Module MA3427: Michaelmas Term 2010 Problems

Geodesic Congruences and Jacobi Fields

1. (a) *Let ∇ be a smooth torsion-free affine connection on a smooth manifold M, and let Q and X be smooth vector fields on M. Suppose that $\nabla_Q Q = 0$ and $[Q, X] = 0$. Prove that*

$$\nabla_Q \nabla_Q X + R(X, Q)Q = 0,$$

where R is the curvature of the connection ∇.

The affine connection ∇ is torsion-free, and therefore

$$\nabla_Q X = \nabla_X Q + [Q, X] = \nabla_X Q.$$

It follows that

$$\begin{aligned} \nabla_Q \nabla_Q X + R(X, Q)Q &= \nabla_Q \nabla_X Q + R(X, Q)Q \\ &= \nabla_X \nabla_Q Q - \nabla_{[X,Q]} Q = 0, \end{aligned}$$

as required.

(b) *Let ∇ be a smooth torsion-free affine connection on a smooth manifold M, and let Q and X be smooth vector fields on M. Suppose that $\nabla_Q Q = 0$ and $[Q, X] = 0$. Let $\gamma\colon I \to M$ be an integral curve for the vector field Q, and let $V(t) = X_{\gamma(t)}$ for all $t \in I$. Explain why γ is a geodesic in M, and also explain why the vector field $V\colon I \to TM$ is a Jacobi field along the curve γ.*

It follows from the definitions of integral curves of vector fields, and of covariant derivatives of vector fields along curves, that

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \frac{D}{dt}\Bigl(Q_{\gamma(t)}\Bigr) = \nabla_{Q_{\gamma(t)}} Q = 0.$$

Thus $\gamma\colon I \to M$ is a geodesic. Also

$$\begin{aligned} \frac{D^2 V(t)}{dt^2} &= \frac{D}{dt}\Bigl(\nabla_{\gamma'(t)} X\Bigr) = \frac{D}{dt}\Bigl(\nabla_Q X\Bigr) = \frac{D}{dt}\Bigl(\nabla_Q X\Bigr)_{\gamma(t)} \\ &= \nabla_{Q_{\gamma(t)}}(\nabla_Q X) \end{aligned}$$

and therefore

$$\frac{D^2V(t)}{dt^2} + R(V(t), \gamma'(t))\gamma'(t) = (\nabla_Q \nabla_Q X + R(X, Q)Q)_{\gamma(t)} = 0.$$

Hence the vector field V along γ satisfies the Jacobi equation, and is thus a Jacobi field along γ.

Constant Curvature Metrics and the Expanding Universe

2. *Let N be a Riemannian manifold with metric tensor g^N, let I be an open interval in $\mathbb{R}$, let $M = N \times I$, and let $\pi: M \to N$ and $\iota_t: N \to M$ be defined for all $t \in I$ so that*

$$\pi(p, t) = p, \quad \textit{and} \quad \iota_t(p) = (p, t)$$

for all $p \in N$ and $t \in I$. Let Q be the vector field $\dfrac{\partial}{\partial t}$ on M whose integral curves are the curves $\gamma_p: I \to M$, where $\gamma_p(t) = (p, t)$ for all $p \in N$ and $t \in I$. Also, for each smooth vector field X on N, let X° be the smooth vector field on M defined such that $X^\circ_{(p,t)} = \iota_{t*} X_p$ for all $p \in N$ and $t \in I$.

Let $a: I \to (0, +\infty)$ be a positive smooth real-valued function on the open interval I, let q be a real constant, and let g be the Riemannian or pseudo-Riemannian metric on M characterized by the properties that $g(Q, Q) = q$, $g(Q, X^\circ) = 0$ and

$$g_{(p,t)}(X^\circ, Y^\circ) = a(t)^2 g^N_p(X, Y)$$

for all smooth vector fields X and Y on N (where g^N denotes the metric tensor of the Riemannian manifold N), so that

$$g = q\, dt \otimes dt + a(t)^2\, \pi^* g^N,$$

where $(\pi^* g^N)(U, V) = g^N(\pi_* U, \pi_* V)$ for all vector fields U and V on M. Also let ∇ and R^M denote the Levi-Civita connection and Riemann curvature tensor respectively on M determined by the metric tensor g.

(a) *Explain why $[Q, X^\circ] = 0$ and $[X^\circ, Y^\circ] = [X, Y]^\circ$ for all smooth vector fields X on N. [Hint: use Lemma 7.6 to prove that $\pi_*[Q, X^\circ]_{(p,t)} = 0$ and $\sigma_*[Q, X^\circ]_{(p,t)} = 0$ for all $p \in N$ and $t \in I$, where $\pi: M \to N$ and $\sigma: M \to I$ are the projection functions that satisfy $\pi(p, t) = p$ and $\sigma(p, t) = t$ for all $p \in N$ and $t \in I$.]*

The smooth vector fields X° and Q satisfy

$$\pi_* X^\circ_{(p,t)} = X_p, \quad \sigma_* X^\circ_{(p,t)} = 0, \quad \pi_* Q_{(p,t)} = 0, \quad \sigma_* Q_{(p,t)} = \frac{d}{dt}.$$

It follows from Lemma 7.6 that

$$\pi_*[Q, X^\circ]_{(p,t)} = 0, \quad \sigma_*[Q, X^\circ]_{(p,t)} = 0$$

and

$$\iota_{t*}[X, Y]_p = [X^\circ, Y^\circ]_{(p,t)},$$

and thus

$$[Q, X^\circ] = 0 \quad \text{and} \quad [X^\circ, Y^\circ] = [X, Y]^\circ$$

for all smooth vector fields X and Y on N.

(b) *By evaluating $Q[g(Q, Q)]$, $X^\circ[g(Q, Q)]$ and $Q[g(Q, X^\circ)]$, where X is some smooth vector field on N, or otherwise, show that $\nabla_Q Q = 0$ and $g(\nabla_Q X^\circ, Q) = 0$.*

The real-valued functions $g(Q, Q)$ and $g(Q, X^\circ)$ are constant on M. Therefore

$$\begin{aligned} 0 &= Q[g(Q, Q)] = g(\nabla_Q Q, Q) + g(Q, \nabla_Q Q) = 2g(\nabla_Q Q, Q), \\ 0 &= X^\circ[g(Q, Q)] = 2g(\nabla_{X^\circ} Q, Q) \end{aligned}$$

and

$$0 = Q[g(Q, X^\circ)] = g(\nabla_Q Q, X^\circ) + g(Q, \nabla_Q X^\circ).$$

But $\nabla_Q X^\circ = \nabla_{X^\circ} Q$, because $[Q, X^\circ] = 0$. It follows that

$$g(\nabla_Q Q, Q) = 0 \quad \text{and} \quad g(\nabla_Q Q, X^\circ) = 0$$

for all smooth vector fields X° on N, and therefore $\nabla_Q Q = 0$.

(c) *Show that*

$$\begin{aligned} g(\nabla_{X^\circ} Y^\circ, Q) &= g(\nabla_{Y^\circ} X^\circ, Q) \\ &= -g(\nabla_{X^\circ} Q, Y^\circ) \\ &= -g(\nabla_Q X^\circ, Y^\circ) \\ &= -\tfrac{1}{2} Q[g(X^\circ, Y^\circ)] \\ &= -H(t)\, g(X^\circ, Y^\circ) \end{aligned}$$

for all smooth vector fields X and Y on N, where

$$H(t) = \frac{1}{a(t)}\frac{da(t)}{dt} = \frac{d(\log(a(t))}{dt},$$

and hence show that

$$\nabla_{X^\circ} Q = \nabla_Q X^\circ = H(t)\, X^\circ$$

for all smooth vector fields X on N.

Let X and Y be smooth vector fields on N. Then $[X^\circ, Y^\circ] = [X, Y]^\circ$, and therefore

$$\begin{aligned} & g(\nabla_{X^\circ} Y^\circ, Q) - g(\nabla_{Y^\circ} X^\circ, Q) \\ &\quad = \quad g([X^\circ, Y^\circ], Q) = g([X, Y]^\circ, Q) = 0. \end{aligned}$$

Also $g(Y^\circ, Q) = 0$, and therefore

$$\begin{aligned} 0 = X^\circ[g(Y^\circ, Q)] &= g(\nabla_{X^\circ} Y^\circ, Q) + g(Y^\circ, \nabla_{X^\circ} Q) \\ &= g(\nabla_{X^\circ} Y^\circ, Q) + g(\nabla_{X^\circ} Q, Y^\circ). \end{aligned}$$

Therefore

$$g(\nabla_{X^\circ} Y^\circ, Q) = -g(\nabla_{X^\circ} Q, Y^\circ) = -g(\nabla_Q X^\circ, Y^\circ),$$

and thus

$$\begin{aligned} Q[g(X^\circ, Y^\circ)] &= g(\nabla_Q X^\circ, Y^\circ) + g(X^\circ, \nabla_Q Y^\circ) \\ &= -g(\nabla_{X^\circ} Y^\circ + \nabla_{Y^\circ} X^\circ, Q) \\ &= 2g(\nabla_Q X^\circ, Y^\circ), \end{aligned}$$

and therefore

$$\begin{aligned} g(\nabla_Q X^\circ, Y^\circ) &= \frac{1}{2}\frac{d}{dt}\left(g(X^\circ, Y^\circ)\right) = \frac{1}{2}\frac{d}{dt}\left(a(t)^2 g^N(X, Y)\right) \\ &= a(t)\,\frac{d(a(t)}{dt}\, g^N(X, Y) = \frac{1}{a(t)}\frac{da(t)}{dt} g(X^\circ, Y^\circ) \\ &= H(t)\, g(X^\circ, Y^\circ). \end{aligned}$$

Also $g(\nabla_Q X^\circ, Q) = 0$, by (b). It follows that $\nabla_Q X^\circ = H(t) X^\circ$, as required.

(d) *Show that*

$$
\begin{aligned}
R^M(X^\circ, Q)Q &= -\left(\frac{dH(t)}{dt} + H(t)^2\right) X^\circ \\
&= -\frac{1}{a(t)}\,\frac{d^2a(t)}{dt^2}\, X^\circ,
\end{aligned}
$$

and thus

$$
R^M(W^\circ, Q, X^\circ, Q) = -\left(\frac{dH(t)}{dt} + H(t)^2\right) g(W^\circ, X^\circ).
$$

for all smooth vector fields W and X on N. [Hint: use the result of problem 1(a).]

Let X be a smooth vector fields on N. The equation

$$
\nabla_Q \nabla_Q X^\circ + R^M(X^\circ, Q)Q = 0,
$$

by problem 1(a). But

$$
\nabla_Q \nabla_Q X^\circ = \nabla_Q (H(t) X^\circ) = \frac{dH(t)}{dt} X^\circ + H(t)^2 X^\circ,
$$

where

$$
\begin{aligned}
\frac{dH(t)}{dt} = \frac{d}{dt}\left(\frac{1}{a(t)}\,\frac{da(t)}{dt}\right) &= -\frac{1}{a(t)}^{2}\left(\frac{da(t)}{dt}\right)^2 + \frac{1}{a(t)}\,\frac{d^2a(t)}{dt^2} \\
&= -H(t)^2 + \frac{1}{a(t)}\,\frac{d^2a(t)}{dt^2}.
\end{aligned}
$$

(e) *Prove that*

$$
g(\nabla_{X^\circ} Y^\circ, Z^\circ) = a(t)^2 g^N(\nabla^N_X Y, Z) \circ \pi = g((\nabla^N_X Y)^\circ, Z^\circ)
$$

for all vector smooth fields X, Y and Z on N, where ∇^N denotes the Levi-Civita connection determined by the metric tensor g^N on N, and apply this result, together with the results of (c) in order to show that

$$
\nabla_{X^\circ} Y^\circ = (\nabla^N_X Y)^\circ - \frac{H(t)}{q} g(X^\circ, Y^\circ)\, Q
$$

for all smooth vector fields X, Y and Z on N.

It follows from Lemma 9.3 that

$$\begin{aligned} 2g^N(\nabla^N_X Y, Z) &= X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X,Y)] \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \\ 2g(\nabla_{X^\circ} Y^\circ, Z^\circ) &= X^\circ[g(Y^\circ, Z^\circ)] + Y^\circ[g(X^\circ, Z^\circ)] \\ &\quad - Z^\circ[g(X^\circ, Y^\circ)] + g([X^\circ, Y^\circ], Z^\circ) \\ &\quad - g([X^\circ, Z^\circ], Y^\circ) - g([Y^\circ, Z^\circ], X^\circ) \end{aligned}$$

But

$$\begin{aligned} X^\circ[g(Y^\circ, Z^\circ)] &= X^\circ[a(t)^2(g(Y,Z)\circ\pi)] = a(t)^2 X^\circ[(g(Y,Z)\circ\pi)] \\ &= a(t)^2 X[(g(Y,Z))]\circ\pi, \end{aligned}$$

and

$$g([X^\circ, Y^\circ], Z^\circ) = g([X,Y]^\circ, Z^\circ) = a(t)^2 g([X,Y],Z)\circ\pi.$$

Therefore

$$g(\nabla_{X^\circ} Y^\circ, Z^\circ) = a(t)^2 g^N(\nabla^N_X Y, Z)\circ\pi = g((\nabla^N_X Y)^\circ, Z^\circ)$$

for all smooth vector fields X, Y and Z. It follows that

$$\nabla_{X^\circ} Y^\circ = (\nabla_X Y)^\circ + fQ,$$

for some real-valued function f on M. But it then follows from (c) that

$$qf = g(fQ, Q) = g(\nabla_{X^\circ} Y^\circ, Q) = -g(\nabla_{X^\circ} Q, Y^\circ) = -H(t) g(X^\circ, Y^\circ).$$

The result follows.

(f) *Use the definition of the Riemann curvature tensor and the results of previous parts of this question to show that*

$$\begin{aligned} R^M(X^\circ, Y^\circ)Z^\circ &= (R^N(X,Y)Z)^\circ \\ &\qquad - \frac{H(t)^2}{q}\Big(g(Z^\circ, Y^\circ)X^\circ - g(Z^\circ, X^\circ)Y^\circ\Big) \end{aligned}$$

and thus

$$\begin{aligned} &R^M(W^\circ, Z^\circ, X^\circ, Y^\circ) \\ &\quad = a(t)^2\, R^N(W,Z,X,Y)\circ\pi \\ &\qquad - \frac{H(t)^2}{q}\Big(g(W^\circ, X^\circ)g(Z^\circ, Y^\circ) - g(W^\circ, Y^\circ)g(Z^\circ, X^\circ)\Big) \end{aligned}$$

and

$$R^M(Q, Z^\circ, X^\circ, Y^\circ) = 0$$

for all smooth vector fields X, Y, Z and W on N, where R^N denotes the Riemann curvature tensor determined by the metric tensor g^N on N.

It follows from (e) and (c) that

$$\begin{aligned}
\nabla_{X^\circ}\nabla_{Y^\circ}Z^\circ &= \nabla_{X^\circ}\left((\nabla^N_Y Z)^\circ - \frac{H(t)}{q} g(Y^\circ, Z^\circ)\, Q\right) \\
&= (\nabla^N_X \nabla^N_Y Z)^\circ - \frac{H(t)}{q} g(X^\circ, (\nabla^N_Y Z)^\circ) \\
&\quad - \frac{H(t)}{q} X^\circ[g(Y^\circ, Z^\circ)]\, Q - \frac{H(t)^2}{q} g(Y^\circ, Z^\circ)\, X^\circ \\
&= (\nabla^N_X \nabla^N_Y Z)^\circ - \frac{H(t)}{q} g(X^\circ, (\nabla^N_Y Z)^\circ) \\
&\quad - \frac{H(t)}{q} g(\nabla_{X^\circ} Y^\circ, Z^\circ)\, Q - \frac{H(t)}{q} g(Y^\circ, \nabla_{X^\circ} Z^\circ)\, Q \\
&\quad - \frac{H(t)^2}{q} g(Y^\circ, Z^\circ)\, X^\circ \\
&= (\nabla^N_X \nabla^N_Y Z)^\circ - \frac{H(t)}{q} g((\nabla^N_X Y)^\circ, Z^\circ)\, Q \\
&\quad - \frac{H(t)}{q} g(X^\circ, (\nabla^N_Y Z)^\circ) - \frac{H(t)}{q} g(Y^\circ, (\nabla^N_X Z)^\circ)\, Q \\
&\quad - \frac{H(t)^2}{q} g(Y^\circ, Z^\circ)\, X^\circ
\end{aligned}$$

Therefore

$$\begin{aligned}
R^M(X^\circ, Y^\circ)Z^\circ &= \nabla_{X^\circ}\nabla_{Y^\circ}Z^\circ - \nabla_{Y^\circ}\nabla_{X^\circ}Z^\circ - \nabla_{[X^\circ, Y^\circ]}Z^\circ \\
&= (\nabla^N_X\nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z)^\circ \\
&\quad - \frac{H(t)}{q} g((\nabla^N_X Y)^\circ - (\nabla^N_Y X)^\circ - [X^\circ, Y^\circ], Z^\circ)\, Q \\
&\quad - \frac{H(t)^2}{q}\Big(g(Y^\circ, Z^\circ)\, X^\circ - g(X^\circ, Z^\circ)\, Y^\circ\Big)
\end{aligned}$$

But

$$(\nabla^N_X Y)^\circ - (\nabla^N_Y X)^\circ - [X^\circ, Y^\circ] = (\nabla^N_X Y - \nabla^N_Y X - [X, Y])^\circ = 0,$$

because the Levi-Civita connection ∇^N on N is torsion-free. Therefore

$$\begin{aligned}
&R^M(X^\circ, Y^\circ)Z^\circ \\
&\quad = (R^N(X, Y)Z)^\circ - \frac{H(t)^2}{q}\Big(g(Y^\circ, Z^\circ)\, X^\circ - g(X^\circ, Z^\circ)\, Y^\circ\Big)
\end{aligned}$$

for all smooth vector fields X, Y and Z on N. The remaining identities now follow from the relevant definitions.

3. *Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, let g be the standard flat metric on $\mathbb{R}^{n+1}$, defined such that*

$$g\left(\sum_{j=1}^{n+1} v^j \frac{\partial}{\partial x^j},\ \sum_{j=1}^{n+1} w^j \frac{\partial}{\partial x^j}\right) = \sum_{j=1}^{n+1} v^j w^j$$

for all $(v^1, \ldots, v^{n+1}), (w^1, \ldots, w^{n+1}) \in \mathbb{R}^{n+1}$, where $(x^1, x^2, \ldots, x^{n+1})$ is the standard Cartesian coordinate system on $\mathbb{R}^{n+1}$, and let g^S denote the Riemannian metric on S^n obtained on restricting the standard flat metric g on $\mathbb{R}^{n+1}$ to the tangent spaces of S^n. Let $\pi\colon \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to S^n$ be the radial projection map, defined so that $\pi(\curvearrowleft) = |\mathbf{x}|^{-1}\curvearrowleft$ for all $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, and let $r\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be the function defined such that $r(\mathbf{x}) = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^{n+1}$, so that $r(\mathbf{x})$ is the Euclidean distance from a point $\mathbf{x}$ of $\mathbb{R}^{n+1}$ to the origin.

(a) *Explain why*

$$g(\mathbf{U}, \mathbf{V}) = \langle dr, \mathbf{U}\rangle \langle dr, \mathbf{V}\rangle + r^2 g^S(\pi_*\mathbf{U}, \pi_*\mathbf{V})$$

for all points $\mathbf{x}$ of $\mathbb{R}^{n+1}$ and tangent vectors $\mathbf{U}$ and $\mathbf{V}$ to $\mathbb{R}^{n+1}$ at $\mathbf{x}$, so that

$$g = dr \otimes dr + r^2 \pi^* g^S.$$

Let $\mathbf{U}$ and $\mathbf{V}$ be tangent vectors to $\mathbb{R}^{n+1}$ at some point $\mathbf{x}$ of $\mathbb{R}^{n+1}$, and let $\hat{\mathbf{r}}$ denote the unit vector pointing radially outwards at $\mathbf{x}$ whose Cartesian components are $(r^{-1}x_1,\ r^{-1}x_2,\ \ldots, r^{-1}x^{n+1})$, where $r = |\mathbf{x}|$. Then

$$\mathbf{U} = \langle dr, \mathbf{U}\rangle\, \hat{\mathbf{r}} + \mathbf{U}^\perp, \quad \mathbf{V} = \langle dr, \mathbf{V}\rangle\, \hat{\mathbf{r}} + \mathbf{V}^\perp,$$

where $\mathbf{U}^\perp$ and $\mathbf{V}^\perp$ are the components of $\mathbf{U}$ and $\mathbf{V}$ respectively that are perpendicular to the radial vector $\hat{\mathbf{r}}$. Then

$$\begin{aligned} g(\mathbf{U}, \mathbf{V}) &= \langle dr, \mathbf{U}\rangle \langle dr, \mathbf{V}\rangle + g(\mathbf{U}^\perp, \mathbf{V}^\perp) \\ &= \langle dr, \mathbf{U}\rangle \langle dr, \mathbf{V}\rangle + r^2 g^S(\pi_*\mathbf{U}^\perp, \pi_*\mathbf{V}^\perp) \\ &= \langle dr, \mathbf{U}\rangle \langle dr, \mathbf{V}\rangle + r^2 g^S(\pi_*\mathbf{U}, \pi_*\mathbf{V}), \end{aligned}$$

as required.

(b) *By applying the results of problem 2, show that the Riemann curvature tensor R^S of the Riemannian metric g^S on S^n satisfies the identity*

$$R^S(W, Z, X, Y) = g^S(W, X) g^S(Z, Y) - g^S(W, Y) g^S(Z, X).$$

The Riemann curvature tensor of the flat metric g on $\mathbb{R}^{n+1}$ is zero. The required result therefore follows on applying problem 2(f).

4. *A Riemannian manifold M with metric tensor g is said to be a* space of constant curvature K_0, *where K_0 is some real constant, if the Riemann curvature tensor R determined by the metric tensor g satisfies the identity*

$$R(W, Z, X, Y) = K_0 \left(g(W, X) g(Z, Y) - g(W, Y) g(Z, X) \right).$$

Question 3 shows that the unit sphere in $\mathbb{R}^{n+1}$, with the usual Riemannian metric, is a space of constant curvature $+1$. Moreover Lemma 9.9 ensures that a Riemannian manifold is a space of constant curvature K_0 if and only if all sectional curvatures of M are equal to K_0.

Let N be a Riemannian manifold of constant curvature K_N, with metric tensor g^N and Riemann curvature tensor R^N, let $M = N \times I$, where I is an open interval in $\mathbf{R}$, and let g be the Riemannian metric on R defined as described in problem 2, with $q = 1$, so that

$$g = dt \otimes dt + a(t)^2 \, \pi^* g^N,$$

where $\pi \colon M \to N$ is the projection function defined such that $\pi(p, t) = p$, the smooth real-valued function t on M corresponds to the projection map $(p, t) \mapsto t$ from M to I, and $a \colon I \to \mathbb{R}$ is a smooth everywhere positive function. Also let $Q = \dfrac{\partial}{\partial t}$, so that $Q_{(p,t)}$ is the velocity vector of the smooth curve $s \mapsto (p, s)$ at each point (p, t) of M. And, for each smooth vector field X on N, let X° denote the smooth vector field on M defined such that $g(Q, X^\circ) = 0$ and $\pi_* X^\circ_{(p,t)} = X_p$ for all $(p, t) \in M$.

(a) *Let P be a 2-dimensional vector subspace of the tangent space $T_{(p,t)}$ to M at some point (p, t) of M. Use the results of problem 2, show that the sectional curvature $K_M(P)$ of M in the plane P satisfies*

$$\begin{aligned} K_M(P) &= -\left(\frac{dH(t)}{dt} + H(t)^2 \right) \\ &= -\frac{1}{a(t)} \frac{d^2 a(t)}{dt^2} \end{aligned}$$

if the plane P contains $Q_{(p,t)}$, and that

$$\begin{aligned} K_M(P) &= \frac{K_N}{a(t)^2} - H(t)^2 \\ &= \frac{1}{a(t)^2} \left(K_N - \left(\frac{da(t)}{dt} \right)^2 \right) \end{aligned}$$

if the plane P is orthogonal to $Q_{(p,t)}$, where

$$H(t) = \frac{1}{a(t)} \frac{da(t)}{dt}.$$

Now $g(Q,Q) = 1$. It follows from problem 2 that

$$\begin{aligned} R^M(W^\circ, Q, X^\circ, Q) &= -\left(\frac{dH(t)}{dt} + H(t)^2\right) g(W^\circ, X^\circ) \\ &= -\left(\frac{dH(t)}{dt} + H(t)^2\right) g(W^\circ, X^\circ) g(Q,Q). \end{aligned}$$

for all smooth vector fields X on M (using the notation of that question, with $q = 1$). This immediately yields the required result when the plane P in $T_{(p,t)}M$ contains the tangent vector $Q_{(p,t)}$.
Also

$$\begin{aligned} & a(t)^2 R^N(W,Z,X,Y) \circ \pi \\ &= a(t)^2 K_N \Big(g^N(W,X) g^N(Z,Y) - g^N(W,Y) g^N(Z,X)\Big) \circ \pi \\ &= \frac{K_N}{a(t)^2} \Big(g(W^\circ, X^\circ) g(Z^\circ, Y^\circ) - g(W^\circ, Y^\circ) g(Z^\circ, X^\circ)\Big) \end{aligned}$$

for all smooth vector fields W, X, Y and Z on M. It follows from problem 2(f) that

$$\begin{aligned} & R^M(W^\circ, Z^\circ, X^\circ, Y^\circ) \\ &= \left(\frac{K_N}{a(t)^2} - H(t)^2\right) \Big(g(W^\circ, X^\circ) g(Z^\circ, Y^\circ) \\ &\qquad - g(W^\circ, Y^\circ) g(Z^\circ, X^\circ)\Big) \end{aligned}$$

when the plane P is orthogonal to $Q_{(p,t)}$.

(b) *Show that if the function a satisfies the differential equation*

$$K_N - \left(\frac{da(t)}{dt}\right)^2 = K_M\, a(t)^2,$$

where K_N and K_M are real constants, then

$$\frac{d^2 a(t)}{dt^2} = -K_M\, a(t).$$

Hence or otherwise show that, in the case where $K_M > 0$ and $K_N > 0$, the Riemannian manifold M is a space of constant curvature K_M if and only if

$$a(t) = \sqrt{\frac{K_N}{K_M}}\, \sin\Bigl(\sqrt{K_M}(t - t_0)\Bigr),$$

where t_0 is an arbitrary real constant. Show also that, in the case where $K_M < 0$ and $K_N > 0$, the Riemannian manifold M is a space of constant curvature K_M if and only if

$$a(t) = \sqrt{-\frac{K_N}{K_M}}\, \sinh\Bigl(\sqrt{-K_M}(t - t_0)\Bigr),$$

where t_0 is an arbitrary real constant. And also show that, in the case where $K_M < 0$ and $K_N < 0$, the Riemannian manifold M is a space of constant curvature K_M if and only if

$$a(t) = \sqrt{\frac{K_N}{K_M}}\, \cosh\Bigl(\sqrt{-K_M}(t - t_0)\Bigr),$$

The second differential equation satisfied by the function $a(t)$ follows on differentiating the first. If M is a space of constant curvature M then both differential equations must be satisfied, and therefore the function $a(t)$ is of the specified form determined by the signs of K_M and K_N. Conversely suppose that these differential equations are satisfied. Let R_0 be the covariant tensor of degree 4 defined by

$$R_0(V_1, V_2, V_3, V_4) = K_M\Bigl(g(V_1, V_3)g(V_2, V_4) - g(V_1, V_4)g(V_2, V_3)\Bigr)$$

for all $V_1, V_2, V_3, V_4 \in T_{(p,t)}M$. Then

$$R^M(W^\circ, Z^\circ, X^\circ, Y^\circ) = R_0(W^\circ, Z^\circ, X^\circ, Y^\circ)$$

and

$$R^M(W^\circ, Q, X^\circ, Q) = R_0(W^\circ, Q, X^\circ, Q).$$

Moreover it follows from the results of problem 2(f) that

$$R^M(V_1, V_2, V_3, V_4) = R_0(V_1, V_2, V_3, V_4) = 0$$

when one of the vectors V_1, V_2, V_3, V_4 is parallel to Q and the remaining vectors are orthogonal to Q. And the basic identities satisfied by the Riemann curvature tensor ensure that the same is true when at least three of these vectors V_1, V_2, V_3, V_4 are parallel to Q. It follows that $R = R_0$, and thus M is a space of constant curvature K_M, as required.

5. **(Friedman-Robertson-Walker Metrics)** *Let N be a Riemannian manifold of constant curvature K_N, with metric tensor g^N and Riemann curvature tensor R^N, where*

$$\begin{aligned} R^N(W,Z,X,Y) &= K_N\Big(g^N(W,X)g^N(Z,Y) \\ &\qquad - g^N(W,Y)g^N(Z,X)\Big) \end{aligned}$$

for all vector fields W, X, Y and Z on N. Let $M = N \times I$, where I is an open interval in $\mathbf{R}$, and let g be the Riemannian metric on R defined as described in problem 2, with $q = -c^2$, where c is a real constant (representing the speed of light in General Relativity), so that

$$g = -c^2\, dt \otimes dt + a(t)^2\, \pi^* g^N,$$

where $\pi\colon M \to N$ is the projection function defined such that $\pi(p,t) = p$, the smooth real-valued function t on M corresponds to the projection map $(p,t) \mapsto t$ from M to I, and $a\colon I \to \mathbb{R}$ is a smooth everywhere positive function. Also let $Q = \dfrac{\partial}{\partial t}$, so that $Q_{(p,t)}$ is the velocity vector of the smooth curve $s \mapsto (p,s)$ at each point (p,t) of M. And, for each smooth vector field X on N, let X° denote the smooth vector field on M defined such that $g(Q, X^\circ) = 0$ and $\pi_* X^\circ_{(p,t)} = X_p$ for all $(p,t) \in M$.

(a) *Show that*

$$\begin{aligned} R^M(W^\circ, Q, X^\circ, Q) &= -\left(\frac{dH(t)}{dt} + H(t)^2\right) g(W^\circ, X^\circ) \\ &= \frac{1}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right) g(W^\circ, X^\circ) g(Q,Q) \end{aligned}$$

and

$$\begin{aligned} & R^M(W^\circ, Z^\circ, X^\circ, Y^\circ) \\ &\quad = \left(\frac{K_N}{a(t)^2} + \frac{H(t)^2}{c^2}\right)\Big(g(W^\circ, X^\circ)g(Z^\circ, Y^\circ) \\ &\qquad\qquad - g(W^\circ, Y^\circ)g(Z^\circ, X^\circ)\Big) \end{aligned}$$

where

$$H(t) = \frac{1}{a(t)}\,\frac{da(t)}{dt}.$$

Verify also that $R(V_1, V_2, V_3, V_4) = 0$ when exactly one of these vector fields V_1, V_2, V_3, V_4 is parallel to Q and the remaining three are orthogonal to Q.

These results follow from problem 2, parts (d) and (f).

Remark Pseudo-Riemannian manifolds whose metric tensor has the structure analysed in problems 2 and 5 are used to model the expansion of the universe, on the assumption that the universe is homogeneous and isotropic (so that mass, energy and pressure are uniformly distributed). This question establishes that the Riemann curvature tensor of M is determined by the two quantities

$$\frac{1}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right) \quad \text{and} \quad \frac{H(t)^2}{c^2} + \frac{k}{a(t)^2},$$

where k is some appropriate constant, which is positive if the universe at any instant of time is a space of constant positive curvature, zero if the universe is flat, and negative if the universe at any instance of time is a space of constant negative curvature. Without loss of generality, we may suppose that k has one of the three values $+1$, 0 and -1. The *Einstein field equations* of General Relativity relate these quantities to appropriate components of the stress-energy tensor. In these models describing the expansion of the universe, the stress-energy tensor is determined by the mass density ρ and the pressure p. Alexander Friedmann discovered in 1922 and 1924 equations that describe the evolution of these cosmological models. Einstein's field equations yield the following equations governing the time evolution of a homogeneous isotropic universe as described by Friedmann's models:

$$\begin{aligned} \frac{1}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right) &= -\frac{4\pi G}{3c^2}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3} \\ \frac{H(t)^2}{c^2} + \frac{k}{a(t)^2} &= \frac{8\pi G}{3c^2}\rho + \frac{\Lambda}{3} \end{aligned}$$

Here the constant c is the speed of light, the function H is the *Hubble parameter* (whose current value is the *Hubble constant* that expresses the current rate of expansion of the universe), G is the gravitational constant, and Λ is the cosmological constant (regretfully introduced by Einstein in the hope of obtaining static solutions to the field equations) which represents *dark energy* in the universe. The constant k represents the curvature of our model universe N, and indeed we may choose N so that k has one of the three values $+1$, 0 and -1. In the case where $k = +1$, the universe at time t is compact, and is isometric to a three-dimensional sphere of radius $a(t)$. In the case where $k = 0$, the universe is flat at all times. In the case where $k = -1$ the universe at time t is non-compact, and is isometric to a three-dimensional hyperbolic space of curvature $-1/a(t)^2$.

The cosmological models discussed here were later studied by Georges Lemaître in 1927 and in the 1930s by Howard Percy Robertson and Arthur Geoffrey Walker. Riemannian metrics with the structure described in problem 5 are often referred to as *Robertson-Walker metrics*.

We now discuss in more detail how to derive Friedmann's equations from Einstein's field equations. Note that if E_0, E_1, E_2 and E_3 are smooth vector fields over an open subset of M that constitute a Lorentzian moving frame for M, so that the vector fields E_0, E_1, E_2, E_3 are mutually orthogonal and satisfy

$$g(E_0, E_0) = -1, \quad g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$

and if this moving frame is chosen such that $E_0 = Q$ and E_1, E_2, E_3 are orthogonal to Q, then the Riemann curvature tensor R of space-time satisfies

$$\begin{aligned} R(E_1, E_2, E_1, E_2) &= R(E_2, E_3, E_2, E_3) = R(E_3, E_1, E_3, E_1) \\ &= \frac{H(t)^2}{c^2} + \frac{k}{a(t)^2} \\ R(E_0, E_1, E_0, E_1) &= R(E_0, E_2, E_0, E_2) = R(E_0, E_3, E_0, E_3) \\ &= -\frac{1}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right) \\ &= -\frac{1}{c^2 a(t)}\frac{d^2 a(t)}{dt^2} \end{aligned}$$

Moreover $R(E_\alpha, E_\beta, E_\gamma, E_\delta) = 0$ unless $\alpha \neq \beta$ and either $\gamma = \alpha$ and $\delta = \beta$ or else $\gamma = \beta$ and $\delta = \alpha$. The *Einstein field equations* require that the curvature of the space-time manifold M satisfy the identity

$$\operatorname{Ric}(X, Y) - \frac{1}{2} g(X, Y) S = \frac{8\pi G}{c^4} T(X, Y) - \Lambda\, g(X, Y)$$

for all vector fields X and Y on M, where T is the *stress-energy tensor*, in covariant form, determined by the matter, electromagnetic fields etc. in the universe, Ric is the *Ricci curvature tensor* of M, defined such that

$$\begin{aligned} \operatorname{Ric}(X, Y) &= -R(E_0, X, E_0, Y) + R(E_1, X, E_1, Y) \\ &\quad + R(E_2, X, E_2, Y) + R(E_3, X, E_3, Y), \end{aligned}$$

and S is the *scalar curvature*, defined such that

$$\begin{aligned} S &= -\operatorname{Ric}(E_0, E_0) + \operatorname{Ric}(E_1, E_1) + \operatorname{Ric}(E_2, E_2) + \operatorname{Ric}(E_3, E_3) \\ &= 2\Big(-R(E_0, E_1, E_0, E_1) - R(E_0, E_2, E_0, E_2) \\ &\quad - R(E_0, E_3, E_0, E_3) + R(E_1, E_2, E_1, E_2) \\ &\quad + R(E_2, E_3, E_2, E_3) + R(E_3, E_1, E_3, E_1)\Big) \end{aligned}$$

The Einstein field equations may also be presented in the form

$$\mathrm{Ein}(X,Y) = \frac{8\pi G}{c^4} T(X,Y) - \Lambda\, g(X,Y)$$

where Ein denotes the *Einstein tensor*, defined such that

$$\mathrm{Ein}(X,Y) = \mathrm{Ric}(X,Y) - \frac{1}{2} g(X,Y) S$$

for all vector fields X and Y on space-time. Then

$$\begin{aligned}
\mathrm{Ein}(E_0,E_0) &= R(E_1,E_2,E_1,E_2) + R(E_2,E_3,E_2,E_3) \\
&\qquad + R(E_3,E_1,E_3,E_1) \\
\mathrm{Ein}(E_1,E_1) &= R(E_0,E_2,E_0,E_2) + R(E_0,E_3,E_0,E_3) \\
&\qquad - R(E_2,E_3,E_2,E_3) \\
\mathrm{Ein}(E_2,E_2) &= R(E_0,E_3,E_0,E_3) + R(E_0,E_1,E_0,E_1) \\
&\qquad - R(E_3,E_1,E_3,E_1) \\
\mathrm{Ein}(E_3,E_3) &= R(E_0,E_1,E_0,E_1) + R(E_0,E_2,E_0,E_2) \\
&\qquad - R(E_1,E_2,E_1,E_2)
\end{aligned}$$

Thus, for the Friedmann metric,

$$\begin{aligned}
\mathrm{Ric}(E_0,E_0) &= -\frac{3}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right), \\
\mathrm{Ric}(E_1,E_1) &= \mathrm{Ric}(E_2,E_2) = \mathrm{Ric}(E_3,E_3) \\
&= \frac{1}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right) + \frac{2k}{a(t)^2} + \frac{2H(t)^2}{c^2}, \\
\mathrm{Ein}(E_0,E_0) &= \frac{3k}{a(t)^2} + \frac{3H(t)^2}{c^2}, \\
\mathrm{Ein}(E_1,E_1) &= \mathrm{Ein}(E_2,E_2) = \mathrm{Ein}(E_3,E_3) \\
&= -\frac{2}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right) - \frac{H(t)^2}{c^2} - \frac{k}{a(t)^2}, \\
S &= \frac{6}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right) + \frac{6k}{a(t)^2} + \frac{6H(t)^2}{c^2}.
\end{aligned}$$

Also $\mathrm{Ric}(E_\alpha,E_\beta) = \mathrm{Ein}(E_\alpha,E_\beta) = 0$ when $\alpha \neq \beta$.

Now if the universe is filled with a perfect homogeneous fluid with mass density ρ and pressure p, where the world-lines of the particles are integral curves for the vector field E_0 (so that E_0 represents the four-velocity of the particles in the fluid) then

$$T(E_0, E_0) = \rho c^2 \quad \text{and} \quad T(E_1, E_1) = T(E_2, E_2) = T(E_3, E_3) = p.$$

It follows from the Einstein field equations that

$$\begin{aligned} \frac{H(t)^2}{c^2} + \frac{k}{a(t)^2} &= \frac{1}{3}\mathrm{Ein}(E_0, E_0) = \frac{8\pi G}{3c^4} T(E_0, E_0) - \frac{\Lambda}{3}\, g(E_0, E_0) \\ &= \frac{8\pi G}{3c^2}\rho + \frac{\Lambda}{3}, \\ \frac{1}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right) &= \frac{1}{6}S - \frac{1}{3}\mathrm{Ein}(E_0, E_0) \\ &= -\frac{4\pi G}{3c^2}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}. \end{aligned}$$

We have thus obtained the equations govening the expansion of the universe in Friedmann's models. We can eliminate the mass density ρ from these two equations to obtain the equation

$$\begin{aligned} &\frac{2}{c^2 a(t)}\frac{d^2 a(t)}{dt^2} + \frac{1}{c^2 a(t)^2}\left(\frac{da(t)}{dt}\right)^2 + \frac{k}{a(t)^2} \\ &\quad = \frac{2}{c^2}\left(\frac{dH(t)}{dt} + H(t)^2\right) + \frac{H(t)^2}{c^2} + \frac{k}{a(t)^2} \\ &\quad = -\frac{8\pi G p}{c^4} + \Lambda. \end{aligned}$$

The Schwarzschild and Reissner-Nordström Metrics

6. *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g and Levi-Civita connection ∇, and let $E_1, E_2, \ldots, E_n$ be smooth vector fields on M, where $g(E_j, E_k)$ is a constant function for all j and k.*

   (a) *Explain why*

   $$\begin{aligned} g(\nabla_{E_i} E_j, E_k) &= -g(E_j, \nabla_{E_i} E_k) \\ &= g([E_k, E_i], E_j) - g(\nabla_{E_k} E_i, E_j) \end{aligned}$$

   for $i, j, k = 1, 2, \ldots, n$.

The function $g(E_j, E_k)$ is constant on M, and therefore

$$0 = E_i[g(E_j, E_k)] = g(\nabla_{E_i} E_j, E_k) + g(E_j, \nabla_{E_i} E_k).$$

Moreover

$$g([E_k, E_i], E_j) = g(\nabla_{E_k} E_i, E_j) - g(\nabla_{E_i} E_k, E_j),$$

because the Levi-Civita connection ∇ is torsion-free. The results follow.

(b) *Show that*

$$g(\nabla_{E_i} E_j, E_k) + g(\nabla_{E_j} E_i, E_k) = g([E_k, E_i], E_j) + g(E_i, [E_k, E_j]),$$

and therefore

$$g(\nabla_{E_i} E_j, E_k) = \tfrac{1}{2}\Big(g([E_k, E_i], E_j) + g(E_i, [E_k, E_j]) + g([E_i, E_j], E_k)\Big).$$

It follows from (a) that

$$\begin{aligned} & g(\nabla_{E_i} E_j, E_k) + g(\nabla_{E_j} E_i, E_k) \\ &\quad = g([E_k, E_i], E_j) - g(\nabla_{E_k} E_i, E_j) + g([E_k, E_j], E_i) - g(E_i, \nabla_{E_k} E_j) \\ &\quad = g([E_k, E_i], E_j) + g([E_k, E_j], E_i) - E_k[g(E_i, E_j)] \\ &\quad = g([E_k, E_i], E_j) + g([E_k, E_j], E_i). \end{aligned}$$

Therefore

$$\begin{aligned} & g(\nabla_{E_i} E_j, E_k) \\ &\quad = \tfrac{1}{2}\Big(g(\nabla_{E_i} E_j, E_k) + g(\nabla_{E_j} E_i, E_k) + g([E_i, E_j], E_k)\Big) \\ &\quad = \tfrac{1}{2}\Big(g([E_k, E_i], E_j) + g(E_i, [E_k, E_j]) + g([E_i, E_j], E_k)\Big), \end{aligned}$$

as required.

(c) *Show that*

$$g(\nabla_{E_i} E_j, E_i) = g([E_i, E_j], E_i) \quad \textit{and} \quad g(\nabla_{E_i} E_j, E_j) = 0.$$

This follows directly from (b), or may be verified directly using the identities $E_i[g(E_j, E_j)] = 0$ and $[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_j} E_i$,

(d) *Suppose that there exist smooth real-valued functions u_{ij} for $i \neq j$ such that $[E_i, E_j] = u_{ij} E_j - u_{ji} E_i$, so that $[E_i, E_j]$ is in the linear span of the vectors E_i and E_j at each point of M for $i, j = 1, 2, \ldots, n$. Show that $g(\nabla_{E_i} E_j, E_k) = 0$ whenever the indices i, j and k are distinct.*

This result follows directly from the formula for $g(\nabla_{E_i} E_j, E_k)$ given in (b).

7. *Consider a pseudo-Riemannian metric g on a space-time M that takes the form*

$$g = -\mu(r)^2\, dt^2 + \kappa(r)^2\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2,$$

on the domain U of a local coordinate system t, r, θ, φ, where (in accordance with traditional notation)

$$dt^2 = dt \otimes dt, \quad dr^2 = dr \otimes dr, \quad d\theta^2 = d\theta \otimes d\theta, \quad d\varphi^2 = d\varphi \otimes d\varphi,$$

and where ψ and κ are smooth real-valued functions defined on $\{r \in \mathbb{R} : r > r_0\}$ for some $r_0 > 0$. Let E_t, E_r, E_θ and E_φ be the smooth vector fields on U defined such that

$$E_t = \frac{1}{\mu(r)}\frac{\partial}{\partial t}, \quad E_r = \frac{1}{\kappa(r)}\frac{\partial}{\partial r}, \quad E_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}, \quad E_\varphi = \frac{1}{r\,\sin\theta}\frac{\partial}{\partial\varphi},$$

so that the vector fields E_t, E_r, E_θ and E_φ are mutually orthogonal, and

$$g(E_t, E_t) = -1$$

and

$$g(E_r, E_r) = g(E_\theta, E_\theta) = g(E_\varphi, E_\varphi) = 1.$$

(a) *Show that*

$$\begin{aligned}
[E_r, E_t] &= -\frac{1}{\kappa(r)\mu(r)}\frac{d\mu(r)}{dr}\,E_t, \\
[E_r, E_\theta] &= -\frac{1}{r\kappa(r)}\,E_\theta, \\
[E_r, E_\varphi] &= -\frac{1}{r\kappa(r)}\,E_\varphi, \\
[E_t, E_\theta] &= 0, \\
[E_t, E_\varphi] &= 0, \\
[E_\theta, E_\varphi] &= -\frac{1}{r\tan\theta}\,E_\varphi.
\end{aligned}$$

These follow by direct calculation using Lemma 7.6.

(b) *Using the results of (a) and of problem 6, or otherwise, show that*

$$\nabla_{E_r} E_t = \nabla_{E_r} E_r = \nabla_{E_r} E_\theta = \nabla_{E_r} E_\varphi = 0,$$

$$\nabla_{E_t} E_\theta = \nabla_{E_\theta} E_t = \nabla_{E_t} E_\varphi = \nabla_{E_\varphi} E_t = \nabla_{E_\theta} E_\varphi = 0,$$

$$\nabla_{E_t} E_r = \frac{1}{\kappa(r)\mu(r)} \frac{d\mu(r)}{dr} E_t, \quad \nabla_{E_t} E_t = \frac{1}{\kappa(r)\mu(r)} \frac{d\mu(r)}{dr} E_r,$$

$$\nabla_{E_\theta} E_r = \frac{1}{r\kappa(r)} E_\theta, \quad \nabla_{E_\theta} E_\theta = -\frac{1}{r\kappa(r)} E_r,$$

$$\nabla_{E_\varphi} E_r = \frac{1}{r\kappa(r)} E_\varphi, \quad \nabla_{E_\varphi} E_\theta = \frac{1}{r\tan\theta} E_\varphi,$$

$$\nabla_{E_\varphi} E_\varphi = -\frac{1}{r\kappa(r)} E_r - \frac{1}{r\tan\theta} E_\theta.$$

It follows from (a) that if X and Y are distinct vector fields chosen from the list $E_t, E_r, E_\theta, E_\varphi$ then $[X, Y] = hX + kY$ for some real-valued functions h and k. It then follows from the results of problem 6 that

$$g(\nabla_X Y, X) = g([X, Y], X), \quad g(\nabla_X X, Y) = -g([X, Y], X),$$

$$g(\nabla_X Y, Y) = 0 \quad \text{and} \quad g(\nabla_X Y, Z) = 0$$

for all mutually orthogonal vector fields X, Y and Z chosen from the list $E_t, E_r, E_\theta, E_\varphi$. The stated results then follow from (a).

(c) *Show that*

$$R(E_r, E_\theta) E_\varphi = R(E_\theta, E_\varphi) E_r = R(E_\varphi, E_r) E_\theta = 0,$$

$$R(E_t, E_\theta) E_\varphi = R(E_\theta, E_\varphi) E_t = R(E_\varphi, E_t) E_\theta = 0,$$

$$R(E_t, E_r) E_\varphi = R(E_r, E_\varphi) E_t = R(E_\varphi, E_t) E_r = 0,$$

$$R(E_t, E_r) E_\theta = R(E_r, E_\theta) E_t = R(E_\theta, E_t) E_r = 0.$$

It follows from (b) that

$$\nabla_{E_r} E_\theta = \nabla_{E_r} E_\varphi = \nabla_{E_t} E_\theta = \nabla_{E_t} E_\varphi = \nabla_{E_\theta} E_\varphi = 0,$$

$$\nabla_{E_r} E_t = \nabla_{E_\theta} E_t = \nabla_{E_\varphi} E_t = 0.$$

Also $[E_r, E_\theta]$ is parallel to E_θ, $[E_r, E_\varphi]$ and $[E_\theta, E_\varphi]$ are parallel to E_φ, $[E_t, E_r]$ is parallel to E_t, and $[E_t, E_\theta] = 0$, and therefore

$$\nabla_{[E_r, E_\theta]} E_t = \nabla_{[E_r, E_\varphi]} E_t = \nabla_{[E_\theta, E_\varphi]} E_t = \nabla_{[E_r, E_\theta]} E_\varphi = 0,$$

$$\nabla_{[E_t,E_r]} E_\theta = \nabla_{[E_t,E_r]} E_\varphi = \nabla_{[E_t,E_\theta]} E_\varphi = 0.$$

It then follows from the definition of the Riemann curvature tensor that

$$R(E_r, E_\theta)E_t = R(E_r, E_\varphi)E_t = R(E_\theta, E_\varphi)E_t = R(E_r, E_\theta)E_\varphi = 0,$$

$$R(E_t, E_r)E_\theta = R(E_t, E_r)E_\varphi = R(E_t, E_\theta)E_\varphi = 0.$$

Also

$$\begin{aligned} R(E_r, E_\varphi)E_\theta &= \nabla_{E_r}\nabla_{E_\varphi} E_\theta - \nabla_{[E_r,E_\varphi]} E_\theta \\ &= \nabla_{E_r}\left(\frac{1}{r\tan\theta} E_\varphi\right) + \frac{1}{r\kappa(r)} \nabla_{E_\varphi} E_\theta \\ &= \frac{1}{\kappa(r)} \frac{d}{dr}\left(\frac{1}{r\tan\theta}\right) E_\varphi + \frac{1}{r^2\kappa(r)\,\tan\theta} E_\varphi \\ &= 0. \end{aligned}$$

Note that this latter result can also be justified as following from the identity $R(E_r, E_\theta)E_\varphi = 0$ on symmetry grounds, since a rotation through an angle of $\pi/2$ about a radial axis will preserve the metric tensor, and will send the vectors E_θ and E_φ to E_φ and $-E_\theta$ respectively at points that lie on the axis of rotation.

Also

$$\begin{aligned} R(E_t, E_\varphi)E_\theta &= \nabla_{E_t}\nabla_{E_\varphi} E_\theta = \nabla_{E_t}\left(\frac{1}{r\tan\theta} E_\varphi\right) = \frac{1}{r\tan\theta}\,\nabla_{E_t} E_\varphi \\ &= 0, \\ R(E_t, E_\theta)E_r &= \nabla_{E_t}\nabla_{E_\theta} E_r - \nabla_{E_\theta}\nabla_{E_t} E_r - \nabla_{[E_t,E_\theta]} E_r \\ &= \nabla_{E_t}\left(\frac{1}{r\kappa(r)}\, E_\theta\right) - \nabla_{E_\theta}\left(\frac{1}{\kappa(r)\mu(r)}\,\frac{d\mu(r)}{dr}\, E_t\right) \\ &= \frac{1}{r\kappa(r)}\,\nabla_{E_t} E_\theta - \frac{1}{\kappa(r)\mu(r)}\,\frac{d\mu(r)}{dr}\,\nabla_{E_\theta} E_t \\ &= 0, \\ R(E_t, E_\varphi)E_r &= \nabla_{E_t}\nabla_{E_\varphi} E_r - \nabla_{E_\varphi}\nabla_{E_t} E_r - \nabla_{[E_t,E_\varphi]} E_r \\ &= \nabla_{E_t}\left(\frac{1}{r\kappa(r)}\, E_\varphi\right) - \nabla_{E_\varphi}\left(\frac{1}{\kappa(r)\mu(r)}\,\frac{d\mu(r)}{dr}\, E_t\right) \\ &= \frac{1}{r\kappa(r)}\,\nabla_{E_t} E_\varphi - \frac{1}{\kappa(r)\mu(r)}\,\frac{d\mu(r)}{dr}\,\nabla_{E_\varphi} E_t \\ &= 0, \\ R(E_\theta, E_\varphi)E_r &= \nabla_{E_\theta}\nabla_{E_\varphi} E_r - \nabla_{E_\varphi}\nabla_{E_\theta} E_r - \nabla_{[E_\theta,E_\varphi]} E_r \end{aligned}$$

$$
\begin{aligned}
&= \nabla_{E_\theta}\left(\frac{1}{r\kappa(r)}E_\varphi\right) - \nabla_{E_\varphi}\left(\frac{1}{r\kappa(r)}E_\theta\right) \\
&\qquad + \frac{1}{r\tan\theta}\nabla_{E_\varphi}E_r \\
&= \frac{1}{r\kappa(r)}\left(\nabla_{E_\theta}E_\varphi - \nabla_{E_\varphi}E_\theta\right) + \frac{1}{r^2\kappa(r)\,\tan\theta}E_\varphi \\
&= \frac{1}{r\kappa(r)}\left[E_\theta, E_\varphi\right] + \frac{1}{r^2\kappa(r)\,\tan\theta}E_\varphi \\
&= 0.
\end{aligned}
$$

Alternatively, these last four identities follow from previously proved identities and the Bianchi Identity. Indeed

$$
\begin{aligned}
R(E_t, E_\varphi)E_\theta &= -R(E_\varphi, E_\theta)E_t - R(E_\theta, E_t)E_\varphi = 0, \\
R(E_t, E_\theta)E_r &= -R(E_\theta, E_r)E_t - R(R_r, E_t)E_\theta = 0, \\
R(E_t, E_\varphi)E_r &= -R(E_\varphi, E_r)E_t - R(R_r, R_t)E_\varphi = 0, \\
R(E_r, E_\theta)E_\varphi &= -R(E_\theta, E_\varphi)E_r - R(E_\varphi, E_r)E_\theta = 0.
\end{aligned}
$$

(d) *Explain why*

$$
\begin{aligned}
R\left(\frac{\partial}{\partial t}, E_r\right)E_r &= -\nabla_{E_r}\nabla_{E_r}\frac{\partial}{\partial t} \\
R\left(\frac{\partial}{\partial \theta}, E_r\right)E_r &= -\nabla_{E_r}\nabla_{E_r}\frac{\partial}{\partial \theta} \\
R\left(\frac{\partial}{\partial \varphi}, E_r\right)E_r &= -\nabla_{E_r}\nabla_{E_r}\frac{\partial}{\partial \varphi}.
\end{aligned}
$$

Hence or otherwise, show that

$$
\begin{aligned}
R\left(E_t,\, E_r\right)E_r &= -\frac{1}{2\kappa(r)\mu(r)}\frac{d}{dr}\left(\frac{1}{\kappa(r)\mu(r)}\frac{d}{dr}\left(\mu(r)^2\right)\right)E_t \\
R\left(E_\theta,\, E_r\right)E_r &= -\frac{1}{2r}\frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right)E_\theta, \\
R\left(E_\varphi,\, E_r\right)E_r &= -\frac{1}{2r}\frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right)E_\varphi.
\end{aligned}
$$

The vector fields $\dfrac{\partial}{\partial t}$ $\dfrac{\partial}{\partial \theta}$ and $\dfrac{\partial}{\partial \varphi}$ satisfy

$$
\left[E_r, \frac{\partial}{\partial t}\right] = \left[E_r, \frac{\partial}{\partial \theta}\right] = \left[E_r, \frac{\partial}{\partial \varphi}\right]
$$

It follows that these vector fields satisfy the Jacobi equation along the geodesics whose velocity vector at each point of M is E_r. Indeed

$$\begin{aligned}\nabla_{E_r}\nabla_{E_r}\frac{\partial}{\partial t} &= \nabla_{E_r}\nabla_{\frac{\partial}{\partial t}}E_r = R\left(E_r, \frac{\partial}{\partial t}\right)E_r + \nabla_{\frac{\partial}{\partial t}}\nabla_{E_r}E_r \\ &= -R\left(\frac{\partial}{\partial t}, E_r\right)E_r.\end{aligned}$$

and similarly

$$\begin{aligned}\nabla_{E_r}\nabla_{E_r}\frac{\partial}{\partial \theta} &= -R\left(\frac{\partial}{\partial \theta}, E_r\right)E_r, \\ \nabla_{E_r}\nabla_{E_r}\frac{\partial}{\partial \varphi} &= -R\left(\frac{\partial}{\partial \varphi}, E_r\right)E_r.\end{aligned}$$

Now

$$\nabla_{E_r}\nabla_{E_r}\frac{\partial}{\partial t} = \nabla_{E_r}\nabla_{E_r}(\mu(r)E_t) = \nabla_{E_r}(E_r[\mu(r)]E_t) = E_r[E_r[\mu(r)]]E_t,$$

because $\nabla_{E_r}E_t = 0$. Similarly

$$\nabla_{E_r}\nabla_{E_r}\frac{\partial}{\partial \theta} = E_r[E_r[r]]E_\theta \quad \text{and} \quad \nabla_{E_r}\nabla_{E_r}\frac{\partial}{\partial \varphi} = E_r[E_r[r\,\sin\theta]]E_\varphi.$$

It follows that

$$\begin{aligned}R\,(E_t,\, E_r)\,E_r &= -\frac{1}{\mu(r)}\,E_r[E_r[\mu(r)]]\,E_t \\ &= -\frac{1}{\kappa(r)\mu(r)}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)}\frac{d\mu(r)}{dr}\right)\,E_t \\ &= -\frac{1}{2\kappa(r)\mu(r)}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)\mu(r)}\frac{d}{dr}\left(\mu(r)^2\right)\right)\,E_t \\ R\,(E_\theta,\, E_r)\,E_r &= -\frac{1}{r}\,E_r[E_r[r]]\,E_\theta \\ &= -\frac{1}{r\kappa(r)}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)}\right)\,E_\theta \\ &= -\frac{1}{2r}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right)\,E_\theta, \\ R\,(E_\varphi,\, E_r)\,E_r &= -\frac{1}{r\,\sin\theta}\,E_r[E_r[r\,\sin\theta]]\,E_\varphi \\ &= -\frac{1}{r\kappa(r)}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)}\right)\,E_\varphi. \\ &= -\frac{1}{2r}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right)\,E_\varphi.\end{aligned}$$

(d) *Show (e.g., by direct calculation using the results of (a) and (b) that*

$$
\begin{aligned}
R(E_\theta, E_\varphi)E_\theta &= \left(\frac{1}{r^2\kappa(r)^2} - \frac{1}{r^2}\right) E_\varphi \\
R(E_\theta, E_\varphi)E_\varphi &= \left(\frac{1}{r^2} - \frac{1}{r^2\kappa(r)^2}\right) E_\theta \\
R(E_t, E_\theta)E_\theta &= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\frac{d}{dr}\Big(\mu(r)^2\Big) E_t \\
R(E_t, E_\varphi)E_\varphi &= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\frac{d}{dr}\Big(\mu(r)^2\Big) E_t \\
R(E_t, E_\theta)E_t &= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\frac{d}{dr}\Big(\mu(r)^2\Big) E_\theta \\
R(E_t, E_\varphi)E_t &= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\frac{d}{dr}\Big(\mu(r)^2\Big) E_\varphi
\end{aligned}
$$

$$
\begin{aligned}
R(E_\theta, E_\varphi)E_\theta &= \nabla_{E_\theta}\nabla_{E_\varphi}E_\theta - \nabla_{E_\varphi}\nabla_{E_\theta}E_\theta - \nabla_{[E_\theta,E_\varphi]}E_\theta \\
&= \nabla_{E_\theta}\left(\frac{1}{r\tan\theta}\, E_\varphi\right) + \nabla_{E_\varphi}\left(\frac{1}{r\kappa(r)}\, E_r\right) \\
&\qquad + \frac{1}{r\tan\theta}\,\nabla_{E_\varphi}E_\theta \\
&= E_\theta\left[\frac{1}{r\tan\theta}\right] E_\varphi \\
&\quad + \frac{1}{r^2\kappa(r)^2}E_\varphi + \frac{1}{r^2\tan^2\theta}\, E_\varphi \\
&= \left(-\frac{1}{r^2\sin^2\theta} + \frac{1}{r^2\kappa(r)^2} + \frac{1}{r^2\tan^2\theta}\right) E_\varphi \\
&= \left(\frac{1}{r^2\kappa(r)^2} - \frac{1}{r^2}\right) E_\varphi \\
R(E_\theta, E_\varphi)E_\varphi &= \nabla_{E_\theta}\nabla_{E_\varphi}E_\varphi - \nabla_{E_\varphi}\nabla_{E_\theta}E_\varphi - \nabla_{[E_\theta,E_\varphi]}E_\varphi \\
&= -\nabla_{E_\theta}\left(\frac{1}{r\kappa(r)}\, E_r + \frac{1}{r\tan\theta}\, E_\theta\right) + \frac{1}{r\tan\theta}\,\nabla_{E_\varphi}E_\varphi \\
&= \frac{1}{r^2\sin^2\theta}E_\theta - \frac{1}{r\kappa(r)}\,\nabla_{E_\theta}E_r - \frac{1}{r\tan\theta}\,\nabla_{E_\theta}E_\theta \\
&\qquad + \frac{1}{r\tan\theta}\,\nabla_{E_\varphi}E_\varphi
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{r^2\sin^2\theta} - \frac{1}{r^2\kappa(r)^2} - \frac{1}{r^2\tan^2\theta}\right) E_\theta \\
&= \left(\frac{1}{r^2} - \frac{1}{r^2\kappa(r)^2}\right) E_\theta
\end{aligned}
$$

$$
\begin{aligned}
R(E_t, E_\theta)E_\theta &= \nabla_{E_t}\nabla_{E_\theta}E_\theta - \nabla_{E_\theta}\nabla_{E_t}E_\theta - \nabla_{[E_t,E_\theta]}E_\theta \\
&= -\nabla_{E_t}\left(\frac{1}{r\kappa(r)}\, E_r\right) \\
&= -\frac{1}{r\kappa(r)^2\mu(r)}\,\frac{d\mu(r)}{dr}E_t \\
&= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\,\frac{d}{dr}\left(\mu(r)^2\right) E_t \\
R(E_t, E_\varphi)E_\varphi &= \nabla_{E_t}\nabla_{E_\varphi}E_\varphi - \nabla_{E_\varphi}\nabla_{E_t}E_\varphi - \nabla_{[E_t,E_\varphi]}E_\varphi \\
&= -\nabla_{E_t}\left(\frac{1}{r\kappa(r)}\, E_r - \frac{1}{r\tan\theta}E_\theta\right) \\
&= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\,\frac{d}{dr}\left(\mu(r)^2\right) E_t \\
R(E_t, E_\theta)E_t &= \nabla_{E_t}\nabla_{E_\theta}E_t - \nabla_{E_\theta}\nabla_{E_t}E_t - \nabla_{[E_t,E_\theta]}E_t \\
&= -\nabla_{E_\theta}\left(\frac{1}{\kappa(r)\mu(r)}\,\frac{d\mu(r)}{dr}\, E_r\right) \\
&= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\,\frac{d}{dr}\left(\mu(r)^2\right) E_\theta \\
R(E_t, E_\varphi)E_t &= \nabla_{E_t}\nabla_{E_\varphi}E_t - \nabla_{E_\varphi}\nabla_{E_t}E_t - \nabla_{[E_t,E_\varphi]}E_t \\
&= -\nabla_{E_\varphi}\left(\frac{1}{\kappa(r)\mu(r)}\,\frac{d\mu(r)}{dr}\, E_r\right) \\
&= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\,\frac{d}{dr}\left(\mu(r)^2\right) E_\varphi
\end{aligned}
$$

Note that general properties of the Riemann curvature tensor ensure that

$$
\begin{aligned}
g(E_\varphi, R(E_\theta, E_\varphi)E_\theta) + g(E_\theta, R(E_\theta, E_\varphi)E_\varphi) &= 0. \\
g(E_t, R(E_t, E_\theta)E_\theta) + g(E_\theta, R(E_t, E_\theta)E_t) &= 0. \\
g(E_t, R(E_t, E_\varphi)E_\varphi) + g(E_\varphi, R(E_t, E_\varphi)E_t) &= 0.
\end{aligned}
$$

(see Proposition 9.7, property (iii)). Moreover the metric tensor g is invariant under rotations about an plane of symmetry along which the coordinate functions θ and φ are constant. A rotation through an angle of $\frac{\pi}{2}$ about a point p on this plane of symmetry fixes the tangent vectors $(E_t)_p$ and $(E_r)_p$ at that point, whilst it sends $(E_\theta)_p$ and $(E_\varphi)_p$ to $(E_\varphi)_p$ and $-(E_\theta)_p$ respectively. It follows that

$$g(E_t, R(E_t, E_\theta)E_\theta) = g(E_t, R(E_t, E_\varphi)E_\varphi).$$

Therefore

$$R(E_\theta, E_\varphi, E_\theta, E_\varphi) = -R(E_\varphi, E_\theta, E_\theta, E_\varphi)$$

and

$$\begin{aligned} R(E_t, E_\theta, E_t, E_\theta) &= R(E_t, E_\varphi, E_t, E_\varphi) = -R(E_\theta, E_t, E_t, E_\theta) \\ &= -R(E_\varphi, E_t, E_t, E_\varphi). \end{aligned}$$

Thus symmetry considerations together with basic properties of the Riemann curvature tensor ensure that these six components of the Riemann curvature tensor are determined by the values of

$$R(E_\theta, E_\varphi, E_\theta, E_\varphi) \quad \text{and} \quad R(E_t, E_\theta, E_t, E_\theta).$$

8. **(The Schwarzschild and Reissner-Nordström Metrics)** *Let the four-dimensional pseudo-Riemannian manifold M, the smooth real-valued functions κ and μ, the metric tensor g, the smooth local coordinates t, r, θ and φ and the smooth vector fields E_t, E_r, E_θ and E_φ be defined as in problem 7, so that*

$$g = -\mu(r)^2\, dt^2 + \kappa(r)^2\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2,$$

and

$$E_t = \frac{1}{\mu(r)}\frac{\partial}{\partial t}, \quad E_r = \frac{1}{\kappa(r)}\frac{\partial}{\partial r}, \quad E_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad E_\varphi = \frac{1}{r\,\sin\theta}\frac{\partial}{\partial \varphi}.$$

The Ricci tensor *of M is the covariant tensor* Ric *of degree 2 on M defined such that, given $X_p, Y_p \in T_pM$, the quantity* $\mathrm{Ric}(X_p, Y_p)$ *is the trace of the linear operator on T_pM that sends $V_p \in T_pM$ to $R(V_p, Y_p)X_p$.*

(a) *Explain why*

$$\begin{aligned} \mathrm{Ric}(X, Y) &= -R(E_t, X, E_t, Y) + R(E_r, X, E_r, Y) \\ &\quad + R(E_\theta, X, E_\theta, Y) + R(E_\varphi, X, E_\varphi, Y) \end{aligned}$$

Let $X_p, Y_p, V_p \in T_pM$. Then Let X, Y and V be vector fields on M. Then

$$\begin{aligned} R(V,Y)X &= -R(E_t, X, V, Y)\, E_t + R(E_r, X, V, Y)\, E_r \\ &\quad + R(E_\theta, X, V, Y)\, E_\theta + R(E_\varphi, X, V, Y)\, E_\varphi, \end{aligned}$$

because the vector fields E_t, E_r, E_θ and E_φ are orthogonal and satisfy

$$g(E_t, E_t) = -1, \quad g(E_r, E_r) = g(E_\theta, E_\theta) = g(E_\varphi, E_\varphi) = 1.$$

It follows that the diagonal elements of the matrix representing the linear operator $V_p \mapsto R(V_pY)X$ with respect to the basis $E_t, E_r, E_\theta, E_\varphi$ are

$$\begin{gathered} -R(E_t, X, E_t, Y), \quad R(E_r, X, E_r, Y), \\ R(E_\theta, X, E_\theta, Y), \quad R(E_\varphi, X, E_\varphi, Y), \end{gathered}$$

and therefore the trace of the matrix representing the linear operator $V_p \mapsto \mathrm{Ric}(V_p, Y)X$ is the sum of these four real numbers, as required.

(b) *Using the results of problem 7, show that*

$$\begin{aligned} \mathrm{Ric}(E_t, E_t) &= \frac{1}{2\kappa(r)\mu(r)}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)\mu(r)}\,\frac{d}{dr}\Bigl(\mu(r)^2\Bigr)\right) \\ &\quad + \frac{1}{r\kappa(r)^2\mu(r)^2}\,\frac{d}{dr}\Bigl(\mu(r)^2\Bigr), \\ \mathrm{Ric}(E_r, E_r) &= -\frac{1}{2\kappa(r)\mu(r)}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)\mu(r)}\,\frac{d}{dr}\Bigl(\mu(r)^2\Bigr)\right) \\ &\quad - \frac{1}{r}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right), \\ \mathrm{Ric}(E_\theta, E_\theta) &= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\,\frac{d}{dr}\Bigl(\mu(r)^2\Bigr) \\ &\quad - \frac{1}{2r}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right) \\ &\quad + \frac{1}{r^2} - \frac{1}{r^2\kappa(r)^2}, \\ \mathrm{Ric}(E_\varphi, E_\varphi) &= -\frac{1}{2r\kappa(r)^2\mu(r)^2}\,\frac{d}{dr}\Bigl(\mu(r)^2\Bigr) \\ &\quad - \frac{1}{2r}\,\frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right) \\ &\quad + \frac{1}{r^2} - \frac{1}{r^2\kappa(r)^2}, \end{aligned}$$

$$
\begin{aligned}
\operatorname{Ric}(E_t, E_r) &= \operatorname{Ric}(E_t, E_\theta) = \operatorname{Ric}(E_t, E_\varphi) = \operatorname{Ric}(E_r, E_\theta) \\
&= \operatorname{Ric}(E_r, E_\varphi) = \operatorname{Ric}(E_\theta, E_\varphi) \\
&= 0.
\end{aligned}
$$

The properties of the Riemann curvature tensor ensure that

$$R(E_t, E_t, E_t, E_t) = 0.$$

It therefore follows from problem 7, parts (b) and (d), that

$$
\begin{aligned}
\operatorname{Ric}(E_t, E_t) &= R(E_r, E_t, E_r, E_t) + R(E_\theta, E_t, E_\theta, E_t) \\
&\quad + R(E_\varphi, E_t, E_\varphi, E_t) \\
&= \frac{1}{2\kappa(r)\mu(r)} \frac{d}{dr}\left(\frac{1}{\kappa(r)\mu(r)} \frac{d}{dr}\left(\mu(r)^2\right)\right) \\
&\quad + \frac{1}{r\kappa(r)^2\mu(r)^2} \frac{d}{dr}\left(\mu(r)^2\right), \\
\operatorname{Ric}(E_r, E_r) &= -R(E_t, E_r, E_t, E_r) + R(E_\theta, E_r, E_\theta, E_r) \\
&\quad + R(E_\varphi, E_r, E_\varphi, E_r) \\
&= -\frac{1}{2\kappa(r)\mu(r)} \frac{d}{dr}\left(\frac{1}{\kappa(r)\mu(r)} \frac{d}{dr}\left(\mu(r)^2\right)\right) \\
&\quad - \frac{1}{r} \frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right), \\
\operatorname{Ric}(E_\theta, E_\theta) &= -R(E_t, E_\theta, E_t, E_\theta) + R(E_r, E_\theta, E_r, E_\theta) \\
&\quad + R(E_\varphi, E_\theta, E_\varphi, E_\theta) \\
&= -\frac{1}{2r\kappa(r)^2\mu(r)^2} \frac{d}{dr}\left(\mu(r)^2\right) - \frac{1}{2r} \frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right) \\
&\quad + \frac{1}{r^2} - \frac{1}{r^2\kappa(r)^2}, \\
\operatorname{Ric}(E_\varphi, E_\varphi) &= -R(E_t, E_\varphi, E_t, E_\varphi) + R(E_r, E_\varphi, E_r, E_\varphi) \\
&\quad + R(E_\theta, E_\varphi, E_\theta, E_\varphi) \\
&= -\frac{1}{2r\kappa(r)^2\mu(r)^2} \frac{d}{dr}\left(\mu(r)^2\right) - \frac{1}{2r} \frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right) \\
&\quad + \frac{1}{r^2} - \frac{1}{r^2\kappa(r)^2}.
\end{aligned}
$$

Also it follows from problem 7, part (c), that if X and Y are distinct vectors from the list $E_t, E_r, E_\theta, E_\varphi$ then $\operatorname{Ric}(X, Y) = 0$.

(c) *Verify that*

$$\mathrm{Ric}(E_t, E_t) = -\mathrm{Ric}(E_r, E_r) = \mathrm{Ric}(E_\theta, E_\theta) = \mathrm{Ric}(E_\varphi, E_\varphi) = \frac{r_Q^2}{r^4}$$

in the special case where

$$\frac{\mu(r)^2}{c^2} = \frac{1}{\kappa(r)^2} = 1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}$$

for some positive real constants c and r_S, and thus

$$\begin{aligned} g &= -c^2 \left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right) dt^2 + \frac{1}{1 - \dfrac{r_S}{r} + \dfrac{r_Q^2}{r^2}}\, dr^2 \\ &\qquad + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2. \end{aligned}$$

If the functions μ and κ determining the metric tensor satisfy the equation $\mu(r)\kappa(r) = c$, where c is some positive constant, then the equations for the components of the Ricci curvature reduce to the following equations:—

$$\begin{aligned} \mathrm{Ric}(E_t, E_t) &= \frac{1}{2c^2}\frac{d^2}{dr^2}\Bigl(\mu(r)^2\Bigr) + \frac{1}{c^2 r}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr), \\ \mathrm{Ric}(E_r, E_r) &= -\frac{1}{2c^2}\frac{d^2}{dr^2}\Bigl(\mu(r)^2\Bigr) - \frac{1}{c^2 r}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr), \\ &= -\mathrm{Ric}(E_t, E_t) \\ \mathrm{Ric}(E_\theta, E_\theta) &= -\frac{1}{c^2 r}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr) + \frac{1}{r^2}\left(1 - \frac{\mu(r)^2}{c^2}\right), \\ \mathrm{Ric}(E_\varphi, E_\varphi) &= \mathrm{Ric}(E_\theta, E_\theta) \end{aligned}$$

Thus if

$$\mu(r)^2 = \frac{c^2}{\kappa(r)^2} = c^2\left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right)$$

then

$$\begin{aligned} \frac{1}{2c^2}\frac{d^2}{dr^2}\Bigl(\mu(r)^2\Bigr) &= -\frac{r_S}{r^3} + \frac{3r_Q^2}{r^4}, \\ \frac{1}{c^2 r}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr) &= \frac{r_S}{r^3} - \frac{2r_Q^2}{r^4}, \\ \frac{1}{r^2}\left(1 - \frac{\mu(r)^2}{c^2}\right) &= \frac{r_S}{r^3} - \frac{r_Q^2}{r^4}, \end{aligned}$$

and therefore

$$\mathrm{Ric}(E_t, E_t) = -\mathrm{Ric}(E_r, E_r) = \mathrm{Ric}(E_\theta, E_\theta) = \mathrm{Ric}(E_\varphi, E_\varphi) = \frac{r_Q^2}{r^4},$$

as required.

(d) *Suppose that the Ricci curvature of the metric tensor g satisfies the identities*

$$\mathrm{Ric}(E_t, E_t) = -\mathrm{Ric}(E_r, E_r) = \mathrm{Ric}(E_\theta, E_\theta) = \mathrm{Ric}(E_\varphi, E_\varphi).$$

Prove that the functions μ and κ determining the metric tensor satisfy the identities

$$\frac{\mu(r)^2}{c^2} = \frac{1}{\kappa(r)^2} = 1 - \frac{r_S}{r} + \frac{A}{r^2}$$

for some constants c, r_S and A.

It follows from (b) that

$$\begin{aligned}
&\mathrm{Ric}(E_t, E_t) + \mathrm{Ric}(E_r, E_r) \\
&\quad = \frac{1}{r\kappa(r)^2\mu(r)^2}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr) - \frac{1}{r}\frac{d}{dr}\left(\frac{1}{\kappa(r)^2}\right) \\
&\quad = \frac{1}{r\kappa(r)^2\mu(r)^2}\left(\frac{1}{\mu(r)^2}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr) + \frac{1}{\kappa(r)^2}\frac{d}{dr}\Bigl(\kappa(r)^2\Bigr)\right) \\
&\quad = \frac{1}{r\kappa(r)^2\mu(r)^2}\frac{d}{dr}\Bigl(\log(\kappa(r)^2\mu(r)^2)\Bigr).
\end{aligned}$$

Thus if $\mathrm{Ric}(E_r, E_r) = -\mathrm{Ric}(E_t, E_t)$ then $\kappa(r)\mu(r) = c$ for some real constant c. But then

$$\begin{aligned}
\mathrm{Ric}(E_t, E_t) &= -\mathrm{Ric}(E_r, E_r) = \frac{1}{2c^2}\frac{d^2}{dr^2}\Bigl(\mu(r)^2\Bigr) + \frac{1}{c^2 r}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr), \\
\mathrm{Ric}(E_\theta, E_\theta) &= \mathrm{Ric}(E_\varphi, E_\varphi) = -\frac{1}{c^2 r}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr) + \frac{1}{r^2}\left(1 - \frac{\mu(r)^2}{c^2}\right).
\end{aligned}$$

Thus if

$$\mathrm{Ric}(E_t, E_t) = -\mathrm{Ric}(E_r, E_r) = \mathrm{Ric}(E_\theta, E_\theta) = \mathrm{Ric}(E_\varphi, E_\varphi).$$

then

$$\frac{1}{2c^2}\frac{d^2}{dr^2}\Bigl(\mu(r)^2\Bigr) + \frac{1}{c^2 r}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr) = -\frac{1}{c^2 r}\frac{d}{dr}\Bigl(\mu(r)^2\Bigr) + \frac{1}{r^2}\left(1 - \frac{\mu(r)^2}{c^2}\right),$$

and therefore

$$r^2 \frac{d^2}{dr^2}\Bigl(\mu(r)^2\Bigr) + 4r \frac{d}{dr}\Bigl(\mu(r)^2\Bigr) + 2\mu(r)^2 - 2c^2 = 0.$$

On solving this differential equation for the function $\mu(r)^2 - c^2$ we find that

$$\mu(r)^2 - c^2 = -\frac{c^2 r_S}{r} + \frac{c^2 A}{r^2}.$$

where c, r_S and A are real constants, and thus

$$\mu(r)^2 = c^2 \left(1 - \frac{r_S}{r} + \frac{A}{r^2}\right),$$

as required.

Remark Einstein published his Theory of General Relativity in 1915. A month later Karl Schwarzschild found the first non-flat exact solution to the Einstein field equations. It describes the geometry of space-time away from a stationary non-rotating uncharged star or black hole. The Schwarzschild metric takes the form

$$g = -c^2 \left(1 - \frac{r_S}{r}\right) dt^2 + \frac{1}{1 - \dfrac{r_S}{r}}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2.$$

The calculations of problem 8 establish that this metric has zero Ricci tensor. This is the basic requirement for satisfying the Einstein field equations in the absence of matter and energy.

The metric that describes the geometry of spacetime around a charged non-rotating spherically symmetric body was obtained by Hans Reissner in 1916 and Gunnar Nordström in 1918. Suppose that there is no magnetic field and that the electric field points radially outwards. Let e denote the electric field strength. It can be shown that if $E_t, E_r, E_\theta, E_\varphi$ is a Lorenzian moving frame, where E_t is pointed in the time direction and E_r is pointed in the radial direction, then the Ricci curvature of the metric tensor must satisfy

$$\mathrm{Ric}(E_t, E_t) = ke^2, \quad \mathrm{Ric}(E_r, E_r) = -ke^2,$$
$$\mathrm{Ric}(E_\theta, E_\theta) = ke^2, \quad \mathrm{Ric}(E_\varphi, E_\varphi) = ke^2,$$

where k is an appropriate physical constant. It then follows from the results of problem 8 that the metric tensor takes the form

$$\begin{aligned} g \;=\; & -c^2 \left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right) dt^2 + \frac{1}{1 - \dfrac{r_S}{r} + \dfrac{r_Q^2}{r^2}}\, dr^2 \\ & + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2, \end{aligned}$$

obtained by Reissner and Nordström. Moreover the electric field strength e is proportional to r^{-2}. This requirement that the electric field strength be proportional to r^{-2} is what one would expect on generalizing Maxwell's equations to curved spacetimes, since it ensures that the surface integral of the electric field over the sphere of area $4\pi r^2$ over which the coordinates t and r are constant is independent of the parameter r that determines the intrinsic curvature and area of the sphere, and this is what one would expect if the surface integral of the electric field over the sphere is to be proportional to the charge enclosed within the sphere.