p \in W$ and $f|W\colon\colon W \to Y$ is continuous. Let N be a neighbourhood of $f(p)$ in Y. Then $(f|W)^{-1}(N)$ is a neighbourhood of p in W. It follows from the definition of the subspace topology on W that there exists an open set E in X for which $p \in E$ and $f(E \cap W) \subset N$. But then $E \cap W$ is an open set in X, because both E and W are open sets in X. It follows that $f^{-1}(N)$ is an open neighbourhood of p in X. We have thus shown that the function f is continuous at p. It then follows from Proposition 1.20 that $f\colon X \to Y$ is continuous, as required. ∎

1.11 The Pasting Lemma

We now show that, if a topological space X is the union of a finite collection of closed sets, and if a function from X to some topological space is continuous on each of these closed sets, then that function is continuous on X. The names *Pasting Lemma* and *Gluing Lemma* are both used to refer to this result.

Lemma 1.24 (Pasting Lemma) *Let X and Y be topological spaces, let $f: X \to Y$ be a function from X to Y, and let $X = A_1 \cup A_2 \cup \cdots \cup A_k$, where $A_1, A_2, \ldots, A_k$ are closed sets in X. Suppose that the restriction of f to the closed set A_i is continuous for $i = 1, 2, \ldots, k$. Then $f: X \to Y$ is continuous.*

Proof Let p be a point of X, and let N be a neighbourhood of $f(p)$. The continuity of the restriction of f to each closed set A_i ensures the existence of open sets W_i for $i = 1, 2, \ldots, k$ such that $W_i \cap A_i = \emptyset$ whenever $p \notin A_i$ and $f(W_i \cap A_i) \subset N$ whenever $p \in A_i$. Let

$$W = W_1 \cap W_2 \cap \cdots \cap W_k$$

Then W is an open set in X, and $p \in W$. Moreover if $x \in W$ then there exists some integer i between 1 and k for which $x \in A_i$ and $p \in A_i$. Then $x \in W_i \cap A_i$, and therefore $f(x) \in N$. We conclude from this that the function f is continuous at each point p of X. It follows that the function f is continuous on X (see Proposition 1.20). ∎

Alternative Proof A function $f: X \to Y$ is continuous if and only if $f^{-1}(G)$ is closed in X for every closed set G in Y (Lemma 1.19). Let G be an closed set in Y. Then $f^{-1}(G) \cap A_i$ is closed in the subspace topology on A_i for $i = 1, 2, \ldots, k$, because the restriction of f to A_i is continuous for each i. But A_i is closed in X, and therefore a subset of A_i is closed in A_i if and only if it is closed in X (see Lemma 1.15). Therefore $f^{-1}(G) \cap A_i$ is closed in X for $i = 1, 2, \ldots, k$. Now $f^{-1}(G)$ is the union of the sets $f^{-1}(G) \cap A_i$ for $i = 1, 2, \ldots, k$. It follows that $f^{-1}(G)$, being a finite union of closed sets, is itself closed in X. It now follows from Lemma 1.19 that $f: X \to Y$ is continuous. ∎

Example Let Y be a topological space, and let $\alpha: [0, 1] \to Y$ and $\beta: [0, 1] \to Y$ be continuous functions defined on the interval $[0, 1]$, where $\alpha(1) = \beta(0)$. Let $\gamma: [0, 1] \to Y$ be defined by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now $\gamma|[0, \frac{1}{2}] = \alpha \circ \rho$ where $\rho: [0, \frac{1}{2}] \to [0, 1]$ is the continuous function defined by $\rho(t) = 2t$ for all $t \in [0, \frac{1}{2}]$. Thus $\gamma|[0, \frac{1}{2}]$ is continuous, being a composition of two continuous functions. Similarly $\gamma|[\frac{1}{2}, 1]$ is continuous. The subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed in $[0, 1]$, and $[0, 1]$ is the union of these two subintervals. It follows from Lemma 1.24 that $\gamma: [0, 1] \to Y$ is continuous.

Example Let X be the surface of a closed cube in $\mathbb{R}^3$ and let $f: X \to Y$ be a function mapping X into a topological space Y. The topological space X is the union of the six square faces of the cube, and each of these faces is a closed subset of X. The Pasting Lemma Lemma 1.24 ensures that the function f is continuous if and only if its restrictions to each of the six faces of the cube is continuous on that face.

We now present a couple of examples to show that the conclusions of the Pasting Lemma (Lemma 1.24) do not follow when the conditions stated in that lemma are relaxed.

Example Let $f: \mathbb{R} \to \mathbb{R}$ be defined so that

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases}$$

and let $A_1 = \{x \in \mathbb{R} : x \leq 0\}$ and $A_2 = \{x \in \mathbb{R} : x > 0\}$. The restriction of the function f to each of the subsets A_1 and A_2 of $\mathbb{R}$ is continuous on that subset, but the function f itself is not continuous on $\mathbb{R}$. This does not contradict the Pasting Lemma because the subset A_2 of $\mathbb{R}$ is not closed in $\mathbb{R}$.

Example Let

$$X = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{Z} \text{ and } n > 0 \right\},$$

and let $f: X \to \mathbb{R}$ be defined so that $f(0) = 0$ and $f(1/n) = n$ for all positive integers n. For each $x \in X$, the set $\{x\}$ is a closed subset of X, and the restriction of f to each of these one-point subsets is continuous on that subset. But the function f itself is not continuous on X. This does not contradict the Pasting Lemma because the number of these one-point closed subsets of X is infinite.

1.12 Continuous Functions between Metric Spaces

The following proposition shows that the definition of continuity for functions between topological spaces is consistent with the standard definition of continuity for functions between metric spaces that is expressed directly in terms of distance functions on those metric spaces.

Proposition 1.25 *Let X and Y be metric spaces with distance functions d_X and d_Y respectively, let $f: X \to Y$ be a function from X to Y, and let p be a point of X. Then the following two conditions are equivalent:*

(i) given any neighbourhood N of $f(p)$ in Y, there exists a neighbourhood M of p in X for which $f(M) \subset N$;

(ii) given any positive real number ε, there exists some positive real number δ such that $d_Y(f(x), f(p)) < \varepsilon$ for all points x of X for which $d(x,p) < \delta$.

(iii) the function $f: X \to Y$ is continuous at p.

Proof Suppose that, given any neighbourhood N of $f(p)$ in Y, there exists a neighbourhood M of p for which $f(M) \subset N$. Let some positive real number ε be given. Then the open ball $B_Y(f(p), \varepsilon)$ of radius ε about the point $f(p)$ is a neighbourhood of $f(p)$ in Y. It follows that there exists a neighbourhood M of p for which $f(M) \subset B_Y(f(p), \varepsilon)$. There then exists some positive real number δ such that $B_X(p, \delta) \subset M$ (see Lemma 1.8). If $x \in X$ satisfies $d_X(x,p) < \delta$ then $x \in M$ and therefore $f(x) \in B_Y(f(p), \varepsilon)$. But then $d_Y(f(x), f(p)) < \varepsilon$. Thus (i) implies (ii).

Conversely suppose that, given any positive real number ε, there exists some positive real number δ such that $d_Y(f(x), f(p)) < \varepsilon$ for all points x of X for which $d(x,p) < \delta$. Let N be a neighbourhood of $f(p)$. Then there exists some positive real number ε for which $B_Y(f(p), \varepsilon) \subset N$, where $B_Y(f(p), \varepsilon)$ denotes the open ball of radius ε about the point $f(p)$. There then exists some positive real number δ for which $f(B_X(p, \delta)) \subset B_Y(f(p), \varepsilon)$, where $B_X(p, \delta)$ denotes the open ball of radius δ about the point p. Let $M = B_X(p, \delta)$. Then M is a neighbourhood of p in X and $f(M) \subset N$. Thus (ii) implies (i).

The equivalence of (i) and (iii), for functions between general topological spaces, was proved in Lemma 1.21. This completes the proof. ∎

1.13 Homeomorphisms

Definition Let X and Y be topological spaces. A function $h: X \to Y$ is said to be a *homeomorphism* if and only if the following conditions are satisfied:

- the function $h: X \to Y$ is both injective and surjective (so that the function $h: X \to Y$ has a well-defined inverse $h^{-1}: Y \to X$),
- the function $h: X \to Y$ and its inverse $h^{-1}: Y \to X$ are both continuous.

Two topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism $h: X \to Y$ from X to Y.

If $h: X \to Y$ is a homeomorphism between topological spaces X and Y then h induces a one-to-one correspondence between the open sets of X and the open sets of Y. Thus the topological spaces X and Y can be regarded as being identical as topological spaces.

1.14 Bases for Topologies

Proposition 1.26 *Let X be a set, let β be a collection of subsets of X, and let τ be the collection consisting of the empty set, together with all subsets of X that are unions of sets belonging to the collection β. Then τ is a topology on X if and only if the following conditions are satisfied:—*

(i) the set X is the union of the subsets belonging to the collection β;

(ii) given subsets $B_1, B_2 \in \beta$, and given any point p of $B_1 \cap B_2$, there exists some $B \in \beta$ such that $p \in B$ and $B \subset B_1 \cap B_2$.

Proof First suppose that τ is a topology on X. Then $X \in \tau$. But any subset of X that belongs to τ is a union of sets belonging to β. Therefore X is a union of subsets belonging to the collection β, and thus condition (i) is satisfied.

Moreover the intersection of any two open subsets of a topological space is required to be open. Thus if τ is a topology on X, and if $B_1, B_2 \in \beta$, then $B_1, B_2 \in \tau$ and therefore $B_1 \cap B_2 \in \tau$. It follows that $B_1 \cap B_2$ is a union of subsets of X that belong to β, and therefore, given any $p \in B_1 \cap B_2$, there exists $B \in \beta$ such that $p \in B$ and $B \subset B_1 \cap B_2$. Thus condition (ii) is satisfied.

Conversely we must prove that if the collection β of subsets of a set X satisfies conditions (i) and (ii) then the collection τ of unions of sets belonging to β is a topology on X.

The empty set belongs to τ. Condition (i) ensures that the whole set X belongs to τ. It follows directly from the definition of τ that any union of sets belonging to τ is a union of sets belonging to β, and therefore itself belongs to τ.

It therefore only remains to show that the intersection of any finite collection of sets belonging to τ belongs to τ. It suffices to prove that the intersection of two sets belonging to τ belongs to τ. Let $V_1, V_2 \in \tau$, and let $p \in V_1 \cap V_2$. Then V_1 and V_2 are union of sets belonging to β, and therefore there exist $B_1, B_2 \in \beta$ such that $p \in B_1$, $p \in B_2$, $B_1 \subset V_1$, and $B_2 \subset V_2$. Now condition (ii) ensures the existence of $B_p \in \beta$ such that $p \in B_p$ and $B_p \subset B_1 \cap B_2$. Then $B_p \subset V_1 \cap V_2$. It follows that the set $V_1 \cap V_2$ is the union of all subsets B of $V_1 \cap V_2$ that belong to β, and therefore $V_1 \cap V_2$ itself belongs to τ. It then follows by induction on the number of sets involved that the intersection of any finite number of subsets of X belonging to τ must itself belong to τ. Thus τ is a topology on the set X, as required. ∎

Definition Let X be a set. A collection β of subsets of X is said to be a *base* for a topology on X if the following conditions are satisfied:—

(i) the set X is the union of the subsets belonging to the collection β;

(ii) given subsets $B_1, B_2 \in \beta$, and given any point p of $B_1 \cap B_2$, there exists some $B \in \beta$ such that $p \in B$ and $B \subset B_1 \cap B_2$.

If β is a base for a topology on X then the topology generated by β is the topology whose open sets are those subsets of X that are unions of sets belonging to the base β.

Lemma 1.27 *Let X be a set, and let β be a base for a topology on X. A non-empty subset V is open in X with respect to the topology generated by β if and only if, given any point v of V, there exists $B \in \beta$ such that $v \in B$ and $B \subset V$.*

Proof This result follows directly from the fact that the non-empty open sets in X are those subsets of X that are unions of sets belonging to the base β. ∎

Example Let X be a metric space. Then the collection of all open balls of positive radius centred on points of X is a base for the topology on X generated by the distance function on X.

1.15 Product Topologies

The *Cartesian product* $X_1 \times X_2 \times \cdots \times X_n$ of sets $X_1, X_2, \ldots, X_n$ is defined to be the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for $i = 1, 2, \ldots, n$.

The sets $\mathbb{R}^2$ and $\mathbb{R}^3$ are the Cartesian products $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ respectively.

Let $X_1, X_2, X_3, \ldots, X_n$ be topological spaces, and let V_i and W_i be open sets in X_i for $i = 1, 2, \ldots, n$. Then

$$(V_1 \times V_2 \times \cdots \times V_n) \cap (W_1 \times W_2 \times \cdots \times W_n) = E_1 \times E_2 \times \cdots \times E_n,$$

where $E_i = V_i \cap W_i$ for $i = 1, 2, \ldots, n$. The intersection of two open sets in a topological space is always itself open. Therefore E_i is an open set in X_i

for $i = 1, 2, \ldots, n$. It follows from this that if β is the collection of subsets of $X_1 \times X_2 \times \cdots \times X_n$ that are of the form $V_1 \times V_2 \times \cdots \times V_n$, where V_i is open in X_i for $i = 1, 2, \ldots, n$, then β is the base for a topology on $X_1 \times X_2 \times \cdots \times X_n$. This topology is the *product topology* on this Cartesian product of topological spaces. Lemma 1.27 ensures that a non-empty subset W of $X_1 \times X_2 \times \cdots \times X_n$ is open in $X_1 \times X_2 \times \cdots \times X_n$ with respect to this product topology if and only if, given any point $(x_1, x_2, \ldots, x_n)$ of W, there exist open sets $V_1, V_2, \ldots, V_n$ such that $x_i \in V_i$ for $i = 1, 2, \ldots, n$ and

$$V_1 \times V_2 \times \cdots \times V_n \subset W.$$

The definition of the product topology is then encapsulated in the following formal definition.

Definition Let $X_1, X_2, \ldots, X_n$ be topological spaces. The *product topology* on the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is the unique topology on this Cartesian product of sets that satisfies the following criterion:

> a non-empty subset W of the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is open with respect to the product topology if and only if, given any point $(x_1, x_2, \ldots, x_n)$ of W, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $x_i \in V_i$ for $i = 1, 2, \ldots, n$ and
>
> $$V_1 \times V_2 \times \cdots \times V_n \subset W.$$

The following result follows directly from the definition of the product topology.

Lemma 1.28 *Let $X_1, X_2, \ldots, X_n$ be topological spaces, let p be a point of $X_1 \times X_2 \times \cdots \times X_n$, and let N be a subset of $X_1 \times X_2 \times \cdots \times X_n$ for which $p \in N$. Then N is a neighbourhood of p in X if and only if there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ for which $p \in V_1 \times V_2 \cdots \times V_n$ and $V_1 \times V_2 \times \cdots \times V_n \subset N$.*

Lemma 1.29 *Let $X_1, X_2, \ldots, X_n$ and Z be topological spaces. Then a function $f\colon X_1 \times X_2 \times \cdots \times X_n \to Z$ is continuous at a point p of $X_1 \times X_2 \times \cdots \times X_n$ if and only if, and given any open set W in Z containing $f(p)$, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ for which $p \in V_1 \times V_2 \cdots \times V_n$ and $f(V_1 \times V_2 \times \cdots \times V_n) \subset W$.*

Proof Given any neighbourhood N of $f(p)$, there exists an open set W in Y such that $f(p) \in W$ and $W \subset N$. It follows from this that the function f is continuous at p if and only if $f^{-1}(W)$ is a neighbourhood of p in X for all open sets W in Y for which $f(p) \in W$. The result therefore follows on applying Lemma 1.28. ∎

Let $X_1, X_2, \ldots, X_n$ be topological spaces, and let V_i be an open set in X_i for $i = 1, 2, \ldots, n$. It follows directly from the definition of the product topology that $V_1 \times V_2 \times \cdots \times V_n$ is open in $X_1 \times X_2 \times \cdots \times X_n$.

Proposition 1.30 *Let $X = X_1 \times X_2 \times \cdots \times X_n$, where $X_1, X_2, \ldots, X_n$ are topological spaces and X is given the product topology, and for each i, let $p_i \colon X \to X_i$ denote the projection function which sends $(x_1, x_2, \ldots, x_n) \in X$ to x_i. Let $f \colon Z \to X$ mapping a topological space Z into X and let z be a point of Z. Then $f \colon Z \to X$ is continuous at z if and only if $p_i \circ f \colon Z \to X_i$ is continuous at z for $i = 1, 2, \ldots, n$.*

Proof Let V be an open set in X_i. Then

$$p_i^{-1}(V) = X_1 \times \cdots \times X_{i-1} \times V \times X_{i+1} \times \cdots \times X_n,$$

and therefore $p_i^{-1}(V)$ is open in X. Thus $p_i \colon X \to X_i$ is continuous for all i. It follows that if the function $f \colon Z \to X$ is continuous at a point z of Z then the composition functions $p_i \circ f$ are also continuous at z for $i = 1, 2, \ldots, n$ (see Lemma 1.22).

Conversely suppose that $f \colon Z \to X$ is a function with the property that $p_i \circ f$ is continuous at z for $i = 1, 2, \ldots, n$, where $z \in Z$. Let N be a neighbourhood of $f(z)$ in X. Then there exist $V_1, V_2, \ldots, V_n$, where V_i is open in X_i for $i = 1, 2, \ldots, n$, such that $f(z) \in V_1 \times V_2 \times \cdots \times V_n$ and $V_1 \times V_2 \times \cdots \times V_n \subset N$ (see Lemma 1.28). Let

$$W_z = f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \cap \cdots \cap f_n^{-1}(V_n),$$

where $f_i = p_i \circ f$ for $i = 1, 2, \ldots, n$. Then $z \in W_z$, and the continuity of $f_1, f_2, \ldots, f_n$ ensures that W_z is an open set in Z. Moreover $f(z') \in V_1 \times V_2 \times \cdots \times V_n$ for all $z' \in W_z$, and therefore $W_z \subset f^{-1}(N)$. We have thus shown that $f^{-1}(N)$ is a neighbourhood of z for all neighbourhoods N of $f(z)$. It follows that $f \colon Z \to X$ is continuous at z, as required. ∎

Proposition 1.31 *The usual topology on $\mathbb{R}^n$ coincides with the product topology on $\mathbb{R}^n$ obtained on regarding $\mathbb{R}^n$ as the Cartesian product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ of n copies of the real line $\mathbb{R}$.*

Proof We must show that a subset W of $\mathbb{R}^n$ is open with respect to the usual topology if and only if it is open with respect to the product topology.

Let W be a subset of $\mathbb{R}^n$ that is open with respect to the usual topology, and let $\mathbf{q} \in W$. Then there exists some positive real number δ such that $B(\mathbf{q}, \delta) \subset W$, where

$$B(\mathbf{q}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{q}| < \delta\}.$$

Let $J_1, J_2, \ldots, J_n$ be the open intervals in $\mathbb{R}$ defined by

$$J_i = \left\{ t \in \mathbb{R} : q_i - \frac{\delta}{\sqrt{n}} < t < q_i + \frac{\delta}{\sqrt{n}} \right\} \qquad (i = 1, 2, \ldots, n),$$

Then $J_1, J_2, \ldots, J_n$ are open sets in $\mathbb{R}$. Moreover

$$\{\mathbf{q}\} \subset J_1 \times J_2 \times \cdots \times J_n \subset B(\mathbf{q}, \delta) \subset W,$$

since

$$|\mathbf{x} - \mathbf{q}|^2 = \sum_{i=1}^{n} (x_i - q_i)^2 < n \left(\frac{\delta}{\sqrt{n}} \right)^2 = \delta^2$$

for all $\mathbf{x} \in J_1 \times J_2 \times \cdots \times J_n$. This shows that any subset W of $\mathbb{R}^n$ that is open with respect to the usual topology on $\mathbb{R}^n$ is also open with respect to the product topology on $\mathbb{R}^n$.

Conversely suppose that W is a subset of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$, and let $\mathbf{q} \in W$. Then there exist open sets $V_1, V_2, \ldots, V_n$ in $\mathbb{R}$ containing $q_1, q_2, \ldots, q_n$ respectively such that $V_1 \times V_2 \times \cdots \times V_n \subset W$. Now we can find $\delta_1, \delta_2, \ldots, \delta_n$ such that $\delta_i > 0$ and $(q_i - \delta_i, q_i + \delta_i) \subset V_i$ for all i. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \ldots, \delta_n$. Then $\delta > 0$, and

$$B(\mathbf{q}, \delta) \subset V_1 \times V_2 \times \cdots \times V_n \subset W,$$

for if $\mathbf{x} \in B(\mathbf{q}, \delta)$ then $|x_i - q_i| < \delta_i$ for $i = 1, 2, \ldots, n$. This shows that any subset W of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$ is also open with respect to the usual topology on $\mathbb{R}^n$. ∎

The following result is now an immediate corollary of Proposition 1.31 and Proposition 1.30.

Corollary 1.32 *Let X be a topological space and let $f: X \to \mathbb{R}^n$ be a function from X to $\mathbb{R}^n$. Let us write*

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

for all $x \in X$, where the components $f_1, f_2, \ldots, f_n$ of f are functions from X to $\mathbb{R}$. The function f is continuous if and only if its components $f_1, f_2, \ldots, f_n$ are all continuous.

Let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous real-valued functions on some topological space X. We claim that $f+g$, $f-g$ and $f.g$ are continuous. Now it is a straightforward exercise to verify that the sum and product functions

$s: \mathbb{R}^2 \to \mathbb{R}$ and $p: \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x,y) = x + y$ and $p(x,y) = xy$ are continuous, and $f + g = s \circ h$ and $f.g = p \circ h$, where $h: X \to \mathbb{R}^2$ is defined by $h(x) = (f(x), g(x))$. Moreover it follows from Corollary 1.32 that the function h is continuous, and compositions of continuous functions are continuous. Therefore $f + g$ and $f.g$ are continuous, as claimed. Also $-g$ is continuous, and $f - g = f + (-g)$, and therefore $f - g$ is continuous. If in addition the continuous function g is non-zero everywhere on X then $1/g$ is continuous (since $1/g$ is the composition of g with the reciprocal function $t \mapsto 1/t$), and therefore f/g is continuous.

Lemma 1.33 *The Cartesian product $X_1 \times X_2 \times \ldots X_n$ of Hausdorff spaces $X_1, X_2, \ldots, X_n$ is Hausdorff.*

Proof Let $X = X_1 \times X_2 \times \ldots, X_n$, and let u and v be distinct points of X, where $u = (x_1, x_2, \ldots, x_n)$ and $v = (y_1, y_2, \ldots, y_n)$. Then $x_i \neq y_i$ for some integer i between 1 and n. But then there exist open sets U and V in X_i such that $x_i \in U$, $y_i \in V$ and $U \cap V = \emptyset$ (since X_i is a Hausdorff space). Let $p_i: X \to X_i$ denote the projection function. Then $p_i^{-1}(U)$ and $p_i^{-1}(V)$ are open sets in X, since p_i is continuous. Moreover $u \in p_i^{-1}(U)$, $v \in p_i^{-1}(V)$, and $p_i^{-1}(U) \cap p_i^{-1}(V) = \emptyset$. Thus X is Hausdorff, as required. ∎

1.16 Identification Maps and Quotient Topologies

Definition Let X and Y be topological spaces and let $q: X \to Y$ be a function from X to Y. The function q is said to be an *identification map* if and only if the following conditions are satisfied:

- the function $q: X \to Y$ is surjective,
- a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

It follows directly from the definition that any identification map is continuous. Moreover, in order to show that a continuous surjection $q: X \to Y$ is an identification map, it suffices to prove that if V is a subset of Y with the property that $q^{-1}(V)$ is open in X then V is open in Y.

Lemma 1.34 *Let X be a topological space, let Y be a set, and let $q: X \to Y$ be a surjection. Then there is a unique topology on Y for which the function $q: X \to Y$ is an identification map.*

Proof Let τ be the collection consisting of all subsets U of Y for which $q^{-1}(U)$ is open in X. Now $q^{-1}(\emptyset) = \emptyset$, and $q^{-1}(Y) = X$, so that $\emptyset \in \tau$ and $Y \in \tau$.

Let $\{V_\alpha : \alpha \in A\}$ be a collection of subsets of Y indexed by a set A. Then it is a straightforward exercise to verify that

$$\bigcup_{\alpha \in A} q^{-1}(V_\alpha) = q^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right),$$

and

$$\bigcap_{\alpha \in A} q^{-1}(V_\alpha) = q^{-1}\left(\bigcap_{\alpha \in A} V_\alpha\right)$$

(i.e., given any collection of subsets of Y, the union of the preimages of the sets is the preimage of the union of those sets, and the intersection of the preimages of the sets is the preimage of the intersection of those sets). It follows easily from this that unions and finite intersections of sets belonging to τ must themselves belong to τ. Thus τ is a topology on Y, and the function $q: X \to Y$ is an identification map with respect to the topology τ. Clearly τ is the unique topology on Y for which the function $q: X \to Y$ is an identification map. ∎

Definition Let X be a topological space, let Y be a set, and let $q: X \to Y$ be a surjection. The unique topology on Y for which the function q is an identification map is referred to as the *quotient topology* (or *identification topology*) on Y.

Lemma 1.35 *Let X and Y be topological spaces and let $q: X \to Y$ be an identification map. Let Z be a topological space, and let $f: Y \to Z$ be a function from Y to Z. Then the function f is continuous if and only if the composition function $f \circ q: X \to Z$ is continuous.*

Proof Suppose that f is continuous. Then the composition function $f \circ q$ is a composition of continuous functions and hence is itself continuous.

Conversely suppose that $f \circ q$ is continuous. Let U be an open set in Z. Then $q^{-1}(f^{-1}(U))$ is open in X (since $f \circ q$ is continuous), and hence $f^{-1}(U)$ is open in Y (since the function q is an identification map). Therefore the function f is continuous, as required. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, and let $q: [0,1] \to S^1$ be the map that sends $t \in [0,1]$ to $(\cos 2\pi t, \sin 2\pi t)$. Then $q: [0,1] \to S^1$ is an identification map, and therefore a function $f: S^1 \to Z$ from S^1 to some topological space Z is continuous if and only if $f \circ q: [0,1] \to Z$ is continuous.

Example Let S^n be the n-sphere, consisting of all points $\mathbf{x}$ in $\mathbb{R}^{n+1}$ satisfying $|\mathbf{x}| = 1$. Let $\mathbb{R}P^n$ be the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin (i.e., $\mathbb{R}P^n$ is the set of all one-dimensional vector subspaces of $\mathbb{R}^{n+1}$). Let

$q\colon S^n \to \mathbb{R}P^n$ denote the function which sends a point $\mathbf{x}$ of S^n to the element of $\mathbb{R}P^n$ represented by the line in $\mathbb{R}^{n+1}$ that passes through both $\mathbf{x}$ and the origin. Note that each element of $\mathbb{R}P^n$ is the image (under q) of exactly two antipodal points $\mathbf{x}$ and $-\mathbf{x}$ of S^n. The function q induces a corresponding quotient topology on $\mathbb{R}P^n$ such that $q\colon S^n \to \mathbb{R}P^n$ is an identification map. The set $\mathbb{R}P^n$, with this topology, is referred to as *real projective n-dimensional space*. In particular $\mathbb{R}P^2$ is referred to as the *real projective plane*. It follows from Lemma 1.35 that a function $f\colon \mathbb{R}P^n \to Z$ from $\mathbb{R}P^n$ to any topological space Z is continuous if and only if the composition function $f \circ q\colon S^n \to Z$ is continuous.

1.17 Compact Topological Spaces

Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to *cover* A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{V}$ and $\mathcal{W}$ are open covers of some topological space X then $\mathcal{W}$ is said to be a *subcover* of $\mathcal{V}$ if and only if every open set belonging to $\mathcal{W}$ also belongs to $\mathcal{V}$.

Definition A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 1.36 *Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{V}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{V}$ such that $A \subset V_1 \cup V_2 \cup \cdots \cup V_r$.*

Proof A subset B of A is open in A (with respect to the subspace topology on A) if and only if $B = A \cap V$ for some open set V in X. The desired result therefore follows directly from the definition of compactness. ∎

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem*. The proof of this theorem uses the *Least Upper Bound Principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 1.37 (Heine-Borel Theorem in One Dimension) *Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof Let $\mathcal{V}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a,b]$ belongs to at least one of these open sets. We must show that $[a,b]$ is covered by finitely many of these open sets.

Let S be the set of all $\tau \in [a,b]$ with the property that $[a,\tau]$ is covered by some finite collection of open sets belonging to $\mathcal{V}$, and let $s = \sup S$. Now $s \in W$ for some open set W belonging to $\mathcal{V}$. Moreover W is open in $\mathbb{R}$, and therefore there exists some positive real number δ such that $(s-\delta, s+\delta) \subset W$. Moreover $s - \delta$ is not an upper bound for the set S, hence there exists some $\tau \in S$ satisfying $\tau > s - \delta$. It follows from the definition of S that $[a,\tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{V}$.

Let $t \in [a,b]$ satisfy $\tau \leq t < s + \delta$. Then

$$[a,t] \subset [a,\tau] \cup (s-\delta, s+\delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $s \in S$, and moreover $s = b$, since otherwise s would not be an upper bound of the set S. Thus $b \in S$, and therefore $[a,b]$ is covered by a finite collection of open sets belonging to $\mathcal{V}$, as required. ∎

Lemma 1.38 *Let A be a closed subset of some compact topological space X. Then A is compact.*

Proof Let $\mathcal{V}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{V}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{V}$ that belong to this finite subcover. It follows from Lemma 1.36 that A is compact, as required. ∎

Lemma 1.39 *Let $f\colon X \to Y$ be a continuous function between topological spaces X and Y, and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$. Then A is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact. ∎

Lemma 1.40 *Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.*

Proof Let $V_j = \{x \in X : -j < f(x) < j\}$ for all positive integers j. For each integer j the subset V_j of X is the preimage under the continuous map f of the open interval $(-j, j)$, and moreover $(-j, j)$ is open in $\mathbb{R}$. It follows from the continuity of f that V_j is an open set in X for all positive integers j. Moreover the compact topological space X is covered by these open sets. It follows from the compactness of X that there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X = V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let N be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $-N < f(x) < N$ for all $x \in X$. The result follows. ∎

Proposition 1.41 *Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.*

Proof The function $f\colon X \to \mathbb{R}$ is bounded on X (Lemma 1.40). Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. For each positive integer j let $V_j = \{x \in X : f(x) < M - 1/j\}$. Then the set V_j is an open set in X, being the preimage of an open interval in $\mathbb{R}$ under the continuous map f. If $j_1, j_2, \ldots, j_k$ are positive integers then

$$V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k} = V_N$$

where N is the largest of the positive integers $j_1, j_2, \ldots, j_k$. Moreover V_N is a proper subset of X, because $M - 1/N$ is not an upper bound on the values of the function f on X. It follows that X cannot covered by any finite collection of sets from the collection $(V_j : j \in \mathbb{N})$. It then follows from the compactness of X that $(V_j : j \in \mathbb{N})$ is not an open cover of X, and therefore there exists $v \in X$ for which $f(v) = M$. Applying this argument with f replaced by $-f$, we conclude that there also exists $u \in X$ for which $f(u) = m$. Then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required. ∎

1.18 Compact Subsets of Hausdorff Spaces

Proposition 1.42 *Let X be a Hausdorff topological space, and let K be a compact subset of X. Let x be a point of $X \setminus K$. Then there exist open sets V and W in X such that $x \in V$, $K \subset W$ and $V \cap W = \emptyset$.*

Proof For each point $y \in K$ there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since X is a Hausdorff space). But then there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of K such that K is contained in $W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}$, since K is compact. Define

$$V = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \qquad W = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Then V and W are open sets, $x \in V$, $K \subset W$ and $V \cap W = \emptyset$, as required. ∎

Corollary 1.43 *A compact subset of a Hausdorff topological space is closed.*

Proof Let K be a compact subset of a Hausdorff topological space X. It follows immediately from Proposition 1.42 that, for each $x \in X \setminus K$, there exists an open set V_x such that $x \in V_x$ and $V_x \cap K = \emptyset$. But then $X \setminus K$ is equal to the union of the open sets V_x as x ranges over all points of $X \setminus K$, and any set that is a union of open sets is itself an open set. We conclude that $X \setminus K$ is open, and thus K is closed. ∎

Lemma 1.44 *Let $f\colon X \to Y$ be a continuous function from a compact topological space X to a Hausdorff space Y. Then $f(K)$ is closed in Y for every closed set K in X.*

Proof If K is a closed set in X, then K is compact (Lemma 1.38), and therefore $f(K)$ is compact (Lemma 1.39). But any compact subset of a Hausdorff space is closed (Corollary 1.43). Thus $f(K)$ is closed in Y, as required. ∎

Theorem 1.45 *A continuous bijection $f\colon X \to Y$ from a compact topological space X to a Hausdorff space Y is a homeomorphism.*

Proof Let $g\colon Y \to X$ be the inverse of the bijection $f\colon X \to Y$. If U is open in X then $X \setminus U$ is closed in X, and hence $f(X \setminus U)$ is closed in Y (see Lemma 1.44). But $f(X \setminus U) = g^{-1}(X \setminus U) = Y \setminus g^{-1}(U)$. It follows that $g^{-1}(U)$ is open in Y for every open set U in X. Therefore $g\colon Y \to X$ is continuous, and thus $f\colon X \to Y$ is a homeomorphism. ∎

Proposition 1.46 *A continuous surjection $f\colon X \to Y$ from a compact topological space X to a Hausdorff space Y is an identification map.*

Proof Let U be a subset of Y. We claim that $Y \setminus U = f(K)$, where $K = X \setminus f^{-1}(U)$. Clearly $f(K) \subset Y \setminus U$. Also, given any $y \in Y \setminus U$, there exists $x \in X$ satisfying $y = f(x)$, since $f\colon X \to Y$ is surjective. Moreover $x \in K$, since $f(x) \notin U$. Thus $Y \setminus U \subset f(K)$, and hence $Y \setminus U = f(K)$, as claimed.

We must show that the set U is open in Y if and only if $f^{-1}(U)$ is open in X. First suppose that $f^{-1}(U)$ is open in X. Then K is closed in X, and hence $f(K)$ is closed in Y, by Lemma 1.44. It follows that U is open in Y. Conversely if U is open in Y then $f^{-1}(U)$ is open in X, since $f\colon X \to Y$ is continuous. Thus the surjection $f\colon X \to Y$ is an identification map. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, defined by $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and let $q\colon [0, 1] \to S^1$ be defined by $q(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in [0, 1]$. It has been shown that the map q is an identification map. This also follows directly from the fact that $q\colon [0, 1] \to S^1$ is a continuous surjection from the compact space $[0, 1]$ to the Hausdorff space S^1.

1.19 The Lebesgue Lemma and Uniform Continuity

Definition Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists a non-negative real number K such that $d(x, y) \leq K$ for all $x, y \in A$. The smallest real number K with this property is referred to as the *diameter* of A, and is denoted by $\operatorname{diam} A$. (Note that $\operatorname{diam} A$ is the supremum of the values of $d(x, y)$ as x and y range over all points of A.)

Lemma 1.47 (Lebesgue Lemma) *Let (X, d) be a compact metric space and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{V}$.*

Proof Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{V}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{V}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_r$. Then $\delta > 0$. Suppose that A is a subset of X whose diameter is less than δ. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{V}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{V}$, as required. ∎

Let $\mathcal{V}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{V}$ is a positive real number δ such that every subset of X

whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{V}$. The Lebesgue Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

Definition Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f\colon X \to Y$ be a function from X to Y. The function f is said to be *uniformly continuous* on X if and only if, given $\varepsilon > 0$, there exists some positive real number δ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x and x' of X satisfying $d_X(x, x') < \delta$. (The value of δ should be independent of both x and x'.)

Theorem 1.48 *Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.*

Proof Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $f\colon X \to Y$ be a continuous function from X to Y. We must show that f is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}\left(B_Y(y, \frac{1}{2}\varepsilon)\right)$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about y in Y. Now the open ball $B_Y(y, \frac{1}{2}\varepsilon)$ is an open set in Y, and f is continuous. Therefore V_y is open in X for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 1.47) that there exists some positive real number δ such that every subset of X whose diameter is less than δ is a subset of some set V_y. Let x and x' be points of X satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$, which is less than δ. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f\colon X \to Y$ is uniformly continuous, as required. ∎

1.20 Finite Cartesian Products of Compact Spaces

Theorem 1.49 *A Cartesian product of a finite number of compact spaces is itself compact.*

Proof It suffices to prove that the product of two compact topological spaces X and Y is compact, since the general result then follows easily by induction on the number of compact spaces in the product.

Let $\mathcal{V}$ be an open cover of $X \times Y$. Then for each $x \in X$ and $y \in Y$ there exists an open set $V_{x,y}$ in $X \times Y$ belonging to the open cover $\mathcal{V}$ for which $(x, y) \in V_{x,y}$. It then follows from the definition of the product topology on $X \times Y$ that there exist an open set $D_{x,y}$ in X and an open set $E_{x,y}$ in Y such that $x \in D_{x,y}$, $y \in E_{x,y}$ and $D_{x,y} \times E_{x,y} \subset V_{x,y}$.

Now the compactness of the topological space Y ensures that for each $x \in X$ there exists a finite subset $B(x)$ of Y for which $\bigcup_{y \in B(x)} E_{x,y} = Y$. Let $W_x = \bigcap_{y \in B(x)} D_{x,y}$. Then W_x is the intersection of a finite number of open sets in X, and is therefore itself an open set in X. Moreover

$$\begin{aligned} W_x \times Y &\subset \bigcup_{y \in B(x)} (W_x \times E_{x,y}) \subset \bigcup_{y \in B(x)} (D_{x,y} \times E_{x,y}) \\ &\subset \bigcup_{y \in B(x)} V_{x,y}. \end{aligned}$$

It then follows from the compactness of the topological space X that there exists a finite subset A of X for which $\bigcup_{x \in A} W_x = X$. Let

$$C = \{(x, y) : x \in A \text{ and } y \in B(x)\},$$

and, for each $(x, y) \in C$, let $V_{x,y}$ be an open set in $X \times Y$ belonging to the open cover $\mathcal{V}$ for which $D_{x,y} \times E_{x,y} \subset V_{x,y}$. Now C is a finite set, and

$$\begin{aligned} X \times Y &= \bigcup_{x \in A} (W_x \times Y) \subset \bigcup_{x \in A} \bigcup_{y \in B(x)} V_{x,y} \\ &= \bigcup_{(x,y) \in C} V_{x,y}. \end{aligned}$$

Thus $(V_{x,y} : (x, y) \in C)$ is an open cover of $X \times Y$. Moreover it is a finite subcover of the open cover $\mathcal{V}$. We have thus shown that $X \times Y$ is compact, as required. ∎

Theorem 1.50 *Let K be a subset of $\mathbb{R}^n$. Then K is compact if and only if K is both closed and bounded.*

Proof Suppose that K is compact. Then K is closed, since $\mathbb{R}^n$ is Hausdorff, and a compact subset of a Hausdorff space is closed (by Corollary 1.43). For each natural number m, let B_m be the open ball of radius m about the origin, given by $B_m = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < m\}$. Then $\{B_m : m \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^n$. It follows from the compactness of K that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that $K \subset B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_k}$. But then $K \subset B_M$, where M is the maximum of $m_1, m_2, \ldots, m_k$, and thus K is bounded.

Conversely suppose that K is both closed and bounded. Then there exists some real number L such that K is contained within the closed cube C given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem (Theorem 1.37), and C is the Cartesian product of n copies of the compact set $[-L, L]$. It follows from Theorem 1.49 that C is compact. But K is a closed subset of C, and a closed subset of a compact topological space is itself compact, by Lemma 1.38. Thus K is compact, as required. ∎

Let K be a closed bounded subset of $\mathbb{R}^n$. It follows from Theorem 1.48 and Theorem 1.50 that any continuous function $f: K \to \mathbb{R}^k$ is uniformly continuous.

1.21 Connected Topological Spaces

Definition A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed.

Lemma 1.51 *A topological space X is connected if and only if there do not exist non-empty open subsets V and W of X for which both $V \cup W = X$ and $V \cap W = \emptyset$.*

Proof Suppose that X is connected. Let V and W be open subsets of X. If $V \cup W = X$ and $V \cap W = \emptyset$ then $V = X \setminus W$, and thus the subset V of X is both open and closed. It follows from the connectedness of X that either $V = \emptyset$ or else $V = X$. Moreover $W = X$ in the case when $V = \emptyset$, and $W = \emptyset$ in the case when $V = X$. Thus the sets V and W cannot both be non-empty. We conclude that if the topological space X is connected then there cannot exist non-empty open sets V and W for which both $V \cap W = X$ and $V \cap W = \emptyset$.

Conversely let X be a topological space that does not contain non-empty open sets V and W with the property that both $V \cup W = X$ and $V \cap W = \emptyset$. Let V be a subset of X that is both open and closed, and let $W = X \setminus V$. Then the sets V and W are both open in X, $V \cup W = X$ and $V \cap W = \emptyset$. It follows that the open sets V and W cannot both be non-empty, and therefore either $V = \emptyset$ or else $W = \emptyset$, in which case $V = X$. This shows that X is connected, as required. ∎

The following two lemmas are immediate consequences of Lemma 1.51

Lemma 1.52 *A topological space X is connected if and only if it has the following property: if U and V are non-empty open sets in X such that $U \cup V = X$, then $U \cap V$ is non-empty.*

Lemma 1.53 *A topological space X is connected if and only if it has the following property: if U and V are non-empty open sets in X such that $U \cap V = \emptyset$, then $U \cup V$ is a proper subset of X.*

Definition A topological space D is *discrete* if every subset of D is open in D.

Example The set $\mathbb{Z}$ integers with the usual topology is an example of a discrete topological space. Indeed, given any integer n, the set $\{n\}$ is open in $\mathbb{Z}$, because it is the intersection of $\mathbb{Z}$ with the open ball in $\mathbb{R}$ of radius $\frac{1}{2}$ about n. Any non-empty subset S of $\mathbb{Z}$ is the union of the sets $\{n\}$ as n ranges over the elements of S. Therefore every subset of $\mathbb{Z}$ is open in $\mathbb{Z}$, and thus $\mathbb{Z}$, with the usual topology, is a discrete topological space.

Proposition 1.54 *Let X be a topological space, and let D be a discrete topological space with at least two elements. Then X is connected if and only if every continuous function from X to D is constant.*

Proof Suppose that X is connected. Let $f: X \to D$ be a continuous function from X to D, let $d \in f(X)$, and let $Z = f^{-1}(\{d\})$. Now $\{d\}$ is both open and closed in D. It follows from the continuity of $f: X \to D$ that Z is both open and closed in X. Moreover Z is non-empty. It follows from the connectedness of X that $Z = X$, and thus $f: X \to D$ is constant.

Now suppose that X is not connected. Then there exists a non-empty proper subset Z of X that is both open and closed in X. Let d_1 and d_2 be elements of D, where $d_1 \neq d_2$, and let $f: X \to D$ be defined so that

$$f(x) = \begin{cases} d_1 & \text{if } x \in Z; \\ d_2 & \text{if } x \in X \setminus Z. \end{cases}$$

If V is a subset of D then $f^{-1}(V)$ is one of the following four sets: $\emptyset$, Z, $X \setminus Z$, X. It follows that $f^{-1}(V)$ is open in X for all subsets V of D. Therefore $f: X \to D$ is continuous. But the function $f: X \to D$ is not constant, because Z is a non-empty proper subset of X. The result follows. ∎

The following results follow immediately from Proposition 1.54.

Corollary 1.55 *A topological space X is connected if and only if every continuous function $f: X \to \{0, 1\}$ from X to the discrete topological space $\{0, 1\}$ is constant.*

Corollary 1.56 *A topological space X is connected if and only if every continuous function $f\colon X \to \mathbb{Z}$ from X to the set $\mathbb{Z}$ of integers is constant.*

Example Let $X = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$. The topological space X is not connected. Indeed if $f\colon X \to \mathbb{Z}$ is defined by

$$f(x, y) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

then f is continuous on X but is not constant.

Definition A subset I of the set $\mathbb{R}$ of real numbers is said to be an *interval* if $(1 - t)c + td \in I$ for all $c \in I$, $d \in I$ and $t \in [0, 1]$.

Using the Least Upper Bound Property of the real number system one can show that a non-empty subset of the set $\mathbb{R}$ of real numbers is an interval if and only if it can be expressed in one of the following forms: $[a, b]$, $[a, b)$, $(a, b]$, (a, b), $[a, +\infty)$, $(a, +\infty)$, $(-\infty, b]$, $(-\infty, b)$, $(-\infty, \infty)$.

Theorem 1.57 *Every interval in the set $\mathbb{R}$ of real numbers is connected.*

Proof An interval consisting of a single real number is clearly connected.

Throughout the remainder of the proof let I be an interval with more than one element, and let V and W be disjoint non-empty subsets of I that are both open in I. We shall show that $V \cup W$ must then be a proper subset of I.

Now there must exist real numbers c and d belonging to the interval I and satisfying $c < d$ for which c belongs to one of the sets V and W and d belongs to the other. We may suppose, without loss of generality, that $c \in V$ and $d \in W$.

Let $z = \sup([c, d] \cap V)$. If $t \in [c, d] \cap V$ then there exists some positive real number δ such that $(t - \delta, t + \delta) \cap [c, d] \subset V$, and therefore $t \neq z$. It follows that $z \notin V$. Similarly if $t \in [c, d] \cap W$ then there exists some positive real number δ such that $(t - \delta, t + \delta) \cap [c, d] \subset W$, But then $(t - \delta, t + \delta) \cap [c, d] \cap V = \emptyset$, because $V \cap W = \emptyset$, and therefore $t \neq z$. It follows that $z \notin W$.

We have now shown that $z \notin V \cup W$. But $z \in I$. It follows that $V \cup W$ is a proper subset of I. We conclude that the interval I is connected (see Lemma 1.51, see also Lemma 1.53). ∎

Corollary 1.58 *Let $f\colon I \to \mathbb{Z}$ be a continuous integer-valued function defined on an interval I in the real line. Then $f\colon I \to \mathbb{Z}$ is a constant function.*

Proof The result follows directly on combining the results of Corollary 1.56 and Theorem 1.57. ∎

Lemma 1.59 *Let X be a topological space, and let A be a subset of X. Then A is connected (with respect to the subspace topology on A) if and only if, given open sets V and W in X for which $A \cap V \neq \emptyset$, $A \cap W \neq \emptyset$ and $A \subset V \cup W$, the set $A \cap V \cap W$ is non-empty.*

Proof A subset of A is open in the subspace topology if and only if it is of the form $A \cap V$ for some open set V in X. It follows from Lemma 1.52 that A is connected if and only if, given any open sets V and W in X for which $A \cap V \neq \emptyset$, $A \cap W \neq \emptyset$ and $(A \cap V) \cup (A \cap W) = A$, the set $(A \cap V) \cap (A \cap W)$ is the emptyset. Now $(A \cap V) \cup (A \cap W) = A$ if and only if $A \subset V \cup W$, and $(A \cap V) \cap (A \cap W) = \emptyset$ if and only if $A \cap V \cap W = \emptyset$. The result therefore follows directly on applying Lemma 1.52. ∎

Lemma 1.60 *Let X be a topological space and let A be a connected subset of X. Then the closure $\overline{A}$ of A is connected.*

Proof Let V and W be open sets in X for which $V \cap \overline{A} \neq \emptyset$, $W \cap \overline{A} \neq \emptyset$, and $\overline{A} \subset V \cup W$. The definition of the closure of A in X ensures that if A is a subset of a closed subset F of X then $\overline{A}$ is also a subset of F. Now $X \setminus V$ and $X \setminus W$ are closed subsets of X and $\overline{A}$ is not a subset of either $X \setminus V$ or $X \setminus W$. It follows that A is not a subset of either $X \setminus V$ or $X \setminus W$ and therefore $V \cap A \neq \emptyset$ and $W \cap A \neq \emptyset$ (see Lemma 1.6). Also $A \subset V \cup W$. It follows from the connectedness of A that $A \cap V \cap W \neq \emptyset$ (see Lemma 1.59). Therefore $\overline{A} \cap V \cap W \neq \emptyset$. We conclude from this that $\overline{A}$ is connected, as required. ∎

Alternative Proof Let $f\colon \overline{A} \to \mathbb{Z}$ be a continuous function mapping the closure $\overline{A}$ of A into the set $\mathbb{Z}$ of integers. It follows from Corollary 1.56 that the restriction of the function f to the connected set A is constant. Therefore there exists some integer n such that $f(x) = n$ for all $x \in A$.

Let $B = \{x \in \overline{A} : f(x) = n\}$. Now $\{n\}$ is closed in $\mathbb{Z}$. It follows from the continuity of f that the set B is closed in the subspace topology on $\overline{A}$. Therefore $B = \overline{A} \cap F$ for some closed subset F of X. But $\overline{A}$ is itself closed in X. It follows that B is closed in X, and therefore $\overline{A} \subset B$. Thus $B = \overline{A}$, and therefore the continuous function $f\colon \overline{A} \to \mathbb{Z}$ is constant on $\overline{A}$. The required result therefore follows from Corollary 1.56. ∎

Lemma 1.61 *Let $f\colon X \to Y$ be a continuous function between topological spaces X and Y, and let A be a connected subset of X. Then $f(A)$ is connected.*

Proof Let V and W be open sets in Y for which $V \cap f(X) \neq \emptyset$, $W \cap f(X) \neq \emptyset$ and $f(X) \subset V \cup W$. Then $f^{-1}(V) \neq \emptyset$, $f^{-1}(W) \neq \emptyset$ and $X \subset f^{-1}(V) \cup f^{-1}(W)$. It follows from the connectedness of X that $f^{-1}(V) \cap f^{-1}(W) \neq \emptyset$. Let $x \in f^{-1}(V) \cap f^{-1}(W)$. Then $f(x) \in V \cap W$, and therefore $f(X) \cap V \cap W \neq \emptyset$. It follows from Lemma 1.59 that the subset $f(X)$ of Y is connected, as required. ∎

Alternative Proof Let $g\colon f(A) \to \mathbb{Z}$ be any continuous integer-valued function on $f(A)$. Then $g \circ f\colon A \to \mathbb{Z}$ is a continuous integer-valued function on A. It follows from Corollary 1.56 that $g \circ f$ is constant on A. Therefore g is constant on $f(A)$. We deduce from Corollary 1.56 that $f(A)$ is connected. ∎

1.22 Products of Connected Topological Spaces

Lemma 1.62 *The Cartesian product $X \times Y$ of connected topological spaces X and Y is itself connected.*

Proof Let $f\colon X \times Y \to \mathbb{Z}$ be a continuous integer-valued function from $X \times Y$ to Z. Choose $x_0 \in X$ and $y_0 \in Y$. The function $x \mapsto f(x, y_0)$ is continuous on X, and is thus constant. Therefore $f(x, y_0) = f(x_0, y_0)$ for all $x \in X$. Now fix x. The function $y \mapsto f(x, y)$ is continuous on Y, and is thus constant. Therefore

$$f(x, y) = f(x, y_0) = f(x_0, y_0)$$

for all $x \in X$ and $y \in Y$. We deduce from Corollary 1.56 that $X \times Y$ is connected. ∎

We deduce immediately that a finite Cartesian product of connected topological spaces is connected.

1.23 Connected Components of Topological Spaces

Proposition 1.63 *Let X be a topological space. For each $x \in X$, let S_x be the union of all connected subsets of X that contain x. Then*

(i) S_x is connected,

(ii) S_x is closed,

(iii) if $x, y \in X$, then either $S_x = S_y$, or else $S_x \cap S_y = \emptyset$.

Proof Let $f: S_x \to \mathbb{Z}$ be a continuous integer-valued function on S_x, for some $x \in X$. Let y be any point of S_x. Then, by definition of S_x, there exists some connected set A containing both x and y. But then f is constant on A, and thus $f(x) = f(y)$. This shows that the function f is constant on S_x. We deduce that S_x is connected. This proves (i). Moreover the closure $\overline{S_x}$ is connected, by Lemma 1.60. Therefore $\overline{S_x} \subset S_x$. This shows that S_x is closed, proving (ii).

Finally, suppose that x and y are points of X for which $S_x \cap S_y \neq \emptyset$. Let $f: S_x \cup S_y \to \mathbb{Z}$ be any continuous integer-valued function on $S_x \cup S_y$. Then f is constant on both S_x and S_y. Moreover the value of f on S_x must agree with that on S_y, since $S_x \cap S_y$ is non-empty. We deduce that f is constant on $S_x \cup S_y$. Thus $S_x \cup S_y$ is a connected set containing both x and y, and thus $S_x \cup S_y \subset S_x$ and $S_x \cup S_y \subset S_y$, by definition of S_x and S_y. We conclude that $S_x = S_y$. This proves (iii). ∎

Given any topological space X, the connected subsets S_x of X defined as in the statement of Proposition 1.63 are referred to as the *connected components* of X. We see from Proposition 1.63, part (iii) that the topological space X is the disjoint union of its connected components.

Example The connected components of $\{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ are

$$\{(x, y) \in \mathbb{R}^2 : x > 0\} \quad \text{and} \quad \{(x, y) \in \mathbb{R}^2 : x < 0\}.$$

Example The connected components of

$$\{t \in \mathbb{R} : |t - n| < \tfrac{1}{2} \text{ for some integer } n\}.$$

are the sets J_n for all $n \in \mathbb{Z}$, where $J_n = (n - \frac{1}{2}, n + \frac{1}{2})$.

1.24 Path-Connected Topological Spaces

A concept closely related to that of connectedness is *path-connectedness*. Let x_0 and x_1 be points in a topological space X. A *path* in X from x_0 to x_1 is defined to be a continuous function $\gamma: [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Definition A topological space X is said to be *path-connected* if and only if, given any two points x_0 and x_1 of X, there exists a continuous map $\gamma: [0, 1] \to X$ from the closed unit interval $[0, 1]$ to the space X for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Proposition 1.64 *Every path-connected topological space is connected.*

Proof Let X be a path-connected topological space, and let V and W be non-empty subsets of X that are open in X and satisfy $V \cap W = \emptyset$. We show that $V \cup W$ is a proper subset of X.

Now X is path-connected. Therefore there exists a continuous map $\gamma: [0, 1] \to X$ for which $\gamma(0) \in V$ and $\gamma(1) \in W$. Then the preimages $\gamma^{-1}(V)$ and $\gamma^{-1}(W)$ of V and W are open in $[0, 1]$, because the map γ is continuous. Moreover $\gamma^{-1}(V)$ and $\gamma^{-1}(W)$ are non-empty and $\gamma^{-1}(V) \cap \gamma^{-1}(W) = \emptyset$. Now the interval $[0, 1]$ is connected (Theorem 1.57). Therefore $\gamma^{-1}(V) \cup \gamma^{-1}(W)$ must be a proper subset of $[0, 1]$ (see Lemma 1.53).

Let s be a real number satisfying $0 \leq s \leq 1$ that does not belong to either $\gamma^{-1}(V)$ or $\gamma^{-1}(W)$. Then $\gamma(s) \in X \setminus (V \cup W)$. Thus there cannot exist non-empty open subsets V and W of X for which both $V \cap W = \emptyset$ and $V \cup W = X$. It follows that X is connected (see Lemma 1.51). ∎

The topological spaces $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^n$ are all path-connected. Indeed, given any two points of one of these spaces, the straight line segment joining these two points is a continuous path from one point to the other. Also the n-sphere S^n is path-connected for all $n > 0$. We conclude that these topological spaces are connected.

Definition A subset X of a real vector space is said to be *convex* if, given points $\mathbf{u}$ and $\mathbf{v}$ of X, the point $(1-t)\mathbf{u}+t\mathbf{v}$ belongs to X for all real numbers t satisfying $0 \leq t \leq 1$.

Corollary 1.65 *All convex subsets of real vector spaces are connected.*

Remark Proposition 1.64 generalizes the Intermediate Value Theorem of real analysis. Indeed let $f: [a, b] \to \mathbb{R}$ be a continuous real-valued function on an interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. The range $f([a, b])$ is then a path-connected subset of $\mathbb{R}$. It follows from Proposition 1.64 that this set is connected. Let c be a real number that lies strictly between $f(a)$ and $f(b)$ and let

$$V = \{y \in f([a, b]) : y < c\} \quad \text{and} \quad W = \{y \in f([a, b]) : y > c\}.$$

Then V and W are non-empty open subsets of $f([a, b])$, and $V \cap W = \emptyset$. It follows from the connectness of $f([a, b])$ that $V \cup W$ must be a proper subset of $f([a, b])$ (see Lemma 1.53), and therefore $c \in f([a, b])$. Thus the range of the function f contains all real numbers between $f(a)$ and $f(b)$.

Example Let $f: \mathbb{R} \to \mathbb{R}$ be defined so that

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and let

$$X = \{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$

We show that X is a connected set. Let

$$X_+ = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = f(x)\}$$

and

$$X_- = \{(x, y) \in \mathbb{R}^2 : x < 0 \text{ and } y = f(x)\}.$$

Now the restriction of the function f to the set of (strictly) positive real numbers is continuous on the set of positive real numbers. It follows from this that the set X_+ is path-connected. It then follows that the set X_+ is connected (see Proposition 1.64). The connectedness of X_+ can also be verified by noting that it is the image of the connected space $\{x \in \mathbb{R} : x > 0\}$ under a continuous map and is therefore itself connected (see Lemma 1.61). Similarly the set X_- is path-connected, and is therefore connected.

Let $\mathbf{p}_n = (1/\pi n, 0)$ for all natural numbers n. Then $\mathbf{p}_n \in X_+$ for all natural numbers n, and $\mathbf{p}_n \to (0, 0)$ as $n \to +\infty$. It follows that $(0, 0)$ belongs to the closure $\overline{X}_+$ of X_+ in X. Connected components of a topological space are closed (see Proposition 1.63). Thus the connected component of X that includes the connected subset X_+ also contains the point $(0, 0)$. Similarly the connected component of X that includes $X + -$ also contains the point $(0, 0)$. Therefore the unique connected component of X that contains the point $(0, 0)$ is the whole of X and thus X is a connected topological space.

However X is not a path-connected topological space. If $\gamma\colon [0, 1] \to X$ is a continuous map from the closed unit interval $[0, 1]$ into X, and if $\gamma(0) = (0, 0)$, then $\gamma(t) = (0, 0)$ for all $t \in [0, 1]$. Indeed let

$$s = \sup\{t \in [0, 1] : \gamma(t) = (0, 0)\}.$$

It follows from the continuity of γ that $\gamma(s) = (0, 0)$. There then exists some positive real number δ such that $|\gamma(t) - (0, 0)| < \frac{1}{2}$ for all $t \in [0, 1]$ satisfying $|t - s| < \delta$. But $\gamma([0, 1] \cap [s, s + \delta))$ must also be a connected subset of X. It follows that $\gamma(t) = (0, 0)$ for all $t \in [0, 1]$ satisfying $s \leq t < s + \delta$, and therefore $s = 1$ and $\gamma(t) = (0, 0)$ for all $t \in [0, 1]$. (Essentially, the path γ cannot get from $(0, 0)$ to any other point of X because continuity prevents from getting over intervening humps where the function f takes values such as ± 1.) We conclude that the connected topological space X is not path-connected.

1.25 Locally Path-Connected Topological Spaces

Definition A topological space X is said to be *locally connected* if, given any point x of X, and given any open set N in X for which $x \in N$, there exists some connected open set V in X such that $x \in V$ and $V \subset N$.

Definition A topological space X is said to be *locally path-connected* if, given any point x of X, and given any open set N in X for which $x \in N$, there exists some path-connected open set V in X such that $x \in V$ and $V \subset N$.

Every path-connected subset of a topological space is connected. (This follows directly from Proposition 1.64.) Therefore every locally path-connected topological space is locally connected.

Proposition 1.66 *Let X be a connected, locally path-connected topological space. Then X is path-connected.*

Proof Choose a point x_0 of X. Let Z be the subset of X consisting of all points x of X with the property that x can be joined to x_0 by a path. We show that the subset Z is both open and closed in X.

Now, given any point x of X there exists a path-connected open set N_x in X such that $x \in N_x$. We claim that if $x \in Z$ then $N_x \subset Z$, and if $x \notin Z$ then $N_x \cap Z = \emptyset$.

Suppose that $x \in Z$. Then, given any point x' of N_x, there exists a path in N_x from x' to x. Moreover it follows from the definition of the set Z that there exists a path in X from x to x_0. These two paths can be concatenated to yield a path in X from x' to x_0, and therefore $x' \in Z$. This shows that $N_x \subset Z$ whenever $x \in Z$.

Next suppose that $x \notin Z$. Let $x' \in N_x$. If it were the case that $x' \in Z$, then we would be able to concatenate a path in N_x from x to x' with a path in X from x' to x_0 in order to obtain a path in X from x to x_0. But this is impossible, as $x \notin Z$. Therefore $N_x \cap Z = \emptyset$ whenever $x \notin Z$.

Now the set Z is the union of the open sets N_x as x ranges over all points of Z. It follows that Z is itself an open set. Similarly $X \setminus Z$ is the union of the open sets N_x as x ranges over all points of $X \setminus Z$, and therefore $X \setminus Z$ is itself an open set. It follows that Z is a subset of X that is both open and closed. Moreover $x_0 \in Z$, and therefore Z is non-empty. But the only subsets of X that are both open and closed are $\emptyset$ and X itself, since X is connected. Therefore $Z = X$, and thus every point of X can be joined to the point x_0 by a path in X. We conclude that X is path-connected, as required. ∎

1.26 Contractible and Locally Contractible Spaces

Definition A topological space X is said to be *contractible* if there exists a point p of X and a continuous map $H\colon X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) = p$ for all $x \in X$.

Lemma 1.67 *Convex sets in Euclidean spaces are contractible.*

Proof Let X be a convex set in a Euclidean space, and let $\mathbf{p}$ be a point of X. Let Let $H\colon X \times [0,1] \to X$ be defined such that $H(\mathbf{x}, t) = (1-t)\mathbf{x} + t\mathbf{p}$ for all $\mathbf{p} \in X$ and $t \in [0,1]$. Then $H(\mathbf{x}, 0) = \mathbf{x}$ and $H(\mathbf{x}, 1) = \mathbf{p}$ for all $\mathbf{x} \in X$. It follows that the convex set X is contractible, as required. ∎

Corollary 1.68 *Open and closed balls in Euclidean spaces are contractible.*

Lemma 1.69 *Every contractible topological space is path-connected, and is therefore connected.*

Proof Let X be a contractible topological space. Then there exists a point p of X and a continuous map $H\colon X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) = p$ for all $x \in X$.

Let u and v be points of X, and let $\gamma\colon [0,1] \to X$ be defined such that

$$\gamma(t) = \begin{cases} H(u, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ H(v, 2-2t) & \text{if } \frac{1}{2} \leq t \leq 1; \end{cases}$$

(Note that the formulae defining $\gamma(t)$ for $t \leq \frac{1}{2}$ and for $t \geq \frac{1}{2}$ are consistent with each other, because $H(u, 2t) = p = H(v, 2-2t)$ when $t = \frac{1}{2}$.) Now the restrictions $\gamma|[0, \frac{1}{2}]$ and $\gamma|[\frac{1}{2}, 1]$ of the function γ to the closed intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are continuous on those intervals. It follows from the Pasting Lemma (Lemma 1.24) that the function $\gamma\colon [0,1] \to X$ is continuous. Moreover $\gamma(0) = u$ and $\gamma(1) = v$. We conclude from this that the topological space X is path-connected. It then follows from Proposition 1.64 that the topological spaces X is connected, as required. ∎

Example The *Comb Space* X is defined so that

$$X = \{(x,y) \in [0,1] \times [0,1] : y = 0 \text{ or } x = 0 \text{ or } x = n^{-1} \text{ for some } n \in \mathbb{N}\}.$$

This topological space is the union of one horizontal line, from $(0,0)$ to $(1,0)$, and an infinite number of vertical lines. First we show that the Comb Space is contractible. Let $H\colon X \times [0,1] \to X$ be defined such that

$$H((x,y),t) = \begin{cases} (x, (1-2t)y) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ ((2-2t)x, 0) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The Pasting Lemma (Lemma 1.24) ensures $H: X \times [0,1] \to X$ is a continuous map from $X \times [0,1]$ to X. Moreover $H((x,y),0) = (x,y)$ and $H((x,y),1) = (0,0)$ for all $(x,y) \in X$. Thus X is contractible.

Next we show that the Comb Space X is not locally connected. Let v be a real number satisfying $0 < v \leq 1$, and let

$$N = \{(x,y) \in X : y > 0\}.$$

Then N is an open set in X, and $(0,v) \in N$. Let W be an open set in X for which $(0,v) \in W$ and $W \subset N$. Then there exists a positive integer n large enough to ensure that $(n^{-1}, v) \in W$. Let q be a positive real number chosen to ensure that $1/q$ is not an integer and $0 < q < n^{-1}$, and let

$$\begin{aligned} V_1 &= \{(x,y) \in W : x < q\}, \\ V_2 &= \{(x,y) \in W : x > q\}. \end{aligned}$$

Now there is no point (x,y) of W for which $x = q$. It follows that V_1 and V_2 are non-empty open subsets of W for which $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = W$. It follows that W is not connected. We conclude from this that the Comb Space X is not locally connected.

Now suppose that we remove the point $(0,v)$ from the Comb Space X, where $0 < v < 1$. The resultant subset $X \setminus \{(0,v)\}$ of X is then connected but not path-connected. Indeed let $Y = X \setminus \{(0,v)\}$. Then the point $(n^{-1}, 1)$ of Y can be joined to $(0,0)$ by a path in Y, and therefore belongs to the same connected component of Y as the point $(0,0)$. Also every open set in Y containing the point $(0,1)$ contains points $(n^{-1}, 0)$ for sufficiently large positive integers n, and thus the point $(0,1)$ belongs to the closure of the connected component of Y that contains the point $(0,0)$. But connected components of topological spaces are closed (Proposition 1.63). Therefore the point $(0,1)$ belongs to the same connected component of Y as the point $(0,0)$. Moreover every point of Y can be joined by a path to at least one of the points $(0,0)$ and $(0,1)$. It follows that Y is connected. But there is no path in Y from $(0,0)$ to $(0,1)$, and therefore Y is not path-connected.

Definition A topological space X is said to be *locally contractible* if, given any point p of X, and given any open set N for which $p \in N$, there exists a contractible open set W for which $p \in W$ and $W \subset N$.

Definition A topological space is said to be *locally Euclidean* of dimension n if, if, given any point p of X, and given any open set N for which $p \in N$, there exists an open set W satisfying $p \in W$ and $W \subset N$ that is homeomorphic to an open set in $\mathbb{R}^n$.

Lemma 1.70 *Locally Euclidean topological spaces are locally contractible, and are therefore locally path-connected and locally connected.*

Proof The result follows directly on combining the relevant definitions with the result stated in Lemma 1.69. ∎

Definition A *topological manifold* of dimension n is a Hausdorff space with a countable base of open sets that is locally Euclidean of dimension n.

It follows from Lemma 1.70 that topological manifolds are locally contractible, and are therefore locally path-connected and locally connected.