UNIVERSITY OF DUBLIN

SAMPLE

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF MATHEMATICS

JS and SS Mathematics **Trinity Term 2011**
SS Theoretical Physics
SS TSM Mathematics

MODULE MA3429 - SAMPLE PAPER

DAY VENUE TIME

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Credit for module MA3429 will be given for the best 3 questions answered

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. A *tangent vector* to a smooth manifold M at some point p of M is defined to be a operator X_p that associates a real number $X_p[f]$ to each smooth real-valued function f defined throughout some open neigbourhood of p, where this operator satisfies the following conditions:—

   (i) $X_p[\alpha f + \beta g] = \alpha X_p[f] + \beta X_p[g]$ for all real numbers α and β and smooth functions f and g defined around the point p;

   (ii) $X_p[f \cdot g] = X_p[f]\, g(p) + f(p)\, X_p[g]$ for all smooth functions f and g defined around the point p, where $(f \cdot g)(m) = f(m)g(m)$ for all points m close to p;

   (iii) if f and g are smooth real-valued functions defined around p, and if $f|V = g|V$ for some open set V that contains the point p, then $X_p[f] = X_p[g]$.

   Let $\varphi\colon M \to N$ be a smooth map between smooth manifolds M and N, and let p be a point of M.

   (a) Let X_p be a tangent vector at the point p, and let

   $$(\varphi_* X_p)[g] = X_p[g \circ \varphi]$$

   for all smooth real-valued functions g that are defined throughout some open neighbourhood of $\varphi(p)$ in N. Prove that the operator $\varphi_* X_p$ is a tangent vector at $\varphi(p)$.

   (6 marks)

   The *derivative* of the smooth map φ at the point p is defined to be the linear transformation $\varphi_*\colon T_p M \to T_{\varphi(p)} N$ characterized by the property that

   $$(\varphi_* X_p)[g] = X_p[g \circ \varphi]$$

   for all smooth real-valued functions g that are defined throughout some open neighbourhood of $\varphi(p)$ in N.

(Question continues on next page)

Let $(x^1, \ldots, x^n)$ be a smooth coordinate system defined throughout some open neighbourhood U of the point p in M, let $(y^1, \ldots, y^k)$ be a smooth coordinate system defined throughout some open neighbourhood V of the point $\varphi(p)$ in N, and let $F^1, F^2, \ldots, F^k$ be the smooth functions of n real variables, defined throughout some open neighbourhood of the point $(x^1(p), x^2(p), \ldots, x^n(p))$ in $\mathbb{R}^n$, that represent the smooth map φ around p with respect to the coordinate systems on M and N, so that

$$y^j(\varphi(u)) = F^j(x^1(u), x^2(u), \ldots, x^n(u))$$

for $j = 1, 2, \ldots, k$ and for all $u \in U \cap \varphi^{-1}(V)$.

(b) Prove that

$$\varphi_* \left(\sum_{i=1}^{n} a^i \left. \frac{\partial}{\partial x^i} \right|_p \right) = \sum_{j=1}^{k} b^j \left. \frac{\partial}{\partial y^j} \right|_{\varphi(p)},$$

where

$$b^j = \sum_{i=1}^{n} a^i \left. \frac{\partial F^j}{\partial x^i} \right|_p.$$

(9 marks)

(c) Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be the smooth map defined such that $\varphi(u, v) = u^2 + v^2$. Calculate

$$\varphi_* \left(u^3 \left. \frac{\partial}{\partial u} \right|_{(1,0)} + v^3 \left. \frac{\partial}{\partial v} \right|_{(1,0)} \right),$$

expressing this tangent vector to $\mathbb{R}$ in the form $b \left. \dfrac{\partial}{\partial x} \right|_1$, where x is the usual coordinate function on the real line $\mathbb{R}$, and where b is some real number.

(5 marks)

2. (a) Let $\pi_E \colon E \to M$ be a smooth vector bundle over a smooth manifold M, and let D be a differential operator which acts on each smooth vector field X and each smooth section s of the vector bundle to produce a smooth section $D_X s$ of that vector bundle. List the conditions that this differential operator is required to satisfy in order that it be a *connection* on the vector bundle.

(5 marks)

Let D be a smooth connection on a smooth vector bundle $\pi_E \colon E \to M$ of rank r, and let U be an open subset in M which is contained in the domain of a smooth coordinate system $(x^1, x^2, \ldots, x^n)$ for M and over which are defined smooth sections $e_1, e_2, \ldots, e_r$ of the vector bundle whose values $e_1(p), e_2(p), \ldots, e_r(p)$ at each point p of E_p constitute a basis of the fibre E_p of this vector bundle over the point p. Let

$$D_j e_\beta = D_{\frac{\partial}{\partial x^j}} e_\beta = \sum_{\alpha=1}^{r} A^\alpha{}_{\beta j} e_\alpha,$$

for $j = 1, 2, \ldots, n$ and $\beta = 1, 2, \ldots, r$, where each function $A^\alpha{}_{\beta j}$ is a smooth real-valued function on U. Let X be a smooth vector field on U, and let $s \colon U \to E$ be a smooth section of the vector bundle $\pi_E \colon E \to M$ defined over U, and let

$$X = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j} \quad \text{and} \quad s = \sum_{\alpha=1}^{r} f^\alpha e_\alpha,$$

where $v^1, v^2, \ldots, v^n$ and $f^1, f^2, \ldots, f^r$ are smooth real-valued functions on U.

(b) Prove that

$$D_X s = \sum_{j=1}^{n} v^j D_j s,$$

where

$$D_j s = \sum_{\alpha=1}^{r} \left(\frac{\partial f^\alpha}{\partial x^j} + \sum_{\beta=1}^{r} A^\alpha{}_{\beta j} f^\beta \right) e_\alpha.$$

(7 marks)

(Question continues on next page)

(c) What is an *affine connection* on a smooth manifold?

(2 marks)

(d) The *torsion tensor* T of an affine connection ∇ on a smooth manifold M is defined such that

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all smooth vector fields X and Y on M. Prove that $T(X, fY) = f\,T(X, Y)$ for all smooth vector fields X and Y and for all smooth real-valued functions f on M.

(6 marks)

3. Let M be a pseudo-Riemannian manifold with metric tensor g. The Levi-Civita connection ∇ on M is torsion-free, and has the property that

$$X[g(Y,Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for all smooth vector fields X, Y and Z on M. The Riemann curvature tensor R of M satisfies $R(W,X,Y,Z) = g(W, R(X,Y)Z)$ for all smooth vector fields X, Y, Z and W on M, where

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z.$$

(a) Prove that $R(X,W,Y,Z) = -R(W,X,Y,Z)$.

(6 marks)

Let M be the smooth 4-dimensional manifold which is the domain of a smooth coordinate chart (t,z,r,θ), where $0 < r < \pi 0$ and $0 < \theta < 2\pi$ throughout M. Let g be the pseudo-Riemannian metric on M defined as follows:

$$g = -\frac{1}{1+r^2}\, dt \otimes dt + \sin^2 r\, dz \otimes dz + dr \otimes dr + r^2\, d\theta \otimes d\theta.$$

Let E_t, E_z, E_r and E_θ be the smooth vector fields on M defined as follows:

$$E_t = \sqrt{1+r^2}\,\frac{\partial}{\partial t}, \quad E_z = \frac{1}{\sin r}\frac{\partial}{\partial z}, \quad E_r = \frac{\partial}{\partial r}, \quad E_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}.$$

(b) Calculate the values of the Lie brackets

$$[E_t, E_z], \quad [E_t, E_r], \quad [E_t, E_\theta],$$

$$[E_z, E_r], \quad [E_z, E_\theta] \quad \text{and} \quad [E_r, E_\theta],$$

(You should express the values of these Lie brackets as linear combinations of the vector fields E_t, E_r, E_z and E_θ of the form $f_t E_t + f_z E_z + f_r E_r + f_\theta E_\theta$, where f_t, f_z, f_r and f_θ are smooth real-valued functions on M.)

(7 marks)

(Question continues on next page)

(c) Explain why

$$g(\nabla_{E_z} E_t, E_t) = g(\nabla_{E_r} E_t, E_t) = g(\nabla_{E_\theta} E_t, E_t) = 0,$$

where ∇ denotes the Levi-Civita connection on M, and find the values of

$$g(\nabla_{E_t} E_z, E_t), \quad g(\nabla_{E_t} E_r, E_t), \quad g(\nabla_{E_t} E_\theta, E_t),$$

$$g(\nabla_{E_t} E_t, E_z), \quad g(\nabla_{E_t} E_t, E_r) \quad \text{and} \quad g(\nabla_{E_t} E_t, E_\theta).$$

Hence or otherwise, express $\nabla_{E_t} E_t$ as a linear combination of the vector fields E_t, E_r, E_z and E_θ of the form $f_t E_t + f_z E_z + f_r E_r + f_\theta E_\theta$, where f_t, f_z, f_r and f_θ are smooth real-valued functions on M.

(7 marks)

4. Let M be a Riemannian manifold with metric tensor g, and let ∇ and R denote the Levi-Civita connection and Riemann curvature tensor on M. The Levi-Civita connection on M is torsion-free, and satisfies the identity

$$X[g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for all smooth vector fields X, Y and Z on M. Let $E_1, E_2, \ldots, E_n$ be smooth vector fields, defined over some open subset U of M which, at each point of U, constitute a basis of the tangent space to M at that point. Let $\gamma\colon I \to M$ be a smooth curve in M, where I is some open interval in $\mathbb{R}$, and let V and W be smooth vector fields along γ. Then

$$V(t) = \sum_{j=1}^{n} v^j(t)(E_j)_{\gamma(t)}, \quad W(t) = \sum_{j=1}^{n} w^j(t)(E_j)_{\gamma(t)}$$

for all $t \in I$ for which $\gamma(t) \in U$, where v^j and w^j are smooth real-valued functions on the interval I. The covariant derivative $\dfrac{DV(t)}{dt}$ of the vector field V along the curve γ is then determined by the formula

$$\frac{dV(t)}{dt} = \sum_{j=1}^{n} \left(\frac{dv^j(t)}{dt} (E_j)_{\gamma(t)} + v^j(t)\, \nabla_{\gamma'(t)} E_j \right)$$

when $\gamma(t) \in U$. (The covariant derivative of the vector field W is determined by an analogous formula.)

(Question continues on next page)

(a) Prove that

$$\frac{d}{dt} g(V(t), W(t)) = g\left(\frac{DV(t)}{dt}, W(t)\right) + g\left(V(t), \frac{DW(t)}{dt}\right).$$

(10 marks)

(b) What is meant by saying that a smooth curve $\gamma: I \to M$ in the Riemannian manifold M is a *geodesic*?

(2 marks)

Let $\gamma: I \to M$ be a geodesic in the Riemannian The length $|\gamma'(t)|$ of the velocity vector $\gamma'(t)$ of γ at $\gamma(t)$ is defined such that $|\gamma'(t)|^2 = g(\gamma'(t), \gamma'(t))$ for all $t \in I$.

(c) Prove that manifold M then

$$\frac{d}{dt}|\gamma'(t)| = 0$$

when $\gamma: I \to M$ is a geodesic in M.

(3 marks)

(d) Let $V: I \to TM$ be a smooth vector field along the geodesic $\gamma: I \to M$. Suppose that $0 \in I$, $g(V(0), \gamma'(0)) = 0$ and

$$\frac{DV(t)}{dt} = f(t)V(t),$$

where $f: I \to \mathbb{R}$ is a smooth real-valued function on the interval I. Prove that $g(V(t), \gamma'(t)) = 0$ for all $t \in I$.

(5 marks)