Module MA3429: Differential Geometry
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Part III: Sections 8 to 11

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Contents

8 Connections, Curvature, and Torsion

8.1 Connections on Smooth Vector Bundles

Definition Let $\pi_E \colon E \to M$ be a smooth vector bundle over a smooth manifold M. A smooth *connection* on M is a differential operator D which, at each point p of M, associates to each smooth section s of the vector bundle defined around the point p and to each tangent vector X_p at p an element element $D_{X_p} s$ of the fibre E_p of the vector bundle over p, where this differential operator has the following properties:—

(i) $D_{W_p + X_p} s = D_{W_p} s + D_{X_p} s$
for all tangent vectors W_p and X_p at a point p of M, and for all smooth sections s of the vector bundle defined around p;

(iii) $D_{c X_p} s = c\, D_{X_p} s$
for all real numbers c and tangent vectors X_p at a point p of M, and for all smooth sections s of the vector bundle defined around p;

(iii) $D_{X_p}(s + t) = D_{X_p} s + D_{X_p} t$
for all tangent vectors X_p at a point p of M, and for all smooth sections s and t of the vector bundle defined around p;

(iv) $D_X(f\, s) = X[f]\, s + f\, D_X s$
for all tangent vectors X_p at a point p of M, for all smooth real-valued functions f defined around p, and for all smooth sections s of the vector bundle defined around p;

(v) given a smooth vector fields X defined over an open subset U of M, and given a smooth section $s \colon U \to E$ of the vector bundle $\pi_E \colon E \to M$ defined over U, the function that sends points p of U to $D_{X_p} s$ is itself a smooth section of the vector bundle defined over U.

The element $D_{X_p} s$ of the fibre E_p of the vector bundle at a point p of the manifold determined by a tangent vector X_p at p and a smooth section s of the vector bundle defined around p is referred to as the *covariant derivative* of the section s along the tangent vector X_p (with respect to the smooth connection D).

Example Let M be a smooth n-dimensional submanifold of k-dimensional Euclidean space $\mathbb{R}^k$, and let $\pi \colon M \times \mathbb{R}^k \to M$ be the product bundle over M with fibre $\mathbb{R}^k$. The tangent space $T_p M$ at each point p of M can be identified with a vector subspace of $\mathbb{R}^k$. Let

$$TM = \{(p, \mathbf{v}) \in M \times \mathbb{R} : \mathbf{v} \in T_p M\}.$$

and let $\pi_{TM}: TM \to M$ be the function defined such that $\pi_{TM}(p, \mathbf{v}) = p$ for all $(p, \mathbf{v}) \in TM$. Then TM is a smooth submanifold of $M \times \mathbb{R}^k$. $\pi_{TM}: TM \to M$ is a smooth map, and moreover $\pi_{TM}: TM \to M$ is the projection map of a smooth vector bundle over M with total space TM which is a subbundle of the product bundle $\pi: M \times \mathbb{R}^k \to M$. This smooth vector bundle $\pi_{TM}: TM \to M$ can be identified with the tangent bundle of the smooth manifold M.

Let N_pM denote the orthogonal complement of T_pM in $\mathbb{R}^k$, and let

$$NM = \{(p, \mathbf{v}) \in M \times \mathbb{R} : \mathbf{v} \in N_pM\}.$$

and let $\pi_{NM}: NM: M$ be the function defined such that $\pi_{NM}(p, \mathbf{v}) = p$ for all $(p, \mathbf{v}) \in NM$. Then NM is also a smooth submanifold of $M \times \mathbb{R}^k$. $\pi_{NM}: NM \to M$ is a smooth map, and moreover $\pi_{NM}: NM \to M$ is the projection map of a smooth vector bundle over M with total space NM which is also a subbundle of the product bundle $\pi: M \times \mathbb{R}^k \to M$. This smooth vector bundle $\pi_{NM}: NM \to M$ is referred to as the *normal bundle* of the smooth submanifold M of $\mathbb{R}^k$. Its fibre at a point p of M is the vector space consisting of all vectors $\mathbf{v}$ in the ambient Euclidean space $\mathbb{R}^k$ that are orthogonal to the smooth submanifold M. A section of this bundle is a *normal vector field* defined on the submanifold M of $\mathbb{R}^k$. The product bundle over M with fibre $\mathbb{R}^k$ is then the direct sum $TM \oplus NM$ of the tangent bundle TM and the normal bundle NM of M.

Let $\mathbf{V}: M \to \mathbf{R}^k$ be a smooth function from M to $\mathbf{R}^k$. Then there are smooth real-valued functions $V^1, V^2, \ldots, V^k$ on M such that

$$\mathbf{V}(p) = (V^1(p), V^2(p), \ldots, V^k(p))$$

for all $p \in M$. Given any smooth (tangential) vector field $\mathbf{X}$ on M, we define

$$\partial_{\mathbf{X}}\mathbf{V} = (\mathbf{X}[V^1], \mathbf{X}[V^2], \ldots, \mathbf{X}[V^k])$$

for all $p \in M$, where $\mathbf{X}[V^j]$ denotes the smooth real-valued function on M whose value at any point p of M is the directional derivative $\mathbf{X}_p[V^j]$ of the smooth real-valued function V^j along the tangent vector $\mathbf{X}_p$ at p for $j = 1, 2, \ldots, k$. Thus if $y^1, y^2, \ldots, y^n$ are smooth local coordinates defined over some open subset U of M, and if $X = \sum_{i=1}^{n} u^i \frac{\partial}{\partial y^i}$, then

$$\mathbf{X}[V^j] = \sum_{i=1}^{n} u^i \frac{\partial V^j}{\partial y^i}$$

over the open set U. It is easy to verify that the differential operator ∂ defined as above represents a smooth connection on the product bundle $\pi: M \times \mathbb{R}^k \to$

M. Indeed

$$\partial_{\mathbf{X}}(\mathbf{V}+\mathbf{W}) = \partial_{\mathbf{X}}\mathbf{V} + \partial_{\mathbf{X}}\mathbf{W}, \quad \partial_{\mathbf{X}+\mathbf{Y}}\mathbf{V} = \partial_{\mathbf{X}}\mathbf{V} + \partial_{\mathbf{Y}}\mathbf{V}$$

and

$$\partial_{f\,\mathbf{X}}\mathbf{V} = f\,\partial_{\mathbf{X}}\mathbf{V}$$

for all smooth real-valued functions f and smooth vector fields $\mathbf{X}$ and $\mathbf{Y}$ on M, and for all smooth functions $\mathbf{V}$ and $\mathbf{W}$ from M to $\mathbb{R}^k$. Also

$$\begin{aligned}\partial_{\mathbf{X}}(f\mathbf{V}) &= (\mathbf{X}[f\cdot V^1], \ldots, \mathbf{X}[f\cdot V^k]) \\ &= (\mathbf{X}[f]\cdot V^1 + f\cdot\mathbf{X}[V^1], \ldots, \mathbf{X}[f]\cdot V^k + f\cdot\mathbf{X}[V^k]) \\ &= \mathbf{X}[f](V^1, \ldots, V^k) + f\,(\mathbf{X}[V^1], \ldots, \mathbf{X}[V^k]) \\ &= \mathbf{X}[f]\mathbf{V} + f\,\partial_{\mathbf{X}}\mathbf{V}\end{aligned}$$

where

$$\mathbf{V}(p) = (V^1(p), \ldots, V^k(p))$$

for all $p \in M$.

Let $\mathbf{V}\colon M \to TM$ and $\mathbf{X}\colon M \to TM$ be smooth sections of the tangent bundle of M representing smooth tangential vector fields on M, and let $\mathbf{Q}\colon M \to TN$ be a smooth section of the normal bundle of M. These sections $\mathbf{V}$, $\mathbf{X}$ and $\mathbf{Q}$ are then smooth sections of the product bundle over M with fibre $\mathbb{R}^k$, and therefore so are $\partial_{\mathbf{X}}\mathbf{V}$ and $\partial_{\mathbf{X}}\mathbf{Q}$. However $\partial_{\mathbf{X}_p}\mathbf{V}$ does not in general belong to the tangent space T_pM at the point p, and $\partial_{\mathbf{X}_p}\mathbf{Q}$ does not in general belong to the normal bundle at the point p. We can however decompose the smooth sections of the product bundle $\pi\colon M \times \mathbb{R}^k \to M$ represented by $\partial_{\mathbf{X}}\mathbf{V}$ and $\partial_{\mathbf{X}}\mathbf{V}$ into their tangential and normal components, so that

$$\partial_{\mathbf{X}}\mathbf{V} = \nabla_{\mathbf{X}}\mathbf{V} - S(\mathbf{X},\mathbf{V}), \quad \partial_{\mathbf{X}}\mathbf{Q} = D_{\mathbf{X}}\mathbf{Q} + \check{S}(\mathbf{X},\mathbf{Q}),$$

where

$$(\nabla_{\mathbf{X}}\mathbf{V})_p \in T_pM, \quad S(\mathbf{X},\mathbf{V})_p \in N_pM,$$

$$(D_{\mathbf{X}}\mathbf{Q})_p \in N_pM, \quad \text{and} \quad \check{S}(\mathbf{X},\mathbf{Q})_p \in T_pM$$

for all $p \in M$. Then the tangential components $\nabla_{\mathbf{X}}\mathbf{V}$ and $\check{S}(\mathbf{X},\mathbf{Q})$ are smooth sections of the tangent bundle $\pi_{TM}\colon TM \to M$ of M and the normal components $D_{\mathbf{X}}\mathbf{Q}$ and $S(\mathbf{X},\mathbf{V})$ are smooth sections of the normal bundle $\pi_{NM}\colon NM \to M$ of M. Moreover

$$\nabla_{\mathbf{X}}(\mathbf{V}+\mathbf{W}) = \nabla_{\mathbf{X}}\mathbf{V} + \nabla_{\mathbf{X}}\mathbf{W}, \quad \nabla_{\mathbf{X}+\mathbf{Y}}\mathbf{V} = \nabla_{\mathbf{X}}\mathbf{V} + \nabla_{\mathbf{Y}}\mathbf{V},$$

$$\nabla_{f\,\mathbf{X}}\mathbf{V} = f\,\nabla_{\mathbf{X}}\mathbf{V}, \quad \nabla_{\mathbf{X}}(f\mathbf{V}) = \mathbf{X}[f]\mathbf{V} + f\,\nabla_{\mathbf{X}}\mathbf{V},$$

$$S(\mathbf{X}, \mathbf{V} + \mathbf{W}) = S(\mathbf{X}, \mathbf{V}) + S(\mathbf{X}, \mathbf{W}),$$
$$S(\mathbf{X} + \mathbf{Y}, \mathbf{V}) = S(\mathbf{X}, \mathbf{V}) + S(\mathbf{Y}, \mathbf{V}),$$
$$S(f\,\mathbf{X}, \mathbf{V}) = S(\mathbf{X}, f\,\mathbf{V}) = f\,S(\mathbf{X}, \mathbf{V}),$$
$$D_{\mathbf{X}}(\mathbf{Q} + \mathbf{R}) = D_{\mathbf{X}}\mathbf{Q} + D_{\mathbf{X}}\mathbf{R}, \quad D_{\mathbf{X}+\mathbf{Y}}\mathbf{Q} = D_{\mathbf{X}}\mathbf{Q} + D_{\mathbf{Y}}\mathbf{Q},$$
$$D_{f\,\mathbf{X}}\mathbf{Q} = f\,D_{\mathbf{X}}\mathbf{Q}, \quad D_{\mathbf{X}}(f\,\mathbf{Q}) = \mathbf{X}[f]\mathbf{Q} + f\,D_{\mathbf{X}}\mathbf{Q},$$

and

$$\check{S}(\mathbf{X}, \mathbf{Q} + \mathbf{R}) = \check{S}(\mathbf{X}, \mathbf{Q}) + \check{S}(\mathbf{X}, \mathbf{R}),$$
$$\check{S}(\mathbf{X} + \mathbf{Y}, \mathbf{Q}) = \check{S}(\mathbf{X}, \mathbf{Q}) + \check{S}(\mathbf{Y}, \mathbf{Q}),$$
$$\check{S}(f\,\mathbf{X}, \mathbf{Q}) = \check{S}(\mathbf{X}, f\,\mathbf{Q}) = f\,\check{S}(\mathbf{X}, \mathbf{Q}),$$

for all smooth real-valued functions f on M, smooth tangential vector fields $\mathbf{V}$, $\mathbf{W}$, $\mathbf{X}$ and $\mathbf{Y}$ on M, and smooth normal vector fields $\mathbf{Q}$ and $\mathbf{R}$ on M. It follows that the differential operator ∇ is a smooth connection on the tangent bundle $\pi_{TM}: TM \to M$ of M. Moreover

$$\mathbf{X}[\mathbf{V}.\mathbf{W}] = (\nabla_{\mathbf{X}}\mathbf{V}).\mathbf{W} + \mathbf{V}.(\nabla_{\mathbf{X}}\mathbf{W})$$

for all smooth tangential vector fields X, V and W on M. Similarly the differential operator D is a smooth connection on the normal bundle $\pi_{NM}: NM \to M$ of M. Moreover

$$\mathbf{X}[\mathbf{Q}.\mathbf{R}] = (D_{\mathbf{X}}\mathbf{Q}).\mathbf{R} + \mathbf{Q}.(D_{\mathbf{X}}\mathbf{R})$$

for all smooth tangential vector fields X and normal vector fields Q and R on M.

If $\mathbf{V}$ is a smooth tangential vector field and $\mathbf{Q}$ is a smooth normal vector field then $\mathbf{V}$ and $\mathbf{Q}$ are orthogonal at each point of M, and therefore $\mathbf{V}.\mathbf{Q} = 0$. On differentiating this equation we find that

$$0 = \partial_{\mathbf{X}}\mathbf{V}.\mathbf{Q} + \mathbf{V}.\partial_{\mathbf{X}}\mathbf{Q} = -S(\mathbf{X}, \mathbf{V}).\mathbf{Q} + \mathbf{V}.\check{S}(\mathbf{X}, \mathbf{Q}),$$

and thus

$$\mathbf{V}.\check{S}(\mathbf{X}, \mathbf{Q}) = S(\mathbf{X}, \mathbf{V}).\mathbf{Q}$$

for all tangential vector fields $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{V}$ and normal vector fields $\mathbf{Q}$ on M.

An application of Proposition 6.15 shows that the value of $S(\mathbf{X}, \mathbf{V})$ at any point p of M is determined by the values of the tangential vector fields $\mathbf{X}$ and $\mathbf{V}$ at the point p, and therefore the operator S is determined by a smooth section of the smooth vector bundle $NM \otimes T^*M \otimes T^*M$ over M, where $\pi_{T^*M}: T^*M \to M$ is the cotangent bundle of M. Similarly the operator

$\check{S}$ is determined by a smooth section of $TM \otimes T^*M \otimes N^*M$ over M, where $\pi_{N^*M}: N^*M \to M$ is the dual of the normal bundle of M. The operator S is referred to as the *second fundamental tensor* of the submanifold M of $\mathbf{R^k}$.

The smooth connection ∇ on the tangent bundle of the submanifold M of $\mathbb{R}^k$ is the *Levi-Civita connection* of this submanifold. One can prove that it is determined by the inner product on the tangent spaces of the submanifold that is the restriction to these tangent spaces of the scalar product on $\mathbb{R}^k$. Indeed a straightforward if lengthy computation establishes that

$$\begin{aligned}(\nabla_{\mathbf{X}}\mathbf{Y}).\mathbf{Z} &= \frac{1}{2}\Big(\mathbf{X}[\mathbf{Y}.\mathbf{Z}] + \mathbf{Y}[\mathbf{X}.\mathbf{Z}] - \mathbf{Z}[\mathbf{X}.\mathbf{Y}] \\ &\qquad + [\mathbf{X},\mathbf{Y}].\mathbf{Z} - [\mathbf{X},\mathbf{Z}].\mathbf{Y} - [\mathbf{Y},\mathbf{Z}].\mathbf{X}\Big)\end{aligned}$$

for all tangential vector fields $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ on M. This identity is a special case of a more general identity applicable to Riemannian and pseudo-Riemannian manifolds.

Proposition 8.1 *Let D be a smooth connection on a smooth vector bundle $\pi_E: E \to M$ of rank r, and let U be an open subset in M which is contained in the domain of a smooth coordinate system $(x^1, x^2, \ldots, x^n)$ for M and over which are defined smooth sections $e_1, e_2, \ldots, e_r$ of the vector bundle whose values $e_1(p), e_2(p), \ldots, e_r(p)$ at each point p of E_p constitute a basis of the fibre E_p of this vector bundle over the point p. Let*

$$D_j e_\beta = D_{\frac{\partial}{\partial x^j}} e_\beta = \sum_{\alpha=1}^{r} A^\alpha{}_{\beta j} e_\alpha,$$

for $j = 1, 2, \ldots, n$ and $\beta = 1, 2, \ldots, r$, where each function $A^\alpha{}_{\beta j}$ is a smooth real-valued function on U. Let X be a smooth vector field on U, and let $s: U \to E$ be a smooth section of the vector bundle $\pi_E: E \to M$ defined over U, and let

$$X = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j} \quad \textit{and} \quad s = \sum_{\alpha=1}^{r} f^\alpha e_\alpha,$$

where $v^1, v^2, \ldots, v^n$ and $f^1, f^2, \ldots, f^r$ are smooth real-valued functions on U. Then

$$D_X s = \sum_{j=1}^{n} v^j D_j s,$$

where

$$D_j s = \sum_{\alpha=1}^{r} \left(\frac{\partial f^\alpha}{\partial x^j} + \sum_{\beta=1}^{r} A^\alpha{}_{\beta j} f^\beta \right) e_\alpha.$$

Proof Property (v) in the definition of smooth connections ensures that the functions $A^{\alpha\beta j}$ are smooth. It follows from properties (i) and (ii) in the definition of smooth connections that

$$D_X s = \sum_{j=1}^{n} v^j D_j s, \quad \text{where} \quad D_j s = D_{\frac{\partial}{\partial x^j}} s.$$

It then follows from properties (iii) and (iv) in the definition of smooth connections that

$$\begin{aligned} D_j s &= D_j \left(\sum_{\alpha=1}^{r} f^\alpha e_\alpha \right) = \sum_{\alpha=1}^{r} D_j (f^\alpha e_\alpha) \\ &= \sum_{\alpha=1}^{r} \frac{\partial f^\alpha}{\partial x^j} e_\alpha + \sum_{\beta=1}^{r} f^\beta D_j e_\beta \\ &= \sum_{\alpha=1}^{r} \left(\frac{\partial f^\alpha}{\partial x^j} + \sum_{\beta=1}^{r} A^\alpha{}_{\beta j} f^\beta \right) e_\alpha, \end{aligned}$$

as required. ∎

Corollary 8.2 *Let D be a smooth connection on a smooth vector bundle $\pi_E: E \to M$ of rank r, and let U be an open subset in M which is contained in the domain of a smooth coordinate system $(x^1, x^2, \ldots, x^n)$ for M and over which are defined smooth sections $e_1, e_2, \ldots, e_r$ and $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_r$, where the values of the sections $e_1, e_2, \ldots, e_r$ at each point p of U constitute a basis of the fibre E_p of the vector bundle over the point p, and where the values of the sections $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_r$ at each point p of U also constitute a basis of the fibre E_p. Let $A_j: U \to \mathrm{M}_r(\mathbb{R})$ and $\hat{A}_j: U \to \mathrm{M}_r(\mathbb{R})$ the smooth functions from the open set U to the algebra $\mathrm{M}_r(\mathbb{R})$ of $r \times r$ matrices with real coefficients whose values at $p \in U$ are the matrices whose entry in row α and column β are $A^\alpha{}_{\beta j}(p)$ and $\hat{A}^\alpha{}_{\beta j}(p)$ respectively, where*

$$D_j e_\beta = D_{\frac{\partial}{\partial x^j}} e_\beta = \sum_{\alpha=1}^{r} A^\alpha{}_{\beta j} e_\alpha,$$

for $j = 1, 2, \ldots, n$ and $\beta = 1, 2, \ldots, r$, and

$$D_j \hat{e}_\eta = D_{\frac{\partial}{\partial x^j}} \hat{e}_\eta = \sum_{\xi=1}^{r} \hat{A}^\xi{}_{\eta k} \hat{e}_\xi,$$

for $j = 1, 2, \ldots, n$ and $\eta = 1, 2, \ldots, r$. Also let $S: U \to \mathrm{GL}(k, \mathbb{R})$ the smooth function from U to the group $\mathrm{GL}(k, \mathbb{R})$ of non-singular $r \times r$ matrices with *real coefficients whose value at $p \in U$ is the non-singular matrix whose value in row α and column ξ is $S^\alpha{}_\xi(p)$, where*

$$\hat{e}_\xi(p) = \sum_{\alpha=1}^r S^\alpha{}_\xi(p) e_\alpha(p).$$

Then

$$\hat{A}_j = S^{-1} A_j S + S^{-1} \frac{\partial S}{\partial x^j}.$$

Proof It follows from Proposition 8.1 that

$$D_j \hat{e}_\eta = \sum_{\alpha=1}^r \left(\sum_{\beta=1}^r A^\alpha{}_{\beta j} S^\beta{}_\eta + \frac{\partial S^\alpha{}_\eta}{\partial x^j} \right) e_\alpha.$$

Now

$$\sum_{\xi=1}^r (S^{-1})^\xi{}_\alpha \hat{e}_\xi(p) = \sum_{\xi,\beta=1}^r (S^{-1})^\xi{}_\alpha S^\beta_\xi e_\beta(p) = \sum_{\beta=1}^r \delta^\beta_\alpha e_\beta(p) = e_\alpha(p),$$

where $(S^{-1})^\eta_\alpha$ is the function whose value at any point p of U is the entry in row η and column α of the matrix $S^{-1}(p)$ that is the inverse of $S(p)$, and where δ^β_α denotes the Kronecker delta that is equal to 1 when $\alpha = \beta$ but is equal to zero otherwise. It follows that

$$D_j \hat{e}_\eta = \sum_{\alpha,\xi=1}^r (S^{-1})^\xi{}_\alpha \left(\sum_{\beta=1}^r A^\alpha{}_{\beta j} S^\beta{}_\eta + \frac{\partial S^\alpha{}_\eta}{\partial x^j} \right) \hat{e}_\xi = \sum_{\xi=1}^r \hat{A}^\xi{}_{\eta j} \hat{e}_\xi,$$

where

$$\hat{A}^\xi{}_{\eta j} = \sum_{\alpha,\beta=1}^r (S^{-1})^\xi{}_\alpha A^\alpha{}_{\beta j} S^\beta{}_\eta + \sum_{\alpha=1}^r (S^{-1})^\xi{}_\alpha \frac{\partial S^\alpha{}_\eta}{\partial x^j}.$$

Thus

$$\hat{A}_j = S^{-1} A_j S + S^{-1} \frac{\partial S}{\partial x^j},$$

as required. ∎

Corollary 8.3 *Let D be a smooth connection on a smooth vector bundle $\pi_E: E \to M$ of rank r, and let U be an open subset in M over which are defined smooth sections $e_1, e_2, \ldots, e_r$ of the vector bundle $\pi_E: E \to M$ whose values at each point p of E_p constitute a basis of the fibre E_p of this vector* *bundle over the point p. Then there exist smooth 1-forms $\omega^\alpha{}_\beta$ on U for $\alpha, \beta = 1, 2, \ldots, r$ such that*

$$D_{X_p} e_\beta = \sum_{\alpha=1}^{r} \omega^\alpha{}_\beta(X_p) e_\alpha(p)$$

for $\beta = 1, 2, \ldots, r$ and for all tangent vectors X_p at points p of U. If $s\colon U \to E$ is a smooth section of $\pi_E\colon E \to M$ defined over U, and if

$$s = \sum_{\alpha=1}^{n} f^\alpha e_\alpha,$$

where $f^1, f^2, \ldots, f^r$ are smooth real-valued functions on U, then

$$D_{X_p} s = \sum_{\alpha=1}^{r} \left\langle df^\alpha + \sum_{\beta=1}^{r} \omega^\alpha{}_\beta f^\beta, X_p \right\rangle e_\alpha,$$

for all tangent vectors X_p at points p of U. Moreover if $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_r$ are smooth sections of the vector bundle $\pi_E\colon E \to M$ over U whose values at each point p of U also constitute a basis of the fibre E_p of the vector bundle over p, if $\hat{\omega}^\xi{}_\eta$ are smooth 1-forms on U defined such that

$$D_X \hat{e}_\nu = \sum_{\xi=1}^{r} \hat{\omega}^\xi{}_\eta(X) \hat{e}_\xi$$

and if

$$\hat{e}_\xi(p) = \sum_{\alpha=1}^{r} S^\alpha{}_\xi(p) e_\alpha(p),$$

where the values of the smooth real-valued functions $S^\alpha{}_\xi$ at each point p of U are the components of a non-singular $r \times r$ matrix $S(p)$, then

$$\hat{\omega}^\xi{}_\eta = \sum_{\alpha,\beta=1}^{r} (S^{-1})^\xi{}_\alpha \omega^\alpha{}_\beta S^\beta{}_\eta + \sum_{\alpha=1}^{r} (S^{-1})^\xi{}_\alpha \, dS^\alpha{}_\eta.$$

where the values of the smooth real-valued functions $(S^{-1})^\xi{}_\alpha$ at each point p of U are the components of the inverse whose inverse $S^{-1}(p)$ of the matrix $S(p)$. Thus if ω and $\hat{\omega}$ denote the $r \times r$ matrices of smooth 1-forms whose components are the 1-forms $\omega^\alpha{}_\beta$ and $\hat{\omega}^\xi{}_\eta$, then

$$\hat{\omega} = S^{-1} \omega S + S^{-1} \, dS,$$

where dS denotes the differential of the smooth matrix-valued function

$$S\colon U \to \mathrm{GL}(k, \mathbb{R}).$$

Proof Let $\omega^\alpha_\beta = \sum\limits_{j=1}^n A^\alpha{}_{\beta j}\, dx^j$ and $\hat\omega^\xi_\eta = \sum\limits_{j=1}^n \hat A^\xi{}_{\eta j}\, dx^j$ where $A^\alpha{}_{\beta j}$ and $\hat A^\xi{}_{\eta j}$ are defined as in the statement of Corollary 8.2. Then the identities given in the statement of this corollary are restatements of those of Proposition 8.1 and Corollary 8.2. They can also be verified by direct calculation. ∎

8.2 Curvature of Connections on Vector Bundles

Let $\pi_E: E \to M$ be a smooth vector bundle of rank r over a smooth manifold M, and, for each $p \in M$, let E_p denote the fibre of this bundle over the point p. Let U be an open set in M, let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system for M defined over U, and let $e_1, e_2, \ldots, e_r$ be smooth sections of $\pi_E: E \to M$ over U, where, for each point p of U, the elements $e_1(p), e_2(p), \ldots, e_r(p)$ constitute a basis of the real vector space E_p.

Let D be a smooth connection on this vector bundle $\pi_E: E \to M$, and let

$$\partial_j f = \frac{\partial f}{\partial x^j} \quad \text{and} \quad D_j s = D_{\frac{\partial}{\partial x^j}} s$$

for all smooth real-valued functions f and for all smooth sections s of the vector bundle defined over U Then there are smooth functions $A^\alpha{}_{\beta j}$ defined over U such that

$$D_j e_\beta = \sum_{\alpha=1}^r A^\alpha{}_{\beta j} e_\alpha,$$

for $\beta = 1, 2, \ldots, r$. Let X be a smooth vector field on U, and let $s: U \to E$ be a smooth section of the vector bundle $\pi_E: E \to M$ defined over U, and let

$$X = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}, \quad Y = \sum_{j=1}^n w^j \frac{\partial}{\partial x^j} \quad \text{and} \quad s = \sum_{\alpha=1}^n f^\alpha e_\alpha,$$

where $v^1, v^2, \ldots, v^n$ and $f^1, f^2, \ldots, f^r$ are smooth real-valued functions on U. Then

$$D_X s = \sum_{j=1}^n v^j D_j s \quad \text{and} \quad D_Y s = \sum_{j=1}^n w^j D_j s,$$

where

$$D_j s = \sum_{\alpha=1}^r \left(\partial_j f^\alpha + \sum_{\beta=1}^r A^\alpha{}_{\beta j} f^\beta \right) e_\alpha$$

(see Proposition 8.1). Then

$$D_X(D_Y s) \;=\; \sum_{j,k=1}^n v^j D_j(w^k D_k s)$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} v^j D_j \left(w^k \left(\partial_k f^\alpha + \sum_{\gamma=1}^{r} A^\alpha{}_{\gamma\, k} f^\gamma \right) e_\alpha \right) \\
&= \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} v^j (\partial_j w^k) \left(\partial_k f^\alpha + \sum_{\gamma=1}^{r} A^\alpha{}_{\gamma\, k} f^\gamma \right) e_\alpha \\
&\qquad + \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} v^j w^k \partial_j \left(\partial_k f^\alpha + \sum_{\gamma=1}^{r} A^\alpha{}_{\gamma\, k} f^\gamma \right) e_\alpha \\
&\qquad + \sum_{j,k=1}^{n} \sum_{\beta=1}^{r} v^j w^k \left(\partial_k f^\beta + \sum_{\gamma=1}^{r} A^\beta{}_{\gamma\, k} f^\gamma \right) D_j e_\beta \\
&= \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} v^j w^k (\partial_j \partial_k f^\alpha) e_\alpha \\
&\qquad + \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} v^j (\partial_j w^k) \left(\partial_k f^\alpha + \sum_{\gamma=1}^{r} A^\alpha{}_{\gamma\, k} f^\gamma \right) e_\alpha \\
&\qquad + \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} v^j w^k \sum_{\gamma=1}^{r} (\partial_j A^\alpha{}_{\gamma\, k}) f^\gamma e_\alpha \\
&\qquad + \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} v^j w^k \sum_{\gamma=1}^{r} A^\alpha{}_{\gamma\, k} (\partial_j f^\gamma) e_\alpha \\
&\qquad + \sum_{j,k=1}^{n} \sum_{\alpha,\beta=1}^{r} v^j w^k A^\alpha{}_{\beta\, j} \left(\partial_k f^\beta + \sum_{\gamma=1}^{r} A^\beta{}_{\gamma\, k} f^\gamma \right) e_\alpha \\
&= \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} (\partial_j \partial_k f^\alpha) v^j w^k e_\alpha \\
&\qquad + \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} v^j (\partial_j w^k) \left(\partial_k f^\alpha + \sum_{\gamma=1}^{r} A^\alpha{}_{\gamma\, k} f^\gamma \right) e_\alpha \\
&\qquad + \sum_{j,k=1}^{n} \sum_{\alpha,\gamma=1}^{r} \left(A^\alpha{}_{\gamma\, k} \partial_j f^\gamma + A^\alpha{}_{\gamma\, j} \partial_k f^\gamma \right) v^j w^k e_\alpha \\
&\qquad + \sum_{j,k=1}^{n} \sum_{\alpha=1}^{r} \left(\sum_{\gamma=1}^{r} (\partial_j A^\alpha{}_{\gamma\, k}) + \sum_{\beta,\gamma=1}^{r} A^\alpha{}_{\beta\, j} A^\beta{}_{\gamma\, k} \right) v^j w^k f^\gamma e_\alpha
\end{aligned}
$$

We see that the value of $D_X(D_Y s)$ at a point p of U is determined by the values of the components v^j, w^k and f^α of X, Y and s at the point p, the first order partial derivatives of all these components, and the second order partial derivatives of the components f^α of the section s. Now the term

$$\sum_{j,k=1}^{n}\sum_{\alpha=1}^{r}\frac{\partial^2 f^\alpha}{\partial x^j\,\partial x^k}v^j w^k e_\alpha$$

involving the second order partial derivatives of the components f^α of the section s is a symmetric function the vectors X and Y which remains invariant when the vectors X and Y are interchanged. Thus this term is eliminated when we calculate $D_X(D_Y s) - D_Y(D_X s)$. It follows from this that the function that sends X, Y and s to $D_X(D_Y s)-D_Y(D_X s)$ is a first order differential operator whose value at a point p of U is determined by the values of the components v^j, w^k and f^α and their first order partial derivatives at the point p. Moreover the term

$$\sum_{j,k=1}^{n}\sum_{\alpha,\gamma=1}^{r}(A^\alpha{}_{\gamma k}\partial_j f^\gamma + A^\alpha{}_{\gamma j}\partial_k f^\gamma)\,v^j w^k e_\alpha$$

occurring in the expression for $D_X(D_Y s)$ also remains invariant when X and Y are interchanged, and therefore is eliminated when we calculate $D_X(D_Y s)-D_Y(D_X s)$. We find that

$$\begin{aligned}
&D_X(D_Y s) - D_Y(D_X s)\\
&\quad= \sum_{j,k=1}^{n}\sum_{\alpha=1}^{r}\left(v^j\frac{\partial w^k}{\partial x^j} - w^j\frac{\partial v^k}{\partial x^j}\right)\left(\frac{\partial f^\alpha}{\partial x^k} + \sum_{\gamma=1}^{r}A^\alpha{}_{\gamma k}f^\gamma\right)e_\alpha\\
&\qquad+\sum_{j,k=1}^{n}\sum_{\alpha,\gamma=1}^{r}\Bigl((\partial_j A^\alpha{}_{\gamma k}) - (\partial_k A^\alpha{}_{\gamma j})\\
&\qquad\quad+\sum_{\beta=1}^{r}(A^\alpha{}_{\beta j}A^\beta{}_{\gamma k} - A^\alpha{}_{\beta k}A^\beta{}_{\gamma j})\Bigr)v^j w^k f^\gamma e_\alpha.
\end{aligned}$$

Now the term

$$\sum_{j,k=1}^{n}\sum_{\alpha=1}^{r}\left(v^j\frac{\partial w^k}{\partial x^j} - w^j\frac{\partial v^k}{\partial x^j}\right)\left(\frac{\partial f^\alpha}{\partial x^k} + \sum_{\gamma=1}^{r}A^\alpha{}_{\gamma k}f^\gamma\right)e_\alpha$$

is the covariant derivative of the section s with respect to a vector field on U which, when expressed in terms of local coordinates $x^1, x^2, \ldots, x^n$ takes the form

$$\sum_{j,k=1}^{n}\sum_{\alpha=1}^{r}\left(v^j\frac{\partial w^k}{\partial x^j} - w^j\frac{\partial v^k}{\partial x^j}\right)\frac{\partial}{\partial x^k}.$$

This vector field is the Lie bracket $[X, Y]$ of the vector fields X and Y. It follows therefore that

$$D_X(D_Y s) - D_Y(D_X s) - D_{[X,Y]} s = \sum_{j,k=1}^{n} \sum_{\alpha,\gamma=1}^{r} F^{\alpha}{}_{\gamma\, j\, k} v^j w^k f^{\gamma} e_{\alpha},$$

where

$$F^{\alpha}{}_{\gamma\, j\, k} = \frac{\partial A^{\alpha}{}_{\gamma\, k}}{\partial x^j} - \frac{\partial A^{\alpha}{}_{\gamma\, j}}{\partial x^k} + \sum_{\beta=1}^{r} (A^{\alpha}{}_{\beta\, j} A^{\beta}{}_{\gamma\, k} - A^{\alpha}{}_{\beta\, k} A^{\beta}{}_{\gamma\, j}).$$

Now these quantities $F^{\alpha}{}_{\gamma\, j\, k}$ are the components of a smooth section of a vector bundle over M. This vector bundle is the tensor product $E \otimes E^* \otimes T^*M \otimes T^*M$, where E^* is the dual bundle of E and T^*M is the cotangent bundle of the smooth manifold M. Indeed there are smooth sections $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^r$ of the dual bundle $\pi_{E^*}: E^* \to M$ of E over the open set U characterized by the property that $\langle \varepsilon^{\alpha}, e_{\beta} \rangle = \delta^{\alpha}_{\beta}$ at each point of U, where δ^{α}_{β} is the Kronecker delta, equal to 1 when $\alpha = \beta$, but equal to zero otherwise. The values of the smooth sections $\varepsilon^1(p), \varepsilon^2(p), \ldots, \varepsilon^r(p)$ at any point p of U constitute a basis of the fibre E^*_p of the dual bundle at p which is the dual basis corresponding to the basis $e_1(p), e_2(p), \ldots, e_r(p)$ of E_p. Let

$$F_D = \sum_{j,k=1}^{n} \sum_{\alpha,\gamma=1}^{r} F^{\alpha}{}_{\gamma\, j\, k} e_{\alpha} \otimes \varepsilon^{\gamma} \otimes dx^j \otimes dx^k$$

over the open set U. Then F_D represents a smooth section of the smooth vector bundle $E \otimes E^* \otimes T^*M \otimes T^*M$ over U. This section determines a multilinear map

$$(F_D)_p: E_p \times T_pM \times T_pM \to T_pM,$$

which sends (s_p, X_p, Y_p) to $F_D(X_p, Y_p)s_p$ for all $s_p \in E_p$ and $X_p, Y_p \in T_pM$, where

$$F_D(X_p, Y_p)s_p = \sum_{j,k=1}^{n} \sum_{\alpha,\gamma=1}^{r} F^{\alpha}{}_{\gamma\, j\, k} e_{\alpha} \, \langle \varepsilon^{\gamma}, s_p \rangle \langle dx^j, X_p \rangle \, \langle dx^k, Y_p \rangle.$$

Thus if s is a section of $\pi_E: E \to M$ over U, and if X and Y are vector fields on U, where

$$X = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}, \quad Y = \sum_{j=1}^{n} w^j \frac{\partial}{\partial x^j} \quad \text{and} \quad s = \sum_{\alpha=1}^{n} f^{\alpha} e_{\alpha},$$

then

$$F_D(X_p, Y_p)s(p) = \sum_{j,k=1}^{n} \sum_{\alpha,\gamma=1}^{r} F^{\alpha}{}_{\gamma j k} f^{\gamma} v^j w^k \, e_{\alpha}.$$

Our calculations thus show that there is a smooth section F_D of the vector bundle $E \otimes E^* \otimes T^*M \otimes T^*M$ characterized by the property that

$$D_X(D_Y s) - D_Y(D_X s) - D_{[X,Y]} s = F_D(X, Y)s,$$

for all smooth sections of $\pi_E \colon E \to M$ over the open set U, and for all smooth vector fields X and Y on U. This section F_D is the *curvature* of the smooth connection D.

Let us consider in more detail the nature of sections of this vector bundle $E \otimes E^* \otimes T^*M \otimes T^*M$. Now $E \otimes E^*$ can be identified with the smooth vector bundle $\mathrm{End}(E)$ whose fibre $\mathrm{End}(E_p)$ at each point p of M is the algebra of linear operators on the fibre E_p. (Linear operators on a vector space are *endomorphisms* of that vector space.) Indeed elements of $E_p \otimes E_p^*$ are expressed as linear combinations of the form

$$\sum_{\alpha,\gamma=1}^{r} S^{\alpha}{}_{\gamma} \, e_{\alpha}(p) \otimes \varepsilon^{\gamma}(p),$$

with uniquely-determined real coefficients $S^{\alpha}{}_{\gamma}$, where $e_1(p), e_2(p), \ldots, e_r(p)$ is a basis of E_p and $\varepsilon^1(p), \varepsilon^2(p), \ldots, \varepsilon^r(p)$ is the corresponding dual basis of E_p^*. This linear combination of basis elements of $E_p \otimes E_p^*$ corresponds to the linear operator that sends $\sum\limits_{\gamma=1}^{r} c^{\gamma} e_{\gamma}(p)$ to $\sum\limits_{\alpha,\gamma=1}^{r} S^{\alpha}{}_{\gamma} c^{\gamma} e_{\alpha}(p)$ for all $c^1, c^2, \ldots, c^r \in \mathbb{R}$.

The fibre $\bigotimes^2 T_p^*M$ of the vector bundle $\bigotimes^2 T^*M$ at each point p of M is a real vector space whose elements represent bilinear forms on the tangent space T_pM at a point p. This vector space splits as a direct sum of subspaces $S^2 T_p^*M$ and $\bigwedge^2 T_p^*M$, where elements of $S^2 T_p^*M$ represent symmetric bilinear forms on the tangent space T_pM, and where elements of $\bigwedge^2 T_p^*M$ represent skew-symmetric bilinear forms on this tangent space. Suppose that the point p belongs to some open set U in M which is contained in the domain of a smooth coordinate system $(x^1, x^2, \ldots, x^n)$. Then the values at p of the smooth tensor fields

$$dx^j \otimes dx^k + dx^k \otimes dx^j$$

for $j \leq k$ constitute at basis of the real vector space $S^2 T_p^*M$. Similarly the values at p of the smooth tensor fields $dx^j \wedge dx^k$ for $j < k$ constitute a basis of the real vector space $\bigwedge^2 T_p^*M$, where

$$dx^j \wedge dx^k = dx^j \otimes dx^k - dx^k \otimes dx^j.$$

It follows from this that the union of the vector spaces $S^2T_p^*M$ for all points p of M consitutes a smooth submanifold S^2T^*M of $\bigotimes^2 T^*M$ which is the total space of a smooth vector bundle of rank $\frac{1}{2}(n^2+n)$ over M (see Proposition 6.18). Similarly the union of the vector spaces $\bigwedge^2 T_p^*M$ for all points p of M is a smooth submanifold $\bigwedge^2 T^*M$ of $\bigotimes^2 T^*M$ which is the total space of a smooth vector bundle of rank $\frac{1}{2}(n^2-n)$ over M (see Proposition 6.18). Smooth sections of the vector bundle $\bigwedge^2 T^*M$ are smooth *differential forms of degree two* on the smooth manifold M.

The curvature F_D of a smooth connection D on a smooth vector bundle $\pi_E\colon E \to M$ may therefore be regarded as a smooth section of the smooth vector bundle $\operatorname{End}(E) \otimes \bigwedge^2 T^*M$ over M whose fibre $\operatorname{End}(E_p) \otimes \bigwedge^2 T_p^*M$ at each point p of M is a real vector space whose elements represent skew-symmetric bilinear maps from $T_pM \times T_pM$ to the space $\operatorname{End}(E_p)$ of linear operators on E_p.

We have shown the existence and basic properties of the smooth section F_D of the vector bundle $\operatorname{End}(E) \otimes \bigwedge^2 T^*M$ representing the curvature of a smooth connection D on $\pi_E\colon E \to M$ using calculations that involve expressing smooth vector fields around a point p terms of local coordinates $(x^1, x^2, \ldots, x^n)$ around p, and expressing sections of the smooth vector bundle $\pi_E\colon E \to M$ as linear combinations of some chosen basis of sections $s_1, s_2, \ldots, s_r$ of the vector bundle around the point p. The existence and basic properties of the curvature can be established by methods that do not make explicit use of such local coordinate systems bases of local sections of the vector bundle. Indeed we shall make use of Proposition 6.15 as a basic tool to develop, in a more coordinate-free fashion, the theory of the curvature of connections on smooth vector bundles over smooth manifolds.

Let $\tilde{E}, E_1, E_2, \ldots, E_k$ be smooth vector bundles over a smooth manifold M, and let $\mathcal{Q}$ be an operator that, over each open set U on M, assigns to smooth sections $s_1, s_2, \ldots s_k$ of the respective vector bundles $E_1, E_2, \ldots, E_k$ defined over U a smooth section $\mathcal{Q}(s_1, s_2, \ldots, s_k)$ of the vector bundle $\tilde{E}$ defined over this open set U. Suppose that this operator $\mathcal{Q}$ on sections is $\mathbb{R}$-multilinear, and that

$$\mathcal{Q}(f_1 s_1, f_2 s_2, \ldots, f_k s_k) = f_1 \cdot f_2 \cdots f_k\, \mathcal{Q}(s_1, s_2, \ldots, s_k)$$

for all smooth functions $f_1, f_2 \cdots f_k$ on U, and for all $s_1, s_2, \ldots, s_k$, where s_j is a smooth section of the vector bundle E_j defined over U for $j = 1, 2, \ldots, k$. Proposition 6.15 then ensures that there exists a smooth section Q of the vector bundle

$$\tilde{E} \otimes E_1^* \otimes E_2^* \otimes \cdots \otimes E_k^*$$

such that

$$\mathcal{Q}(s_1, s_2, \ldots, s_k) = Q(s_1, s_2, \ldots, s_k)$$

for all $s_1, s_2, \ldots, s_k$, where s_j is a smooth section of the vector bundle E_j over U for $j = 1, 2, \ldots, k$.

Proposition 8.4 *Let D be a smooth connection on a smooth vector bundle $\pi_E\colon E \to M$ over a smooth manifold M, and let $\mathrm{End}(E_p)$ denote the space of linear operators on the fibre E_p of the vector bundle over the point p. Given any smooth section $s\colon U \to X$ of this vector bundle, defined over some open subset U of M, and given any smooth vector fields X and Y on U, let $\mathcal{F}_D(X,Y)s$ denote the smooth section of $\pi_E\colon E \to M$ defined such that*

$$\mathcal{F}_D(X,Y)s = D_X(D_Y s) - D_Y(D_X s) - D_{[X,Y]}s.$$

Then

$$\mathcal{F}_D(X,Y)(fs) = \mathcal{F}_D(fX,Y)s = \mathcal{F}_D(X,fY)s = f\,\mathcal{F}_D(X,Y)s$$

for all smooth real-valued functions f on the open set U, and thus there exists a smooth section F_D of the smooth vector bundle $\mathrm{End}(E) \otimes \bigwedge^2 T^*M$ whose value at each point p of M represents a skew-symmetric bilinear map $(F_D)_p\colon T_pM \times T_pM \to \mathrm{End}(E_p)$ on the tangent space T_pM at each point p of M which is defined such that

$$(F_D)_p(X_p, Y_p)s(p) = (\mathcal{F}_D(X,Y)s)(p) = D_{X_p}(D_Y s) - D_{Y_p}(D_X s) - D_{[X,Y]_p}s.$$

for all smooth vector fields X and Y defined around the point p and for all smooth sections s of $\pi_E\colon E \to M$ defined around p.

Proof Let s be a smooth section of the vector bundle $\pi_E\colon E \to M$ defined over some open set U in M, let X and Y be smooth vector fields on U, and let $f\colon U \to \mathbb{R}$ be a smooth real-valued function on U. Then

$$\begin{aligned} [X,Y][f] &= X[Y[f]] - Y[X[f]], \\ [fX,Y] &= f\,[X,Y] - Y[f]\,X, \\ [X,fY] &= f\,[X,Y] + X[f]\,Y \end{aligned}$$

(see Lemma 7.6). It follows that

$$\begin{aligned} \mathcal{F}_D(fX,Y)s &= D_{fX}(D_Y s) - D_Y(D_{fX} s) - D_{[fX,Y]}s \\ &= f\,D_X(D_Y s) - D_Y(f\,D_X s) - D_{f[X,Y]-Y[f]X}s \\ &= f\,D_X(D_Y s) - f\,D_Y(D_X s) - Y[f]\,D_X s \end{aligned}$$

$$
\begin{aligned}
&\quad - f D_{[X,Y]} s + Y[f] D_X s \\
&= f\, D_X(D_Y s) - f\, D_Y(D_X s) - f D_[X, Y] s \\
&= f\, \mathcal{F}_D(X,Y)s, \\
\mathcal{F}_D(X, fY)s &= -\mathcal{F}_D(fY, X)s = -f\, \mathcal{F}_D(Y,X)s \\
&= f\, \mathcal{F}_D(X,Y)s, \\
\mathcal{F}_D(X,Y)(fs) &= D_X(D_Y(fs)) - D_Y(D_X(fs)] - D_{[X,Y]}(fs) \\
&= D_X(Y[f]\, s + f\, D_Y s) - D_Y(X[f]\, s + f\, D_Y s) \\
&\quad - [X,Y][f]\, s - f\, D_{[X,Y]} s \\
&= Y[f]\, D_X s + X[Y[f]]\, s + f\, D_X(D_Y s) + X[f]\, D_Y s \\
&\quad - X[f]\, D_Y s - Y[X[f]]\, s - f\, D_Y(D_X s) - Y[f]\, D_X s \\
&\quad - [X,Y][f]\, s - f\, D_{[X,Y]} s \\
&= f\,(D_X(D_Y s) - D_Y(D_X s) - D_{[X,Y]} s) \\
&\quad + X[Y][f] - Y[X[f]] - [X,Y][f] \\
&= f\, \mathcal{F}_D(X,Y)s.
\end{aligned}
$$

Moreover it is easy to see that

$$
\begin{aligned}
\mathcal{F}_D(X_1 + X_2, Y)s &= \mathcal{F}_D(X_1, Y)s + \mathcal{F}_D(X_2, Y)s, \\
\mathcal{F}_D(X, Y_1 + Y_2)s &= \mathcal{F}_D(X, Y_1)s + \mathcal{F}_D(X, Y_2)s, \\
\mathcal{F}_D(X,Y)(s_1 + s_2) &= \mathcal{F}_D(X,Y)s_1 + \mathcal{F}_D(X,Y)s_2.
\end{aligned}
$$

for all sections s, s_1, s_2 of the vector bundle and for all smooth vector fields X, X_1, X_2, Y, Y_1 and Y_2 defined over the open set U of M.

It now follows from Proposition 6.15 that the operator $\mathcal{F}_D$ determines a smooth section F_D of the vector bundle $\mathrm{End}(E) \otimes \bigwedge^2 T^*M$ with the required properties. ∎

Definition Let D be a smooth connection defined on a smooth vector bundle $\pi_E\colon E \to M$ over a smooth manifold M. Let $s\colon U \to E$ be a smooth section of this vector bundle, defined over some open set U in M, and let X and Y be smooth vector fields on U. We define the *curvature* F_D of the smooth connection D to be the smooth section of the smooth vector bundle $\mathrm{End}(E) \otimes \bigwedge^2 T^*M$ characterized by the property that

$$F_D(X,Y)s = D_X(D_Y s) - D_Y(D_X s) - D_{[X,Y]} s.$$

for all smooth vector fields X and Y and smooth sections s of $\pi_E\colon E \to M$, where these vector fields and sections are all defined over some open set in M.

The following proposition summarizes the results of calculations presented above.

Proposition 8.5 *Let $\pi_E\colon E \to M$ be a smooth vector bundle of rank r over a smooth manifold M, and, for each $p \in M$, let E_p denote the fibre of this bundle over the point p. Let U be an open set in M, let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system for M defined over U, and let $e_1, e_2, \ldots, e_r$ be smooth sections of $\pi_E\colon E \to M$ over U, where, for each point p of U, the elements $e_1(p), e_2(p), \ldots, e_r(p)$ constitute a basis of the real vector space E_p. Let D be a smooth connection on this vector bundle, and let $A^\alpha{}_{\beta j}$ be the smooth real-valued functions on U defined such that*

$$D_{\frac{\partial}{\partial x^j}} e_\beta = \sum_{\alpha=1}^{r} A^\alpha{}_{\beta j} e_\alpha,$$

Then

$$F_D = \sum_{j,k=1}^{n} \sum_{\alpha,\gamma=1}^{r} F^\alpha{}_{\gamma j k} e_\alpha \otimes \varepsilon^\gamma \otimes dx^j \otimes dx^k$$

where

$$F^\alpha{}_{\gamma j k} = \frac{\partial A^\alpha{}_{\gamma k}}{\partial x^j} - \frac{\partial A^\alpha{}_{\gamma j}}{\partial x^k} + \sum_{\beta=1}^{r} (A^\alpha{}_{\beta j} A^\beta{}_{\gamma k} - A^\alpha{}_{\beta k} A^\beta{}_{\gamma j}).$$

Proposition 8.6 *Let D be a smooth connection defined on a smooth vector bundle $\pi_E\colon E \to M$ over a smooth manifold M, and let F_D be the curvature of D. Let $s\colon M \to E$ be a smooth section of the vector bundle $\pi_E\colon E \to M$, and let X, Y and Z be smooth vector fields on M. Then*

$$\begin{aligned} D_X(F_D(Y,Z)s) &+ D_Y(F_D(Z,X)s) + D_Z(F_D(X,Y)s) \\ &= F_D([Y,Z],X)s + F_D([Z,X],Y)s + F_D([X,Y],Z)s \\ &\quad + F_D(Y,Z)(D_X s) + F_D(Z,X)(D_Y s) + F_D(X,Y)(D_Z s). \end{aligned}$$

Proof It follows from Proposition 8.4 and the definition of connections on smooth bundles that

$$\begin{aligned} D_X(F_D(Y,Z)s) &+ D_Y(F_D(Z,X)s) + D_Z(F_D(X,Y)s) \\ &= D_X(D_Y(D_Z s) - D_Z(D_Y s) - (D_{[Y,Z]} s)) \\ &\quad + D_Y(D_Z(D_X s) - D_X(D_Z s) - (D_{[Z,X]} s)) \\ &\quad + D_Z(D_X(D_Y s) - D_Y(D_X s) - (D_{[X,Y]} s)) \\ &= D_X(D_Y(D_Z s)) - D_X(D_Z(D_Y s)) - D_X(D_{[Y,Z]} s) \end{aligned}$$

$$
\begin{aligned}
&\quad + D_Y(D_Z(D_X s)) - D_Y(D_X(D_Z s)) - D_Y(D_{[Z,X]} s) \\
&\quad + D_Z(D_X(D_Y s)) - D_Z(D_Y(D_X s)) - D_Z(D_{[X,Y]} s) \\
= &\; F_D(Y,Z)(D_X s) + D_{[Y,Z]}(D_X s) - D_X(D_{[Y,Z]} s) \\
&\quad + F_D(Z,X)(D_Y s) + D_{[Z,X]}(D_Y s) - D_Y(D_{[Z,X]} s) \\
&\quad + F_D(X,Y)(D_Z s) + D_{[Z,X]}(D_Z s) - D_Z(D_{[X,Y]} s) \\
= &\; F_D(Y,Z)(D_X s) + F_D([Y,Z],X) s + D_{[[Y,Z],X]} s \\
&\quad + F_D(Z,X)(D_Y s) + F_D([Z,X],Y) s + D_{[[Z,X],Y]} s \\
&\quad + F_D(X,Y)(D_Z s) + F_D([X,Y],Z) s + D_{[[X,Y],Z]} s \\
= &\; F_D(Y,Z)(D_X s) + F_D(Z,X)(D_Y s) + F_D(X,Y)(D_Z s) \\
&\quad + F_D([Y,Z],X) s + F_D([Z,X],Y) s + F_D([X,Y],Z) s \\
&\quad + D_{[[Y,Z],X]+[Z,X],Y]+[[X,Y],Z]} s.
\end{aligned}
$$

But the Lie Bracket satisfies the Jacobi Identity, and therefore

$$[[Y,Z],X] + [Z,X],Y] + [[X,Y],Z] = 0.$$

(see Lemma 7.5). The result follows. ∎

8.3 Induced Connections on Dual Bundles

Proposition 8.7 *Let D be a smooth connection on a smooth vector bundle $\pi_E \colon E \to M$. Then D induces a connection (which we also denote by D) on the dual bundle $\pi_{E^*} \colon E^* \to M$. This connection on the dual bundle is defined such that if φ is a smooth section of $\pi_{E^*} \colon E^* \to M$ defined around some point p of M then*

$$\langle D_{X_p}\varphi, s(p)\rangle = X_p[\langle \varphi, s\rangle] - \langle \varphi, D_{X_p} s\rangle$$

for all smooth sections s of $\pi_E \colon E \to M$ defined around the point p, and for all tangent vectors X_p to M at the point p.

Proof Let X be a smooth vector field, let s be a smooth section of $\pi_E \colon E \to M$, and let f be a smooth real-valued function defined throughout some open set U in M. Then

$$
\begin{aligned}
&X[\langle \varphi, fs\rangle] - \langle \varphi, D_X(fs)\rangle \\
&\quad = X[f \cdot \langle \varphi, s\rangle] - \langle \varphi,\, f\, D_X s + X[f]\, s\rangle \\
&\quad = X[f]\langle \varphi, s\rangle + f\, X[\langle \varphi, s\rangle] \\
&\qquad\quad - f\,\langle \varphi, D_X s\rangle - X[f]\,\langle \varphi, s\rangle \\
&\quad = f\, \left(X[\langle \varphi, s\rangle] - \langle \varphi, D_X s\rangle\right).
\end{aligned}
$$

It follows from a direct application of Proposition 6.15 that there is a well-defined smooth section $D_X\varphi$ of $\pi_{E^*}\colon E^* \to M$ characterized by the property that

$$\langle D_X\varphi, s\rangle = X[\langle\varphi, s\rangle] - \langle\varphi, D_X s\rangle$$

for all smooth sections s of $\pi_E\colon E \to M$ defined around the point p. Moreover this differential operator satisfies the properties required of a smooth connection on the vector bundle $\pi_{E^*}\colon E^* \to M$. ∎

8.4 Induced Connections on Tensor Products of Vector Bundles

Proposition 8.8 *Let $E, E_1, \ldots, E_k$ be smooth vector bundles over a smooth manifold M, and let*

$$\mathcal{M}(E_1, E_2, \ldots, E_k; E)$$

denote the smooth vector bundle $E \otimes E_1^* \otimes E_2^* \otimes \cdots \otimes E_k^*$ whose fibre at each point p of M is the real vector space whose elements are multilinear maps from $(E_1)_p \times (E_2)_p \times \cdots \times (E_k)_p$ to E_p. Let D^E be a smooth connection on the smooth vector bundle E, and let D^{E_j} be a smooth connection on the smooth vector bundle E_j for $j = 1, 2, \ldots, k$. Then these smooth connections D^E and D^{E_j} induce a smooth connection D on the smooth vector bundle $\mathcal{M}(E_1, E_2, \ldots, E_k; E)$ characterized by the property that

$$\begin{aligned}
&(D_{X_p}S)(s_1, s_2, s_3, \ldots, s_k) \\
&\quad = D^E_{X_p}\Bigl(S(s_1, s_2, s_3, \ldots, s_k)\Bigr) \\
&\qquad - S(D^{E_1}_{X_p}s_1, s_2, s_3, \ldots, s_k) - S(s_1, D^{E_2}_{X_p}s_2, s_3, \ldots, s_k) \\
&\qquad - S(s_1, s_2, D^{E_2}_{X_p}s_3, \ldots, s_k) - \cdots - S(s_1, s_2, s_3, \ldots, D^{E_k}_{X_p}s_k)
\end{aligned}$$

for all $s_1, s_2, \ldots, s_k$, where s_j is a smooth section of the smooth vector bundle E_j for $j = 1, 2, \ldots, k$, and where these sections $s_1, s_2, \ldots, s_k$ are all defined over some open set U in M.

Proof Let f be a smooth real-valued function defined on a neighbourhood of a point p of M. Then

$$(D_{f\,X_p}S)(s_1, s_2, s_3, \ldots, s_k) = f\,(D_{X_p}S)(s_1, s_2, s_3, \ldots, s_k).$$

Also

$$\begin{aligned}
D^E_{X_p}\Bigl(S(fs_1, s_2, s_3, \ldots, s_k)\Bigr) &= D^E_{X_p}\Bigl(f\,S(s_1, s_2, s_3, \ldots, s_k)\Bigr) \\
&= X_p[f]\,S(s_1, s_2, s_3, \ldots, s_k) \\
&\qquad + f\,D^E_{X_p}\Bigl(S(s_1, s_2, s_3, \ldots, s_k)\Bigr)
\end{aligned}$$

and

$$S(D^{E_1}_{X_p}(fs_1), s_2, s_3, \ldots, s_k) = X_p[f]\, S(s_1, s_2, s_3, \ldots, s_k) + f\, S(D^{E_1}_{X_p} s_1, s_2, s_3, \ldots, s_k),$$

and therefore

$$(D_{X_p} S)(fs_1, s_2, s_3, \ldots, s_k) = f\,(D_{X_p} S)(s_1, s_2, s_3, \ldots, s_k).$$

Similarly

$$\begin{aligned}
(D_{X_p} S)(s_1, fs_2, s_3, \ldots, s_k) &= f\,(D_{X_p} S)(s_1, s_2, s_3, \ldots, s_k),\\
(D_{X_p} S)(s_1, s_2, fs_3, \ldots, s_k) &= f\,(D_{X_p} S)(s_1, s_2, s_3, \ldots, s_k),\\
&\vdots\\
(D_{X_p} S)(s_1, s_2, s_3, \ldots, fs_k) &= f\,(D_{X_p} S)(s_1, s_2, s_3, \ldots, s_k).
\end{aligned}$$

The required result follows immediately on applying Proposition 6.15. ∎

Proposition 8.9 *Let $E_1, \ldots, E_k$ be smooth vector bundles over a smooth manifold M, and let D_{E_j} be a smooth connection on the smooth vector bundle E_j for $j = 1, 2, \ldots, k$. Then these smooth connections E_j induce a smooth connection D on the smooth vector bundle $E_1 \otimes E_2 \otimes \cdots \otimes E_k$ characterized by the property that*

$$\begin{aligned}
&D_{X_p}(s_1 \otimes s_2 \otimes s_3 \otimes \cdots \otimes s_k)\\
&\quad = D^{E_1}_{X_p} s_1 \otimes s_2 \otimes s_3 \otimes \cdots \otimes s_k + s_1 \otimes D^{E_2}_{X_p} s_2 \otimes s_3 \otimes \cdots \otimes s_k\\
&\qquad + s_1 \otimes s_2 \otimes D^{E_3}_{X_p} s_3 \otimes \cdots \otimes s_k + \cdots + s_1 \otimes s_2 \otimes s_3 \otimes \cdots \otimes D^{E_k}_{X_p} s_k
\end{aligned}$$

for all $s_1, s_2, \ldots, s_k$, where s_j is a smooth section of the smooth vector bundle E_j for $j = 1, 2, \ldots, k$, and where these sections $s_1, s_2, \ldots, s_k$ are all defined over some open set U in M.

Proof At each point p of M the tensor product

$$(E_1)_p \otimes (E_2)_p \otimes \cdots \otimes (E_k)_p$$

of the fibres of the vector bundles can be identified with the space

$$\mathcal{M}((E_1)^*_p, (E_2)^*_p, \ldots, (E_k)^*_p; \mathbb{R})$$

of multilinear maps from $(E^*_1)_p \times (E^*_2)_p \times \cdots \times (E^*_k)_p$ to the field $\mathbb{R}$ of real numbers, where E^*_j is the dual bundle of E_j for $j = 1, 2, \ldots, k$. Thus a smooth section of the tensor product bundle $E_1 \otimes E_2 \otimes \cdots \otimes E_k$ can be represented as a function that assigns to each point p of M a multilinear map

$$S_p \colon (E_1^*)_p \times (E_2^*)_p \times \cdots \times (E_k^*)_p \to \mathbb{R},$$

where the function $S(\varphi_1, \varphi_2, \ldots, \varphi_k)$ sending $p \in M$ to

$$S_p(\varphi_1(p), \varphi_2(p), \ldots, \varphi_k(p))$$

is smooth for all $\varphi_1, \varphi_2, \ldots, \varphi_k$, where φ_j is a smooth section of E_j^* for $j = 1, 2, \ldots, k$. Moreover if the operator $S = s_1 \otimes s_2 \otimes \cdots \otimes s_k$, where s_j is a smooth section of the vector bundle E_j for $j = 1, 2, \ldots, k$, then

$$S(\varphi_1, \varphi_2, \ldots, \varphi_k) = \langle \varphi_1, s_1 \rangle \cdot \langle \varphi_2, s_2 \rangle \cdots \langle \varphi_k, s_k \rangle.$$

It follows from Proposition 8.8 that there is an induced connection D on $E_1 \otimes E_2 \otimes \cdots \otimes E_k$, where

$$\begin{aligned} &(D_X(s_1 \otimes s_2 \otimes \cdots \otimes s_k))(\varphi_1, \varphi_2, \ldots, \varphi_k) \\ &\quad = X[S(\varphi_1, \varphi_2, \ldots, \varphi_k)] \\ &\qquad - S(D_X^{E_1^*}\varphi_1, \varphi_2, \ldots, \varphi_k) - S(\varphi_1, D_X^{E_2^*}\varphi_2, \ldots, \varphi_k) \\ &\qquad - \cdots - S(\varphi_1, \varphi_2, \ldots, D_X^{E_k^*}\varphi_k) \end{aligned}$$

But

$$\begin{aligned} X[S(\varphi_1, \varphi_2, \ldots, \varphi_k)] &= X[\langle \varphi_1, s_1 \rangle \cdot \langle \varphi_2, s_2 \rangle \cdots \langle \varphi_k, s_k \rangle] \\ &= \sum_{j=1}^{k} X[\langle \varphi_j, s_j \rangle] \cdot \prod_{m \neq j} \langle \varphi_m, s_m \rangle \end{aligned}$$

and

$$S(\varphi_1, \ldots, \varphi_{j-1}, D^{E_j^*}\varphi_j, \varphi_{j+1}, \ldots, \varphi_k) = \langle D_X^{E_j^*}\varphi_j, s_j \rangle \prod_{m \neq j} \langle \varphi_m, s_m \rangle.$$

and

$$X[\langle \varphi_j, s_j \rangle] = \langle D_X^{E_j^*}\varphi_j, s_j \rangle + \langle \varphi_j, D_X^{E_j} s_j \rangle$$

for $j = 1, 2, \ldots, k$. It follows that

$$\begin{aligned} &(D_X(s_1 \otimes s_2 \otimes \cdots \otimes s_k))(\varphi_1, \varphi_2, \ldots, \varphi_k) \\ &\quad = \sum_{j=1}^{k} \left(X[\langle \varphi_j, s_j \rangle] - \langle D_X^{E_j^*}\varphi_j, s_j \rangle \right) \prod_{m \neq j} \langle \varphi_m, s_m \rangle \end{aligned}$$

$$= \sum_{j=1}^{k} \langle \varphi_j, D_X^{E_j} s_j \rangle \prod_{m \neq j} \langle \varphi_m, s_m \rangle$$

$$= \sum_{j=1}^{k} (s_1 \otimes \cdots \otimes s_{j-1} \otimes D_X^{E_j} s_j \otimes s_{j+1} \otimes \cdots \otimes s_k)(\varphi_1, \varphi_2, \ldots, \varphi_k)$$

and thus

$$D_X(s_1 \otimes s_2 \otimes \cdots \otimes s_k)) = \sum_{j=1}^{k} s_1 \otimes \cdots \otimes s_{j-1} \otimes D_X^{E_j} s_j \otimes s_{j+1} \otimes \cdots \otimes s_k,$$

as required. ▮

8.5 Affine Connections on Smooth Manifolds

Definition An *affine connection* ∇ on a smooth manifold M is a connection on the tangent bundle $\pi_{TM}: TM \to M$ of M.

Thus an *affine connection* ∇ on a smooth manifold M is a differential operator which, at each point p of M, associates a tangent vector $\nabla_{X_p} Y$ to each smooth vector field Y defined around p and to each tangent vector X_p at p, and which satisfies the following conditions:— of M:

(i) $\nabla_{W_p + X_p} Y = \nabla_{W_p} Y + \nabla_{X_p} Y$
for all tangent vectors W_p and X_p at a point p of M, and for all smooth vector fields Y defined around p;

(iii) $\nabla_{c X_p} Y = c \nabla_{X_p} Y$
for all real numbers c and tangent vectors X_p at a point p of M, and for all smooth vector fields Y defined around p;

(iii) $\nabla_{X_p} (Y + Z) = \nabla_{X_p} Y + \nabla_{X_p} Z$
for all tangent vectors X_p at a point p of M, and for all smooth vector fields Y and Z defined around p;

(iv) $\nabla_X (f\, Y) = X[f]\, Y + f\, \nabla_X Y$
for all tangent vectors X_p at a point p of M, for all smooth real-valued functions f defined around p, and for all smooth vector fields Y defined around p;

(v) given smooth vector fields X and Y defined over a subset U of M, the function that sends points p of U to $\nabla_{X_p} Y$ is itself a smooth vector field defined over U.

The tangent vector $\nabla_{X_p} Y$ at a point p of the manifold determined by a tangent vector X_p at p and a smooth vector field Y defined around p is referred to as the *covariant derivative* of the vector field Y along the tangent vector X_p (with respect to the affine connection ∇).

Let ∇ be an smooth affine connection on a smooth manifold M, and let $\mathcal{T}$ be the $\mathbb{R}$-bilinear operator acting on smooth vector fields on M defined such that if U is an open set in M and if X and Y are smooth vector fields on U, then

$$\mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

If f is a smooth real-valued function and if X and Y are smooth vector fields defined over an open set U in M then

$$[X, f\,Y] = f\,[X, Y] + X[f]\,Y$$

(see Lemma 7.6). Therefore

$$\begin{aligned} \mathcal{T}(X, fY) &= \nabla_X(f\,Y) - \nabla_{f\,Y} X - [X, f\,Y] \\ &= f\,\nabla_X Y + X[f]\,Y - f\,\nabla_Y X - f\,[X, Y] - X[f]\,Y \\ &= f\mathcal{T}(X, Y). \end{aligned}$$

Also

$$\mathcal{T}(fX, Y) = -\mathcal{T}(Y, fX) = -f\,\mathcal{T}(Y, X) = f\,\mathcal{T}(X, Y).$$

It therefore follows from Proposition 6.15 that there is a smooth tensor field T of type $(1, 2)$ on M such that

$$T(X, Y) = \mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all smooth vector fields X and Y that are defined over some open subset of M.

Also let $\mathcal{R}$ be the $\mathbb{R}$-trilinear operator acting on smooth vector fields on M defined such that if U is an open set in M and if X, Y and Z are smooth vector fields on U, then

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

on U. It then follows from Proposition 8.4 that

$$\mathcal{R}(f\,X, Y)Z = \mathcal{R}(X, f\,Y)Z = \mathcal{R}(X, Y)(f\,Z) = f\,R(X, Y)Z$$

for all smooth real-valued functions f and smooth vector fields X, Y, Z on U, and thus there exists a tensor field R of type $(1, 3)$ on M such that

$$R(X, Y)Z = \mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all smooth vector fields X, Y and Z that are defined over some open subset of M.

Definition The *torsion tensor* T and the *curvature tensor* R of a smooth affine connection ∇ are the smooth tensor fields of types $(1,2)$ and $(1,3)$ respectively on M defined such that if U is an open subset of M, and if X, Y and Z are smooth vector fields on U, then

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

An affine connection ∇ on M is said to be *torsion-free* if its torsion tensor is everywhere zero (so that $\nabla_X Y - \nabla_Y X = [X,Y]$ for all smooth vector fields X and Y on M).

Note that the torsion tensor T and the curvature tensor R of a smooth affine connection ∇ on a smooth manifold M satisfy $T(X,Y) = -T(Y,X)$ and $R(X,Y)Z = -R(Y,X)Z$ for all smooth vector fields X, Y and Z on M.

Example Let U be an open set in $\mathbb{R}^n$, and let $\mathbf{X}$ and $\mathbf{Y}$ be smooth vector fields on U. Then

$$\mathbf{X} = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}, \qquad \mathbf{Y} = \sum_{i=1}^n w^i \frac{\partial}{\partial x^i},$$

where $v^1, v^2, \ldots, v^n$ and $w^1, w^2, \ldots, w^n$ are the components of the vector fields $\mathbf{X}$ and $\mathbf{Y}$ with respect to the Cartesian coordinate system $(x^1, x^2, \ldots, x^n)$ on $\mathbb{R}^n$. The *directional derivative* $\partial_{\mathbf{X}}\mathbf{Y}$ of the vector field $\mathbf{Y}$ along the vector field $\mathbf{X}$ is then given by the formula

$$\partial_{\mathbf{X}}\mathbf{Y} = \sum_{i=1}^n \mathbf{X}[w^i] \frac{\partial}{\partial x^i} = \sum_{i,j=1}^n v^j \frac{\partial w^i}{\partial x^j} \frac{\partial}{\partial x^i}$$

(where $\mathbf{X}[w^i]$ denotes the directional derivative of the function w^i along the vector field $\mathbf{X}$). Then the differential operator sending smooth vector fields $\mathbf{X}$ and $\mathbf{Y}$ to $\partial_{\mathbf{X}}\mathbf{Y}$ is an affine connection on U. We refer to this affine connection as the *canonical* (or *usual*) *flat connection* on the open set U. Now

$$\partial_{\mathbf{X}}\mathbf{Y} - \partial_{\mathbf{Y}}\mathbf{X} = \sum_{i=1}^n \sum_{j=1}^n \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} = [\mathbf{X}, \mathbf{Y}]$$

(see Lemma 7.7). Thus the canonical flat connection ∂ on U is torsion-free. Moreover, given any smooth vector field $\mathbf{Z}$ on U with Cartesian compoents

$c^1, c^2, \ldots, c^n$, we see that

$$\begin{aligned} \partial_{\mathbf{X}}\partial_{\mathbf{Y}}\mathbf{Z} - \partial_{\mathbf{Y}}\partial_{\mathbf{X}}\mathbf{Z} &= (\partial_{\mathbf{X}}\partial_{\mathbf{Y}} - \partial_{\mathbf{Y}}\partial_{\mathbf{X}}) \left(\sum_{i=1}^n c^i \frac{\partial}{\partial x^i} \right) \\ &= \sum_{i=1}^n \bigl(\mathbf{X}[\mathbf{Y}[c^i]] - \mathbf{Y}[\mathbf{X}[c^i]] \bigr) \frac{\partial}{\partial x^i} = \sum_{i=1}^n [\mathbf{X}, \mathbf{Y}][c^i] \frac{\partial}{\partial x^i} \\ &= \partial_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}. \end{aligned}$$

We deduce that the curvature tensor of the canonical flat connection ∂ on U is zero everywhere on U.

Example Let M be a smooth n-dimensional submanifold of k-dimensional Euclidean space. The Levi-Civita connection on M is the smooth connection ∇ on the tangent bundle $\pi_{TM}\colon TM \to M$ that is defined such that $(\nabla_{\mathbf{X}}\mathbf{Y}).\mathbf{Z} = (\partial_{\mathbf{X}}\mathbf{Y}).\mathbf{Z}$ for all tangential vector fields $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ on M, where the Cartesian components of $\partial_{\mathbf{X}}\mathbf{Y}$ at a point p of M are the directional derivatives of those of $\mathbf{Y}$ along the tangent vector $\mathbf{X}_p$.

Let f be a smooth function defined on an open set U in $\mathbf{R}^k$, and let $v^1, v^2, \ldots, v^k$ and $w^1, w^2, \ldots, w^k$ be smooth real-valued functions on U that at each point p of $M \cap U$ are the Cartesian components of the tangential vectors $\mathbf{X}_p$ and $\mathbf{Y}_p$, so that

$$\mathbf{X} = (v^1, v^2, \ldots, v^k), \quad \mathbf{Y} = (w^1, w^2, \ldots, w^k)$$

throughout $M \cap U$. Then

$$\mathbf{Y}[f] = \sum_{j=1}^k w^j \frac{\partial f}{\partial x^j}$$

on $M \cap U$, where $x^1, x^2, \ldots, x^k$ are the standard Cartesian coordinate functions on $\mathbb{R}^k$, and therefore

$$\begin{aligned} \mathbf{X}[\mathbf{Y}[f]] &= \sum_{i,j=1}^k v^i \frac{\partial}{\partial x^i} \left(w^j \frac{\partial f}{\partial x^j} \right) \\ &= \sum_{i,j=1}^k \left(v^i \frac{\partial w^j}{\partial x^i} \frac{\partial f}{\partial x^j} + v^i w^j \frac{\partial^2 f}{\partial x^i \partial x^j} \right) \end{aligned}$$

and therefore

$$[\mathbf{X}, \mathbf{Y}][f] = \mathbf{X}[\mathbf{Y}[f]] - \mathbf{Y}[\mathbf{X}[f]]$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{k} \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i} \right) \frac{\partial f}{\partial x^j}$$

$$= \sum_{j=1}^{k} \left(\mathbf{X}[w^j] - \mathbf{Y}[v^j] \right) \frac{\partial f}{\partial x^j}$$

where $[\mathbf{X}, \mathbf{Y}]$ is the Lie bracket of the smooth tangential vector fields $\mathbf{X}$ and $\mathbf{Y}$. It follows that $[\mathbf{X}, \mathbf{Y}]$ is the tangential vector field whose jth Cartesian component on $M \cap U$ is the smooth real-valued function $\mathbf{X}[w^j] - \mathbf{Y}[v^j]$. Thus

$$[\mathbf{X}, \mathbf{Y}] = \partial_{\mathbf{X}} \mathbf{Y} - \partial_{\mathbf{Y}} \mathbf{X}.$$

On taking the orthogonal projection of both sides of this equation onto the tangent space at each point of the submanifold M, we find that

$$[\mathbf{X}, \mathbf{Y}] = \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X}.$$

Thus the Levi-Civita connection ∇ on a smooth submanifold M of some Euclidean space $\mathbb{R}^k$ is torsion-free.

Lemma 8.10 *Let ∇ be an affine connection on a smooth manifold M, let $(x^1, x^2, \ldots, x^n)$ be a smooth smooth coordinate system defined over an open set U in M. Let X and Y be smooth vector fields on U, and let $v^1, v^2, \ldots, v^n$ and $w^1, w^2, \ldots, w^n$ be the components of the vector fields X and Y with respect to the smooth coordinate system, so that*

$$X = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{i=1}^{n} w^i \frac{\partial}{\partial x^i}.$$

Then

$$\nabla_X Y = \sum_{i,j=1}^{n} \left(v^j \frac{\partial w^i}{\partial x^j} + \sum_{k=1}^{n} v^j w^k \Gamma^i_{jk} \right) \frac{\partial}{\partial x^i},$$

on U where the coefficients Γ^i_{jk} are smooth functions defined over U so that

$$\nabla_{\frac{\partial}{\partial x^j}} \left(\frac{\partial}{\partial x^k} \right) = \sum_{i=1}^{n} \Gamma^i_{jk} \frac{\partial}{\partial x^i},$$

for $j, k = 1, 2, \ldots, n$.

Proof This is a special case of Proposition 8.1 and can be verified directly by a straightforward computation. ∎

Proposition 8.11 *Let ∇ be a smooth affine connection on a smooth manifold M, and let T and R be the torsion and curvature tensors respectively of ∇. Let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system for M defined over an open subset U of M, and let*

$$\nabla_{\frac{\partial}{\partial x^j}}\left(\frac{\partial}{\partial x^k}\right) = \sum_{l=1}^{n} \Gamma^l_{jk} \frac{\partial}{\partial x^l},$$

on U. Then

$$T = \sum_{l=1}^{n} (\Gamma^l_{jk} - \Gamma^l_{kj}) \frac{\partial}{\partial x^l} \otimes dx^j \otimes dx^k,$$

and

$$R = \sum_{l=1}^{n} R^l{}_{ijk} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k$$

on U where

$$R^l{}_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial x^j} - \frac{\partial \Gamma^l_{ji}}{\partial x^k} + \sum_{m=1}^{n} (\Gamma^l_{jm}\Gamma^m_{ki} - \Gamma^l_{km}\Gamma^m_{ji}).$$

Moreover if X, Y and Z are smooth vector fields over U, and if

$$X = \sum_{j=1}^{n} u^j \frac{\partial}{\partial x^j}, \qquad Y = \sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k}, \qquad Z = \sum_{i=1}^{n} w^i \frac{\partial}{\partial x^i},$$

then

$$T(X, Y) = \sum_{j,k,l=1}^{n} (\Gamma^l_{jk} - \Gamma^l_{kj}) u^j v^k \frac{\partial}{\partial x^l}$$

and

$$R(X, Y)Z = \sum_{i,j,k,l=1}^{n} R^l{}_{ijk} w^i u^j v^k \frac{\partial}{\partial x^l}.$$

Proof The formula for the components of the torsion tensor follows directly from the definition of that tensor. The formula for the coefficients $R^m{}_{ijk}$ of the curvature tensor is a special case of the formula for the curvature of a smooth connection stated in Proposition 8.5. In order to verify it directly, we note that

$$\left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right] = 0$$

(see Corollary 7.8). Therefore

$$\begin{aligned}
R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^i} &= \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i} \\
&= \sum_{l=1}^{n}\left(\nabla_{\frac{\partial}{\partial x^j}}\left(\Gamma^l_{ki}\frac{\partial}{\partial x^l}\right) - \nabla_{\frac{\partial}{\partial x^k}}\left(\Gamma^l_{ji}\frac{\partial}{\partial x^l}\right)\right) \\
&= \sum_{l=1}^{n}\left(\frac{\partial \Gamma^l_{ki}}{\partial x^j} - \frac{\partial \Gamma^l_{ji}}{\partial x^k}\right)\frac{\partial}{\partial x^l} \\
&\qquad + \sum_{l=1}^{n}\left(\Gamma^l_{ki}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l} - \Gamma^l_{ji}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}\right) \\
&= \sum_{l=1}^{n}\left(\frac{\partial \Gamma^l_{ki}}{\partial x^j} - \frac{\partial \Gamma^l_{ji}}{\partial x^k}\right)\frac{\partial}{\partial x^l} \\
&\qquad + \sum_{l,m=1}^{n}\left(\Gamma^m_{jl}\Gamma^l_{ki} - \Gamma^m_{kl}\Gamma^l_{ji}\right)\frac{\partial}{\partial x^m} \\
&= \sum_{l=1}^{n}\left(\frac{\partial \Gamma^l_{ki}}{\partial x^j} - \frac{\partial \Gamma^l_{ji}}{\partial x^k} + \sum_{m=1}^{n}\left(\Gamma^l_{jm}\Gamma^m_{ki} - \Gamma^l_{km}\Gamma^m_{ji}\right)\right)\frac{\partial}{\partial x^l},
\end{aligned}$$

as required. ∎

8.6 Covariant Derivatives of Tensor Fields

The dual of the tangent bundle $\pi_{TM}\colon TM \to M$ of a smooth manifold is the cotangent bundle $\pi_{T^*M}\colon T^*M \to M$. Smooth sections of this cotangent bundle represent smooth 1-forms on the smooth manifold M. It follows from Proposition 8.7 that an affine connection on M induces a connection on the cotangent bundle of M. The following lemma summarizes the basis properties of this connection.

Lemma 8.12 *Let ∇ be a smooth affine connection on a smooth manifold M. Then the affine connection ∇ on the tangent bundle $\pi_{TM}\colon TM \to M$ induces a smooth connection on the cotangent bundle $\pi_{T^*M}\colon T^*M \to M$. This connection is defined so that if U is an open set in M, if X and Y are smooth vector fields defined over U, and if φ is a smooth 1-form on U, then*

$$X[\langle \varphi, Y\rangle] = \langle \nabla_X \varphi, Y\rangle + \langle \varphi, \nabla_X Y\rangle.$$

If $(x^1, x^2, \ldots, x^n)$ is a smooth coordinate system defined over an open set U

in M, and if

$$\nabla_{\frac{\partial}{\partial x^j}} \left(\frac{\partial}{\partial x^k}\right) = \sum_{l=1}^n \Gamma^l_{jk} \frac{\partial}{\partial x^l}$$

for $j, k = 1, 2, \ldots, n$, then

$$\nabla_{\frac{\partial}{\partial x^j}} dx^l = -\sum_{k=1}^n \Gamma^l_{jk}\, dx^k.$$

Proof The existence and basic properties of the connection on the cotangent bundle of M induced by ∇ follow on applying Proposition 8.7. It follows from the definition of this induced connection that

$$\begin{aligned} \left\langle \nabla_{\frac{\partial}{\partial x^j}} dx^l, \frac{\partial}{\partial x^k} \right\rangle &= \frac{\partial}{\partial x^j} \left\langle dx^l, \frac{\partial}{\partial x^k} \right\rangle - \left\langle dx^l, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \right\rangle \\ &= \frac{\partial}{\partial x^j}(\delta^l_k) - \left\langle dx^l, \sum_{m=1}^n \Gamma^m_{jk} \frac{\partial}{\partial x^m} \right\rangle \\ &= -\Gamma^l_{jk} \end{aligned}$$

where δ^l_k is the Kronecker delta, equal to 1 when $l = k$, and equal to zero otherwise. Thus

$$\nabla_{\frac{\partial}{\partial x^j}} dx^l = -\sum_{k=1}^n \Gamma^l_{jk}\, dx^k,$$

as required. ∎

The following proposition establishes the standard formula for the covariant derivative of a tensor field of type (r, s) on a smooth manifold, when expressed in terms of components with respect to a local coordinate system.

Proposition 8.13 *Let ∇ be a smooth affine connection on a smooth manifold M, and let $(x^1, x^2, \ldots, ^n)$ be a smooth local coordinate system defined over an open subset U of M. Let S be a smooth tensor field of type (r, s) defined over U, where*

$$\begin{aligned} S = \sum_{j_1, j_2, \ldots, j_r = 1}^n \sum_{k_1, k_2, \ldots, k_r = 1}^n & \Big(S^{j_1, j_2, \ldots, j_r}_{k_1, k_2, \ldots, k_s} \frac{\partial}{\partial x^{j_1}} \otimes \frac{\partial}{\partial x^{j_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \\ & \otimes dx^{k_1} \otimes dx^{k_2} \otimes \cdots \otimes dx^{k_s} \Big) \end{aligned}$$

where the components $S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s}$ *of* S *are smooth real-valued functions on* U*. Then the covariant derivative of* S *with respect to the induced connection on the smooth vector bundle over* M *whose sections are tensor fields of type* (r,s) *is determined with respect to the smooth local coordinate system* $(x^1,x^2,\ldots,x^n)$ *by the following formula:—*

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^m}} S &= \sum_{j_1,j_2,\ldots,j_r=1}^{n} \sum_{k_1,k_2,\ldots,k_r=1}^{n} \Bigl(S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s;m} \frac{\partial}{\partial x^{j_1}} \otimes \frac{\partial}{\partial x^{j_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \\ &\qquad \otimes dx^{k_1} \otimes dx^{k_2} \otimes \cdots \otimes dx^{k_s} \Bigr),\end{aligned}$$

where

$$\begin{aligned} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s;m} &= \frac{\partial S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s}}{\partial x^m} \\ &\quad + \sum_{l=1}^{n} \left(\Gamma^{j_1}_{ml} S^{l,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} + \Gamma^{j_2}_{ml} S^{j_1,l,\ldots,j_r}_{k_1,k_2,\ldots,k_s} + \cdots + \Gamma^{j_r}_{ml} S^{j_1,j_2,\ldots,l}_{k_1,k_2,\ldots,k_s} \right) \\ &\quad - \sum_{l=1}^{n} \left(\Gamma^{l}_{mk_1} S^{j_1,j_2,\ldots,j_r}_{l,k_2,\ldots,k_s} + \Gamma^{l}_{mk_2} S^{j_1,j_2,\ldots,j_r}_{k_1,l,\ldots,k_s} + \cdots + \Gamma^{l}_{mk_s} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,l} \right). \end{aligned}$$

Proof We note that

$$S = \sum_{j_1,j_2,\ldots,j_r=1}^{n} \sum_{k_1,k_2,\ldots,k_r=1}^{n} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} P_{j_1,j_2,\ldots,j_r} \otimes Q^{k_1,k_2,\ldots,k_s},$$

where

$$\begin{aligned} P_{j_1,j_2,\ldots,j_r} &= \frac{\partial}{\partial x^{j_1}} \otimes \frac{\partial}{\partial x^{j_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \\ Q^{k_1,k_2,\ldots,k_s} &= dx^{k_1} \otimes dx^{k_2} \otimes \cdots \otimes dx^{k_s} \end{aligned}$$

for $j_1,j_2,\ldots,j_r,k_1,k_2,\ldots,k_s = 1,2,\ldots,n$. Then

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^m}} P_{j_1,j_2,\ldots,j_r} &= \left(\nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^{j_1}} \right) \otimes \frac{\partial}{\partial x^{j_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \\ &\quad + \frac{\partial}{\partial x^{j_1}} \otimes \left(\nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^{j_2}} \right) \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \\ &\quad + \cdots + \frac{\partial}{\partial x^{j_1}} \otimes \frac{\partial}{\partial x^{j_2}} \otimes \cdots \otimes \left(\nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^{j_2}} \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{n}\Big(\Gamma^{l}_{mj_1}\frac{\partial}{\partial x^{l}}\otimes\frac{\partial}{\partial x^{j_2}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_r}} \\
&\quad + \Gamma^{l}_{mj_2}\frac{\partial}{\partial x^{j_1}}\otimes\frac{\partial}{\partial x^{l}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_r}} \\
&\quad + \cdots + \Gamma^{l}_{mj_r}\frac{\partial}{\partial x^{j_1}}\otimes\frac{\partial}{\partial x^{j_2}}\otimes\cdots\otimes\frac{\partial}{\partial x^{l}}\Big) \\
&= \sum_{l=1}^{n}\Big(\Gamma^{l}_{mj_1}P_{l,j_2,\ldots,j_r} + \Gamma^{l}_{mj_2}P_{j_1,l,\ldots,j_r} \\
&\quad + \cdots + \Gamma^{l}_{mj_r}P_{j_1,j_2,\ldots,l}\Big).
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial x^m}} Q^{k_1,k_2,\ldots,k_s} &= \Big(\nabla_{\frac{\partial}{\partial x^m}} dx^{k_1}\Big)\otimes dx^{k_2}\otimes\cdots\otimes dx^{k_s} \\
&\quad + dx^{k_1}\otimes\Big(\nabla_{\frac{\partial}{\partial x^m}} dx^{k_2}\Big)\otimes\cdots\otimes dx^{k_s} \\
&\quad + \cdots + dx^{k_1}\otimes dx^{k_2}\otimes\cdots\otimes\Big(\nabla_{\frac{\partial}{\partial x^m}} dx^{k_s}\Big) \\
&= -\sum_{l=1}^{n}\Big(\Gamma^{k_1}_{ml}\,dx^{l}\otimes dx^{k_2}\otimes\cdots\otimes dx^{k_s} \\
&\quad + \Gamma^{k_2}_{ml}\,dx^{k_1}\otimes dx^{l}\otimes\cdots\otimes dx^{k_s} \\
&\quad + \cdots + \Gamma^{k_s}_{ml}\,dx^{k_1}\otimes dx^{k_2}\otimes\cdots\otimes dx^{l}\Big) \\
&= -\sum_{l=1}^{n}\Big(\Gamma^{k_1}_{ml}Q^{l,k_2,\ldots,k_s} + \Gamma^{k_2}_{ml}Q^{k_1,l,\ldots,k_s} \\
&\quad + \cdots + \Gamma^{k_s}_{ml}Q^{k_1,k_2,\ldots,l}\Big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial x^m}} S &= \sum_{j_1,j_2,\ldots,j_r=1}^{n}\;\sum_{k_1,k_2,\ldots,k_r=1}^{n}\Big(\frac{\partial S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s}}{\partial x^m}P_{j_1,j_2,\ldots,j_r}\otimes Q^{k_1,k_2,\ldots,k_s} \\
&\quad + S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s}\,\nabla_{\frac{\partial}{\partial x^m}}P_{j_1,j_2,\ldots,j_r}\otimes Q^{k_1,k_2,\ldots,k_s} \\
&\quad + S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s}\,P_{j_1,j_2,\ldots,j_r}\otimes\nabla_{\frac{\partial}{\partial x^m}}Q^{k_1,k_2,\ldots,k_s}\Big) \\
&= \sum_{j_1,j_2,\ldots,j_r=1}^{n}\;\sum_{k_1,k_2,\ldots,k_r=1}^{n}\Big(\frac{\partial S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s}}{\partial x^m}P_{j_1,j_2,\ldots,j_r}\otimes Q^{k_1,k_2,\ldots,k_s}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{n} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} \Big(\Gamma^l_{mj_1} P_{l,j_2,\ldots,j_r} + \Gamma^l_{mj_2} P_{j_1,l,\ldots,j_r} \\
&\quad + \cdots + \Gamma^l_{mj_r} P_{j_1,j_2,\ldots,l}\Big) \otimes Q^{k_1,k_2,\ldots,k_s} \\
&-\sum_{l=1}^{n} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} P_{j_1,j_2,\ldots,j_r} \otimes \Big(\Gamma^{k_1}_{ml} Q^{l,k_2,\ldots,k_s} + \Gamma^{k_2}_{ml} Q^{k_1,l,\ldots,k_s} \\
&\quad + \cdots + \Gamma^{k_s}_{ml} Q^{k_1,k_2,\ldots,l}\Big)\Big),
\end{aligned}
$$

Now it follows on relabelling indices of summation that

$$
\begin{aligned}
\sum_{j_1,j_2,\ldots,j_r,l=1}^{n} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} \Gamma^l_{mj_1} P_{l,j_2,\ldots,j_r} &= \sum_{j_1,j_2,\ldots,j_r,l=1}^{n} S^{l,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} \Gamma^{j_1}_{ml} P_{j_1,j_2,\ldots,j_r} \\
\sum_{j_1,j_2,\ldots,j_r,l=1}^{n} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} \Gamma^l_{mj_2} P_{j_1,l,\ldots,j_r} &= \sum_{j_1,j_2,\ldots,j_r,l=1}^{n} S^{j_1,l,\ldots,j_r}_{k_1,k_2,\ldots,k_s} \Gamma^l_{mj_2} P_{j_1,j_2,\ldots,j_r}
\end{aligned}
$$

etc. Therefore

$$
\nabla_{\frac{\partial}{\partial x^m}} S = \sum_{j_1,j_2,\ldots,j_r=1}^{n} \sum_{k_1,k_2,\ldots,k_r=1}^{n} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s;m} P_{j_1,j_2,\ldots,j_r} \otimes Q^{k_1,k_2,\ldots,k_s}
$$

where $S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s;m}$ is defined as in the statement of the proposition. ∎

Example Let ∇ be a smooth affine connection on a smooth manifold M, and let $(x^1, x^2, \ldots, ^n)$ be a smooth local coordinate system defined over an open subset U of M. Let H be a smooth tensor field of type $(0, 2)$ defined over U, where

$$
H = \sum_{j,k=1}^{n} H_{jk}\, dx^j \otimes dx^k.
$$

Then

$$
\nabla_{\frac{\partial}{\partial x^m}} H = \sum_{j,k=1}^{n} H_{jk;m}\, dx^j \otimes dx^k,
$$

where

$$
H_{jk;m} = \frac{\partial H_{jk}}{\partial x^m} - \sum_{l=1}^{n} \left(\Gamma^l_{mj} H_{lk} + \Gamma^l_{mk} H_{jl}\right).
$$

We can verify this identity using the basic method employed in the proof of Proposition 8.13. Now smooth tensor fields of type $(0, 2)$ are by definition smooth sections of the vector bundle $T^*M \otimes T^*M$. The induced connection on this bundle is defined as described in the statement of Proposition 8.9. It follows that

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^m}} H &= \sum_{j,k=1}^{n} \frac{\partial H_{jk}}{\partial x^m}\, dx^j \otimes dx^k \\ &\qquad + \sum_{j,k=1}^{n} H_{jk} \Bigl(\bigl(\nabla_{\frac{\partial}{\partial x^m}} dx^j \bigr) \otimes dx^k + dx^j \otimes \bigl(\nabla_{\frac{\partial}{\partial x^m}} dx^k \bigr) \Bigr)\end{aligned}$$

But it follows from Lemma 8.12 that

$$\nabla_{\frac{\partial}{\partial x^m}} dx^j = -\sum_{q=1}^{n} \Gamma^j_{mq}\, dx^q.$$

Therefore

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^m}} H &= \sum_{j,k=1}^{n} \frac{\partial H_{jk}}{\partial x^m}\, dx^j \otimes dx^k \\ &\qquad - \sum_{j,k,l=1}^{n} \Bigl(\Gamma^j_{ml} H_{jk}\, dx^l \otimes dx^k + \Gamma^k_{ml} H_{jk}\, dx^j \otimes dx^l \Bigr) \\ &= \sum_{j,k=1}^{n} \frac{\partial H_{jk}}{\partial x^m}\, dx^j \otimes dx^k \\ &\qquad - \sum_{j,k,l=1}^{n} \Bigl(\Gamma^l_{mj} H_{lk}\, dx^j \otimes dx^k + \Gamma^l_{mk} H_{jl}\, dx^j \otimes dx^k \Bigr) \\ &= \sum_{j,k=1}^{n} \left(\frac{\partial H_{jk}}{\partial x^m} - \sum_{l=1}^{n} \bigl(\Gamma^l_{mj} H_{lk} + \Gamma^l_{mk} H_{jl} \bigr) \right) dx^j \otimes dx^k \\ &= \sum_{j,k=1}^{n} H_{jk;m}\, dx^j \otimes dx^k.\end{aligned}$$

Example Let ∇ be a smooth affine connection on a smooth manifold M, and let $(x^1, x^2, \ldots, {}^n)$ be a smooth local coordinate system defined over an open subset U of M. Let W be a smooth tensor field of type $(1, 3)$ defined over U, where

$$W = \sum_{i,j,k,l=1}^{n} W^l{}_{ijk} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k.$$

Then

$$\nabla_{\frac{\partial}{\partial x^m}} W = \sum_{i,j,k,l=1}^{n} W^l{}_{ijk;m} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k,$$

where

$$W^l_{ijk;m} = \frac{\partial W^l{}_{ijk}}{\partial x^m} + \sum_{q=1}^{n} \left(\Gamma^l_{mq} W^q{}_{ijk} - \Gamma^q_{mi} W^l{}_{qjk} - \Gamma^q_{mj} W^l{}_{iqk} - \Gamma^q_{mk} W^l{}_{ijq} \right).$$

We can verify this identity using the basic method employed in the proof of Proposition 8.13. Now smooth tensor fields of type $(1,3)$ are by definition smooth sections of the vector bundle $TM \otimes T^*M \otimes T^*M \otimes T^*M$. The induced connection on this bundle is defined as described in the statement of Proposition 8.9. It follows that

$$\begin{aligned} \nabla_X W &= \sum_{i,j,k,l=1}^{n} X[W^l{}_{ijk}] \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k \\ &\quad + \sum_{i,j,k,l=1}^{n} W^l{}_{ijk} \Bigg(\left(\nabla_X \frac{\partial}{\partial x^l} \right) \otimes dx^i \otimes dx^j \otimes dx^k \\ &\quad + \frac{\partial}{\partial x^l} \otimes \left(\nabla_X dx^i \right) \otimes dx^j \otimes dx^k \\ &\quad + \frac{\partial}{\partial x^l} \otimes dx^i \otimes \left(\nabla_X dx^j \right) \otimes dx^k \\ &\quad + \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes \left(\nabla_X dx^k \right) \Bigg) \end{aligned}$$

for all smooth vector fields X on M, and therefore

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^m}} W &= \sum_{i,j,k,l=1}^{n} \frac{\partial W^l{}_{ijk}}{\partial x^m} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k \\ &\quad + \sum_{i,j,k,l,q=1}^{n} \Bigg(\Gamma^q_{ml} W^l{}_{ijk} \frac{\partial}{\partial x^q} \otimes dx^i \otimes dx^j \otimes dx^k \\ &\quad - \Gamma^i_{mq} W^l{}_{ijk} \frac{\partial}{\partial x^l} \otimes dx^q \otimes dx^j \otimes dx^k \\ &\quad - \Gamma^j_{mq} W^l{}_{ijk} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^q \otimes dx^k \\ &\quad - \Gamma^k_{mq} W^l{}_{ijk} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^q \Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,k,l=1}^{n} \frac{\partial W^l{}_{ijk}}{\partial x^m} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k \\
&\quad + \sum_{i,j,k,l,q=1}^{n} \Big(\Gamma^l_{mq} W^q{}_{ijk} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k \\
&\qquad - \Gamma^q_{mi} W^l{}_{qjk} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k \\
&\qquad - \Gamma^q_{mj} W^l{}_{iqk} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k \\
&\qquad - \Gamma^q_{mk} W^l{}_{ijq} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k \Big) \\
&= \sum_{i,j,k,l=1}^{n} W^l{}_{ijk;m} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k .
\end{aligned}
$$

This formula for the covariant derivative of a smooth tensor field of type $(1,3)$ can also be established by applying Proposition 8.8.

8.7 The First Bianchi Identity

Proposition 8.14 (First Bianchi Identity) *Let ∇ be a torsion-free affine connection on a smooth manifold M. Let R denote the curvature operator of ∇. Then*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Proof The connection ∇ is torsion-free, hence $\nabla_X Y - \nabla_Y X = [X,Y]$ for all vector fields X and Y on M. Therefore

$$
\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&\quad = \nabla_X \nabla_Y Z + \nabla_Y \nabla_Z X + \nabla_Z \nabla_X Y \\
&\qquad - \nabla_Y \nabla_X Z - \nabla_Z \nabla_Y X - \nabla_X \nabla_Z Y \\
&\qquad - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\
&\quad = \nabla_X (\nabla_Y Z - \nabla_Z Y) + \nabla_Y (\nabla_Z X - \nabla_X Z) \\
&\qquad + \nabla_Z (\nabla_X Y - \nabla_Y Z) \\
&\qquad - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\
&\quad = \nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y] \\
&\qquad - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\
&\quad = 0. \quad \blacksquare
\end{aligned}
$$

Let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system defined over an open set U in M, and let the smooth real-valued functions $(R^l{}_{ijk})$ be defined such that

$$R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^i} = \sum_{l=1}^{n} R^l{}_{ijk}\frac{\partial}{\partial x^l}.$$

The *First Bianchi Identity* then ensures that

$$R^l{}_{ijk} + R^l{}_{jki} + R^k{}_{kij} = 0.$$

8.8 The Second Bianchi Identity

Let D be a smooth connection on a smooth principal bundle $\pi_E\colon E \to M$. Then the connection D on E and the corresponding connection on the dual bundle $\pi_{E^*}\colon E^* \to M$ of E induce a smooth connection on the bundle $\mathrm{End}(E)$ whose fibre over a point p of M is the space of linear operators on the fibre E_p of the vector bundle E (see Proposition 8.8 and Proposition 8.9). This connection on $\mathrm{End}(E)$ is defined such that

$$(D_X K)(s) = D_X(K(s)) - K(D_X s)$$

for all smooth sections K of $\mathrm{End}(E)$, smooth sections s of E and smooth vector fields X on M. It then follows from Proposition 8.8, or Proposition 8.9), that if ∇ is an affine connection on M, then the connection on $\mathrm{End}(E)$ and the affine connection ∇ on M induce a connection on the vector bundle $\mathrm{End}(E) \otimes \bigwedge^2 T^*M$, where

$$\begin{aligned}(D_X Q)(Y, Z)(s) &= D_X(Q(Y, Z)s) - Q(\nabla_X Y, Z)s - Q(Y, \nabla_X Z)s \\ &\quad - Q(X, Z)(D_X s)\end{aligned}$$

for all smooth sections s of $\pi_E\colon E \to M$ and for all smooth vector fields X, Y and Z on M.

Proposition 8.15 *Let D be a smooth connection defined on a smooth vector bundle $\pi_E\colon E \to M$ over a smooth manifold M, let F_D be the curvature of D, and let ∇ be a torsion-free affine connection on M. Then*

$$(D_X F_D)(Y, Z)s + (D_Y F_D)(Z, X)s + (D_Z F_D)(X, Y)s = 0$$

for all smooth sections s of $\pi_E\colon E \to M$ and smooth vector fields X, Y and Z on M, where

$$\begin{aligned}(D_X F_D)(Y, Z)s &= D_X(F_D(Y, Z)s) - F_D(\nabla_X Y, Z)s - F_D(Y, \nabla_X Z)s \\ &\quad - F_D(X, Y)(D_X s).\end{aligned}$$

Proof It follows from Proposition 8.6 that

$$\begin{aligned} D_X(F_D(Y,Z)s) &+ D_Y(F_D(Z,X)s) + D_Z(F_D(X,Y)s) \\ &= F_D([Y,Z],X)s + F_D([X,Z],Y)s + F_D([X,Y],Z)s \\ &\quad + F_D(Y,Z)(D_X s) + F_D(Z,X)(D_Y s) + F_D(X,Y)(D_Z s). \end{aligned}$$

But $[X,Y] = \nabla_X Y - \nabla_Y X$, etc. because the affine connection ∇ is torsion-free. Also $F_D(X,Y)s = -F_D(Y,X)s$. It follows that

$$\begin{aligned} D_X(F_D(Y,Z)s) &+ D_Y(F_D(Z,X)s) + D_Z(F_D(X,Y)s) \\ &= F_D(\nabla_Y Z, X)s - F_D(\nabla_Z Y, X)s \\ &\quad + F_D(\nabla_Z X, Y)s - F_D(\nabla_X Z, Y)s \\ &\quad + F_D(\nabla_X Y, Z)s - F_D(\nabla_Y X, Z)s \\ &\quad + F_D(Y,Z)(D_X s) + F_D(Z,X)(D_Y s) + F_D(X,Y)(D_Z s) \\ &= F_D(\nabla_X Y, Z)s + F_D(Y, \nabla_X Z)s + F_D(Y,Z)(D_X s) \\ &\quad + F_D(\nabla_Y Z, X)s + F_D(Z, \nabla_Y X)s + F_D(Z,X)(D_Y s) \\ &\quad + F_D(\nabla_Z X, Y)s + F_D(X, \nabla_Z Y)s + F_D(X,Y)(D_Z s), \end{aligned}$$

and thus

$$(D_X F_D)(Y,Z)s + (D_Y F_D)(Z,X)s + (D_Z F_D)(X,Y)s = 0,$$

as required. ∎

We can apply Proposition 8.15 in the special case when the vector bundle E over M is the tangent bundle M, and where the smooth connection is a torsion-free affine connection on M. That proposition then yields the following result.

Corollary 8.16 (Second Bianchi Identity) *Let ∇ be a torsion-free affine connection on a smooth manifold M. Then*

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0$$

for all smooth vector fields X, Y, Z and W on M, where R is the curvature tensor of the connection ∇, and where

$$\begin{aligned} (\nabla_X R)(Y,Z)W &= \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W \\ &\quad - R(X,Y)(\nabla_X W). \end{aligned}$$

9 Riemannian and Pseudo-Riemannian Manifolds

9.1 Riemannian and Pseudo-Riemannian Metrics

Definition Let M be a smooth manifold. A *metric tensor* g on M is a tensor field on M that assigns to each point p of M a non-degenerate symmetric bilinear form g_p on the tangent space T_pM to M at p.

Lemma 9.1 *Let M be a smooth manifold, and let g be a metric tensor on M. Then*

$$\begin{aligned} g(X_p + Y_p, Z_p) &= g(X_p, Z_p) + g(Y_p, Z_p), \\ g(X_p, Y_p + Z_p) &= g(X_p, Y_p) + g(X_p, Z_p), \\ g(c\,X_p, Y_p) &= g(X_p, cY_p) = cg(X_p, Y_p), \\ g(X_p, Y_p) &= g(Y_p, X_p) \end{aligned}$$

for all $p \in M$ and $X_p, Y_p, Z_p \in T_pM$, and for all real numbers c. Moreover, given any non-zero tangent vector X_p at some point p of M, there exists some tangent vector Y_p at p such that $g(X_p, Y_p) \neq 0$. Also, given any element θ_p of the cotangent space T_p^*M of M at p, there exists some tangent vector $\theta_p^\sharp$ which satisfies

$$g(\theta_p^\sharp, Y_p) = \theta_p(Y_p) = \langle \theta_p, Y_p \rangle$$

for all $Y_p \in T_pM$.

Proof The given identities represent the fact that the metric tensor is a symmetric bilinear form on each tangent space. The definition of non-degeneracy for bilinear forms on a vector space requires that, given any tangent vector $X \in T_pM$, there exists some tangent vector $Y \in T_pM$ such that $g(X_p, Y_p) \neq 0$. Let $\lambda_p \colon T_pM \to T_p^*M$ be the linear transformation defined such that $\lambda_p(X_p) = X_p^\flat$, where

$$\langle X_p^\flat, Y_p \rangle = g_p(X_p, Y_p)$$

for all $Y_p \in T_pM$. Then the non-degeneracy of the bilinear form g_p ensures that the linear transformation λ_p is injective. But the domain T_pM and codomain T_p^*M of this linear transformation have the same dimension. It follows from basic linear algebra that $\lambda_p \colon T_pM \to T_p^*M$ is an isomorphism of vector spaces. Let $\theta_p^\sharp = \lambda_p^{-1}(\theta_p)$ for all $\theta_p \in T^*M$. Then

$$\langle \theta_p, Y_p \rangle = \langle \lambda_p(\theta_p^\sharp), Y_p, \rangle = g_p(\theta_p^\sharp, Y_p)$$

for all $Y_p \in T_pM$, as required. ∎

Definition A *Riemannian metric* on a smooth manifold M is a smooth metric tensor that is positive-definite at each point of M.

Definition A *pseudo-Riemannian metric* on a smooth manifold M is a smooth metric tensor that is nondegenerate but that need not be positive-definite at each point of M.

Definition Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g. The *raising* and *lowering operators* at a point p of M are the isomorphisms $\rho: T_p^*M \to T_pM$ and $\lambda: T_pM \to T_p^*M$ between the tangent and cotangent spaces of M at the point p defined such that $\rho(\theta_p) = \theta_p^\sharp$ and $\lambda(X_p) = X_p^\flat$ for all $\theta_p \in T_p^*M$ and $X_p \in T_pM$, where

$$g(\theta_p^\sharp, Y_p) = \langle \theta_p, Y_p \rangle \quad \text{and} \quad \langle X_p^\flat, Y_p \rangle = g(X_p, Y_p).$$

Let M be a Riemannian or pseudo-Riemannian manifold of dimension n, with metric tensor g, and let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system defined over some open set U in M. Let

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

for $i, j = 1, 2, \ldots, n$. Then the components g_{ij} of the metric tensor g are smooth real-valued functions on U. If M is a Riemannian manifold then the values of these components at each point of U are the entries of a positive-definite symmetric matrix. If M is a pseudo-Riemannian manifold then the values of these components at each point of U are the entries of a symmetric matrix that is non-singular but need not be positive definite.

Let X and Y are smooth vector fields defined over the open set U. Then

$$X = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{i=1}^n w^i \frac{\partial}{\partial x^i},$$

where $v^1, a^2, \ldots, a^n$ and $w^1, b^2, \ldots, b^n$ are smooth functions on U, and therefore

$$g(X, Y) = \sum_{i,j=1}^n g_{ij} v^i w^j.$$

Now, at each point p of M, the matrix with entry g_{ij} in the ith row and jth column is invertible. The inverse of this matrix has entries g^{kl}, where $g^{kl} = g^{lk}$ and

$$\sum_{j=1}^n g_{ij} g^{jk} = \delta_i^k,$$

where δ_i^k denotes the Kronecker delta that has the value 1 when $i = k$, but has the value 0 otherwise.

Lemma 9.2 *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, and let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system for M defined over an open subset U of M, let*

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

for $i, j = 1, 2, \ldots, n$, and let g^{kl} be the smooth functions defined on U so that $g^{kl} = g^{lk}$ and $\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_i^k$, where δ_i^k denotes the Kronecker delta. Let $X_p \in T_pM$ and $\theta_p \in T_p^*M$ be elements of the tangent and cotangent spaces to M at some point p of M, and let $X_p^\flat$ and $\theta_p^\sharp$ be defined such that

$$g(\theta_p^\sharp, Y_p) = \langle \theta_p, Y_p \rangle \quad \text{and} \quad \langle X_p^\flat, Y_p \rangle = g(X_p, Y_p).$$

Let

$$X_p = \sum_{j=1}^{n} a^j \left.\frac{\partial}{\partial x^j}\right|_p \quad \text{and} \quad \theta_p = \sum_{k=1}^{n} c_k \, dx_p^k.$$

Then

$$X_p^\flat = \sum_{j,k=1}^{n} g_{jk} a^j \, dx_p^k \quad \text{and} \quad \theta_p^\sharp = \sum_{j,k=1}^{n} g^{jk} c_k \left.\frac{\partial}{\partial x^j}\right|_p.$$

Proof Let

$$Y_p = \sum_{k=1}^{n} b^k \left.\frac{\partial}{\partial x^k}\right|_p.$$

Then

$$\left\langle \sum_{j,k=1}^{n} g_{jk} a^j \, dx_p^k, \, Y_p \right\rangle = \sum_{j,k=1}^{n} g_{jk} a^j b^k = g(X_p, Y_p) = \langle X_p^\flat, Y_p \rangle.$$

This identity holds for all $Y \in T_pM$, and therefore

$$\sum_{j,k=1}^{n} g_{jk} a^j \, dx_p^k = X_p^\flat$$

Also

$$\begin{aligned} g\left(\sum_{j,k=1}^{n} g^{jk} c_k \left.\frac{\partial}{\partial x^j}\right|_p, \, Y_p\right) &= \sum_{i,j,k=1}^{n} g^{jk} c_k b^i g_{ji} = \sum_{i,k=1}^{n} \delta_i^k c_k b^i \\ &= \sum_{k=1}^{n} c_k b^k = \langle \theta_p, Y_p \rangle = g(\theta_p^\sharp, Y_p). \end{aligned}$$

This identity holds for all $Y \in T_pM$, and therefore

$$\sum_{j,k=1}^{n} g^{jk} c_k \left. \frac{\partial}{\partial x^j} \right|_p = \theta_p^\sharp,$$

as required. ∎

Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system for M, defined over some open subset U of M, let g_{ij} be the components of the metric tensor g with respect to this coordinate system, so that

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

and let g^{ij} be the smooth functions on U defined such that $\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_i^k$, where δ_i^k denotes the Kronecker delta.

Let X_p be a tangent vector at some point p of U, and let

$$X_p = \sum_{j=1}^{n} a^j \left. \frac{\partial}{\partial x^j} \right|_p.$$

When using traditional index notation it is customary to denote the components of $X_p^\flat$ by $a_1, a_2, \ldots, a_n$, so that

$$X_p^\flat = \sum_{k=1}^{n} a_k \, dx_p^k.$$

Similarly given an element θ_p of the cotangent space T_p^*M at p, where

$$\theta_p = \sum_{k=1}^{n} c_k \, dx_p^k,$$

it is customary to denote the components of $\theta_p^\sharp$ by $c^1, c^2, \ldots, x^n$, so that

$$\theta_p^\sharp = \sum_{j=1}^{n} c^j \left. \frac{\partial}{\partial x^j} \right|_p.$$

It follows from Lemma 9.2 that

$$a_k = \sum_{j=1}^{n} g_{jk} a^j \quad \text{and} \quad c^j = \sum_{k=1}^{n} g^{jk} c_k.$$

It is also common practice, when using traditional index notation for tensor fields, to adopt analogous operations of raising and lowering indices in order to convert between tensors of different types.

Example Let S be a tensor field S of type $(1,3)$ on a Riemannian or pseudo-Riemannian manifold M. Then S is represented at each point p of M by a trilinear map $S\colon T_pM \times T_pM \times T_pM \to T_pM$. Now the metric tensor g on M determines an isomorphism between the vector space of such trilinear maps and the vector space of quadrilinear forms on T_pM. This isomorphism sends the trilinear map S to the quadrilinear form $S^\flat$, where

$$S^\flat(W_p, X_p, Y_p, Z_p) = g(W_p, R(X_p, Y_p, Z_p))$$

for all $W_p, X_p, Y_p, Z_p \in T_pM$. If

$$S = \sum_{l,i,j,k=1}^{n} S^l{}_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$$

then

$$S^\flat = \sum_{h,i,j,k=1}^{n} S_{hijk}\, dx^h \otimes dx^i \otimes dx^j \otimes dx^k,$$

where

$$S_{hijk} = \sum_{l=1}^{n} g_{hl} S^l{}_{ijk}.$$

Then

$$S^l{}_{ijk} = \sum_{h=1}^{n} g^{lh} S_{hijk},$$

where the functions g^{mh} are defined such that $\sum_{h=1}^{n} g^{mh} g_{hl} = \delta^m_l$.

9.2 The Levi-Civita Connection

Let M be a Riemannian or pseudo-Riemannian manifold, with metric tensor g, and let ∇ be an affine connection on M. We say that ∇ is *compatible with* the metric tensor g if

$$Z[g(X,Y)] = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

for all smooth vector fields X, Y and Z on M. We shall show that on every pseudo-Riemannian manifold there exists a unique torsion-free connection that is compatible with the metric tensor.

Lemma 9.3 *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, and let ∇ be a torsion-free affine connection on M that is compatible with the metric tensor g. Then*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X,Y)] \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned}$$

Proof Let X, Y and Z be smooth vector fields on M. The requirement that ∇ be both torsion-free and compatible with the metric tensor ensures that

$$\begin{aligned} g([X,Y],Z) &= g(\nabla_X Y, Z) - g(\nabla_Y X, Z), \\ g([Y,Z],X) &= g(\nabla_Y Z, X) - g(\nabla_Z Y, X), \\ g([Z,X],Y) &= g(\nabla_Z X, Y) - g(\nabla_X Z, Y), \\ X[g(Y,Z)] &= g(\nabla_X Y, Z) + g(\nabla_X Z, Y), \\ Y[g(Z,X)] &= g(\nabla_Y Z, X) + g(\nabla_Y X, Z), \\ Z[g(X,Y)] &= g(\nabla_Z X, Y) + g(\nabla_Z Y, X). \end{aligned}$$

Thus if

$$\begin{aligned} \chi(X,Y,Z) &= \tfrac{1}{2}\Big(X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X,Y)] \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \Big) \end{aligned}$$

then

$$\begin{aligned} 2\chi(X,Y,Z) &= g(\nabla_X Y, Z) + g(\nabla_X Z, Y) + g(\nabla_Y Z, X) + g(\nabla_Y X, Z) \\ &\quad - g(\nabla_Z X, Y) - g(\nabla_Z Y, X) + g(\nabla_X Y, Z) - g(\nabla_Y X, Z) \\ &\quad + g(\nabla_Z X, Y) - g(\nabla_X Z, Y) - g(\nabla_Y Z, X) + g(\nabla_Z Y, X) \\ &= 2g(\nabla_X Y, Z). \end{aligned}$$

The result follows directly. ∎

Lemma 9.3 shows that a Riemannian or pseudo-Riemannian manifold can have at most one torsion-free affine connection that is compatible with the metric tensor. We now show that there exists a smooth affine connection on any Riemannian or pseudo-Riemannian manifold which is characterized by the identity given in the statement of Lemma 9.3.

Theorem 9.4 *Let (M, g) be a Riemannian or pseudo-Riemannian manifold. Then there exists a unique torsion-free affine connection ∇ on M compatible with the metric tensor g. This connection is characterized by the identity*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X,Y)] \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned}$$

for all smooth vector fields X, Y and Z on M.

Proof Given smooth vector fields X, Y and Z on M, let $\chi(X, Y, Z)$ be the smooth function on M defined by

$$\begin{aligned} \chi(X,Y,Z) &= \tfrac{1}{2}(X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X,Y)] \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)). \end{aligned}$$

Then

$$\begin{aligned} \chi(X_1 + X_2, Y, Z) &= \chi(X_1, Y, Z) + \chi(X_2, Y, Z), \\ \chi(X, Y_1 + Y_2, Z) &= \chi(X, Y_1, Z) + \chi(X, Y_2, Z), \\ \chi(X, Y, Z_1 + Z_2) &= \chi(X, Y, Z_1) + \chi(X, Y, Z_2) \end{aligned}$$

for all smooth vector fields X, X_1, X_2, Y, Y_1, Y_2, Z, Z_1 and Z_2 on M. Now

$$[X, fY] = f\,[X,Y] + X[f]\,Y \quad \text{and} \quad [fX, Y] = f\,[X,Y] - Y[f]\,X$$

for all smooth real-valued functions f and smooth vector fields X and Y on M (see Lemma 7.6). It follows that

$$\begin{aligned} \chi(fX,Y,Z) &= f\,\chi(X,Y,Z) + \tfrac{1}{2}\Big(Y[f]\,g(X,Z) - Z[f]\,g(X,Y) \\ &\qquad - Y[f]\,g(X,Z) + Z[f]\,g(Y,X)\Big) \\ &= f\,\chi(X,Y,Z), \\ \chi(X,fY,Z) &= f\,\chi(X,Y,Z) + \tfrac{1}{2}\Big(X[f]\,g(Y,Z) - Z[f]\,g(X,Y) \\ &\qquad + X[f]\,g(Y,Z) + Z[f]\,g(Y,X)\Big) \\ &= f\,\chi(X,Y,Z) + X[f]\,g(Y,Z), \\ \chi(X,Y,fZ) &= f\,\chi(X,Y,Z) + \tfrac{1}{2}\Big(X[f]\,g(Y,Z) + Y[f]\,g(X,Z) \\ &\qquad - X[f]\,g(Z,Y) - Y[f]\,g(Z,X)\Big) \\ &= f\,\chi(X,Y,Z) \end{aligned}$$

for all smooth real-valued functions f and smooth vector fields X, Y and Z on M. An application of Corollary 6.16 shows that each smooth vector field Y on M determines a well-defined bilinear form $\mu_Y\colon T_pM \times T_pM \to \mathbb{R}$ on the tangent space T_pM at each point p of M characterized by the property that $\chi(X, Y, Z) = \mu_Y(X_p, Z_p)$ for all smooth vector fields X and Z on M. Given a smooth vector field Y defined around a point p of M, and given a tangent vector $X_p \in T_pM$ at p, let $\mu_{X_p,Y}$ be the element of the cotangent space T_p^*M defined such that $\langle \mu_{X_p,Y}, Z_p\rangle = \mu_Y(X_p, Z_p)$ for all $Z_p \in T_pM$, and let $\nabla_{X^p} Y = \mu^\sharp_{X_p,Y}$, so that

$$g(\nabla_{X^p} Y, Z_p) = g(\mu^\sharp_{X_p,Y}, Z_p) = \langle \mu_{X_p,Y}, Z_p\rangle = \mu_Y(X_p, Z_p)$$

for all $Z_p \in T_pM$. Clearly

$$\nabla_{W_p+X_p} Y = \nabla_{W_p} Y + \nabla_{X_p} Y \quad \text{and} \quad \nabla_{c\,X_p} Y = c\,\nabla_{X_p} Y$$

for all $W_p, X_p \in T_pM$ and for all real numbers c. The identity

$$\chi(X, fY, Z) = f\,\chi(X, Y, Z) + X[f]\,g(Y, Z)$$

ensures that

$$\nabla_{X_p}(Y + Z) = \nabla_{X_p} Y + \nabla_{X_p} Z \quad \text{and} \quad \nabla_{X_p}(fY) = X_p[f]\,Y + f\,\nabla_{X_p} Y$$

for all smooth real-valued functions f and smooth vector fields Y and Z defined around the point p. Moreover $g(\nabla_X Y, Z) = \chi(X, Y, Z)$, and therefore $g(\nabla_X Y, Z)$ is a smooth real-valued function, and therefore $\nabla_X Y$ is a smooth vector field, for all smooth vector fields X, Y and Z defined around the point p. We have thus shown that the differential operator ∇ is a smooth affine connection on M.

Let X, Y and Z be smooth vector fields on M. Then

$$\chi(X, Y, Z) - \chi(Y, X, Z) = g([X, Y], Z).$$

It follows that $\nabla_X Y - \nabla_Y X = [X, Y]$. This shows that the affine connection ∇ is torsion-free. Also

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = \chi(X, Y, Z) + \chi(X, Z, Y) = X[g(Y, Z)],$$

and thus the affine connection ∇ is compatible with the metric tensor. Lemma 9.3 guarantees that this torsion-free affine connection compatible with the metric tensor is uniquely determined, as required. ∎

Definition Let M be a Riemannian or pseudo-Riemannian manifold. The *Levi-Civita connection* on M is the unique smooth torsion-free affine connection on M that is compatible with the metric tensor on M.

Example Let M be a smooth n-dimensional submanifold of k-dimensional Euclidean space $\mathbb{R}^k$. Given (tangential) vector fields X and Y on M, we decompose the directional derivative $\partial_X Y$ of Y along X as $\partial_X Y = \nabla_X Y - S(X, Y)$, where $\nabla_X Y$ is tangential to M and $S(X, Y)$ is orthogonal to M. Then ∇ is a torsion-free affine connection on M. Now the restriction to the tangent spaces of M of the standard scalar product $\langle ., . \rangle$ on $\mathbb{R}^k$ defines a Riemannian metric g on M. Moreover

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = \langle \partial_X Y, Z \rangle + \langle Y, \partial_X Z \rangle = X[\langle Y, Z \rangle] = X[g(Y, Z)]$$

for all vector fields X, Y and Z on M that are everywhere tangential to M. We conclude that the affine connection ∇ is the Levi-Civita connection of the Riemannian manifold M.

Corollary 9.5 *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, and let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system for M defined over an open subset U of M, let*

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

for $i, j = 1, 2, \ldots, n$, and let g^{kl} be the smooth functions defined on U so that $g^{kl} = g^{lk}$ and $\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_i^k$, where δ_i^k denotes the Kronecker delta. Let ∇ be the Levi-Civita connection on M. Then

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \sum_{i=1}^{n} \Gamma^i_{jk} \frac{\partial}{\partial x^i}$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m} \right).$$

Moreover $\Gamma^i_{jk} = \Gamma^i_{kj}$ for all i, j and k, and if X and Y are smooth vector fields on U, and if

$$X = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{i=1}^{n} w^i \frac{\partial}{\partial x^i},$$

then

$$\nabla_X Y = \sum_{i,j=1}^{n} \left(v^j \frac{\partial w^i}{\partial x^j} + \sum_{k=1}^{n} \Gamma^i_{jk} v^j w^k \right) \frac{\partial}{\partial x^i}.$$

Proof The basis vector fields determined by the coordinate system satisfy

$$\left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right] = 0$$

for all j and k. It follows from Theorem 9.4 that

$$g\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}\right) = \frac{1}{2}\left(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m}\right).$$

A straightforward application of Lemma 9.2 yields the formula for the quantities Γ^i_{jk}. Then

$$\Gamma^i_{jk} - \Gamma^i_{kj} = \left\langle dx^i, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\right\rangle = \left\langle dx^i, \left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right]\right\rangle = 0.$$

The formula giving an expression for $\nabla_X Y$ in terms the components of the vector fields X and Y in a local coordinates system is a particular case of the more general result proved in Lemma 8.10. ∎

The quantities Γ^i_{jk} that represent the Levi-Civita connection of a Riemannian or pseudo-Riemannian manifold with respect to a smooth local coordinate system on that manifold are known as *Christoffel symbols*, and can be calculated from the components of the metric tensor and their first derivatives according to the formula given in Corollary 9.5

9.3 The Riemann Curvature Tensor

Definition Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g. The *Riemann curvature tensor* R of M is defined by the formula

$$R(W, Z, X, Y) = g(W, R(X, Y)Z),$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all smooth vector fields X, Y and Z on M, where ∇ denotes the Levi-Civita connection on M.

The Riemann curvature tensor on a Riemannian or pseudo-Riemannian manifold M is thus the curvature tensor of the Levi-Civita connection on M. The value of $R(W, Z, X, Y)$ at a point p of M is determined by the values of the vector fields W, X, Y and Z at that point.

The following proposition is a special case of Proposition 8.11.

Proposition 9.6 . *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system for M defined over an open subset U of M, and let Γ^i_{jk} denote the Christoffel symbols that represent the Levi-Civita connection ∇ determined by the metric tensor g with respect to the smooth local coordinates $x^1, x^2, \ldots, x^n$, so that*

$$\nabla_{\frac{\partial}{\partial x^j}} \left(\frac{\partial}{\partial x^k} \right) = \sum_{l=1}^{n} \Gamma^l_{jk} \frac{\partial}{\partial x^l},$$

on U. Let $R^l{}_{ijk}$ denote the components of the Riemann curvature tensor with respect to this coordinate system, so that

$$R = \sum_{l=1}^{n} R^l{}_{ijk} \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j \otimes dx^k$$

on U. Then

$$R^l{}_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial x^j} - \frac{\partial \Gamma^l_{ji}}{\partial x^k} + \sum_{m=1}^{n} (\Gamma^l_{jm} \Gamma^m_{ki} - \Gamma^l_{km} \Gamma^m_{ji})$$

for $l, i, j, k = 1, 2, \ldots, n$. Moreover if X, Y and Z are smooth vector fields over U, and if

$$X = \sum_{i=1}^{n} u^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}, \qquad Z = \sum_{k=1}^{n} w^k \frac{\partial}{\partial x^k},$$

then

$$R(Y, Z)X = \sum_{i,j,k,l=1}^{n} R^l{}_{ijk} u^i v^j w^k \frac{\partial}{\partial x^l}.$$

Proposition 9.7 *Let M be a Riemannian or pseudo-Riemannian manifold. Then the Riemann curvature tensor R on M satisfies the following identities at each point p of M, and for all $W, X, Y, Z \in T_pM$:—*

(i) $R(W, Z, X, Y) = -R(W, Z, Y, X)$;

(ii) $R(W, X, Y, Z) + R(W, Y, Z, X) + R(W, Z, X, Y) = 0$;

(iii) $R(W, Z, X, Y) = -R(Z, W, X, Y)$;

(iv) $R(W, Z, X, Y) = R(X, Y, W, Z)$.

Proof Property (i) follows directly from the definition of the Riemann curvature tensor, and (ii) corresponds to the First Bianchi Identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

(see Proposition 8.14). Now

$$\begin{aligned} X[Y[g(W,Z)]] &= X\left[g(\nabla_Y W, Z) + g(W, \nabla_Y Z)\right] \\ &= g(\nabla_X \nabla_Y W, Z) + g(\nabla_Y W, \nabla_X Z) \\ &\quad + g(\nabla_X W, \nabla_Y Z) + g(W, \nabla_X \nabla_Y Z), \end{aligned}$$

and hence

$$\begin{aligned} [X,Y][g(W,Z)] &= X[Y[g(W,Z)]] - Y[X[g(W,Z)]] \\ &= g(\nabla_X \nabla_Y W - \nabla_Y \nabla_X W, Z) \\ &\quad + g(W, \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z). \end{aligned}$$

Therefore

$$\begin{aligned} & R(W,Z,X,Y) + R(Z,W,X,Y) \\ &\quad = g(W, R(X,Y)Z) + g(R(X,Y)W, Z) \\ &\quad = [X,Y][g(W,Z)] - g(\nabla_{[X,Y]} W, Z) - g(W, \nabla_{[X,Y]} Z) \\ &\quad = 0. \end{aligned}$$

This proves (iii). Using (i), (ii) and (iii), we see that

$$\begin{aligned} 2R(W,Z,X,Y) &= R(W,Z,X,Y) - R(Z,W,X,Y) \\ &= -R(W,X,Y,Z) - R(W,Y,Z,X) \\ &\quad + R(Z,X,Y,W) + R(Z,Y,W,X) \\ &= (R(X,W,Y,Z) + R(X,Z,W,Y)) \\ &\quad + (R(Y,W,Z,X) + R(Y,Z,X,W)) \\ &= -R(X,Y,Z,W) - R(Y,X,W,Z) \\ &= 2R(X,Y,W,Z). \end{aligned}$$

This proves (iv). ∎

The following result expressed the properties of the Riemann curvature tensor in terms of its components with respect to a smooth local coordinate system on the Riemannian or pseudo-Riemannian manifold. It follows directly from Proposition 9.7

Corollary 9.8 *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, and let the functions R_{hijk} be the components of the Riemann curvature tensor R of M, determined with respect to a smooth local coordinate system $(x^1, x^2, \ldots, x^n)$ for M defined over an open subset U of M, so that*

$$R_{hijk} = g\left(\frac{\partial}{\partial x^h}, R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \right)\frac{\partial}{\partial x^i}\right).$$

Then these components have the following properties:—

(i) $R_{hijk} = -R_{hikj}$;

(ii) $R_{hijk} + R_{hjki} + R_{hkij} = 0$;

(iii) $R_{hijk} = -R_{ihjk}$;

(iv) $R_{hijk} = R_{jkhi}$.

The curvature tensor of a Riemannian or pseudo-Riemannian manifold also satisfies the *Second Bianchi Identity*

$$(\nabla_X R)(Y, Z)W + (\nabla_Y R)(Z, X)W + (\nabla_Z R)(X, Y)W = 0$$

(see Corollary 8.16).

9.4 The Sectional Curvatures of a Riemannian Manifold

Let M be a Riemannian manifold, let p be a point of M, and let P be a two-dimensional vector subspace of the tangent space T_pM to M at p. Let (E_1, E_2) be an orthonormal basis of P. We define the *sectional curvature* $K(P)$ of M in the plane P by the formula

$$K(P) = R(E_1, E_2, E_1, E_2).$$

Note that if X and Y are tangent vectors contained in the plane P then

$$X = a_{11}E_1 + a_{12}E_2, \qquad Y = a_{21}E_1 + a_{22}E_2,$$

for some real numbers a_{11}, a_{12}, a_{21} and a_{22}, and hence

$$\begin{aligned} R(X, Y, X, Y) &= R(X, Y, a_{11}E_1 + a_{12}E_2, a_{21}E_1 + a_{22}E_2) \\ &= (a_{11}a_{22} - a_{12}a_{21})R(X, Y, E_1, E_2) \\ &= (\det A)R(X, Y, E_1, E_2) = (\det A)^2 R(E_1, E_2, E_1, E_2) \\ &= (\det A)^2 K(P), \end{aligned}$$

where A is the matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

In particular, if (X, Y) is any orthonormal basis of P then the matrix A is an orthogonal matrix, and thus $\det A = \pm 1$. It follows that the value of the sectional curvature $K(P)$ does not depend on the choice of the orthonormal basis (E_1, E_2) of P.

Lemma 9.9 *Let M be a Riemannian manifold with metric tensor g, and let p be a point of M. Then the values of the sectional curvatures $K(P)$ for all two-dimensional vector subspaces P of the tangent space T_pM to M at p determine the Riemann curvature tensor at p.*

Proof The calculation given above shows that the sectional curvatures determine the values of $R(X, Y, X, Y)$ for all $X, Y \in T_pM$.

Now suppose that we are given $X, Y, Z \in T_pM$. Using the symmetries of the Riemann curvature tensor listed in Proposition 9.7, we see that

$$\begin{aligned} 2R(X, Y, X, Z) &= R(X, Y, X, Z) + R(X, Z, X, Y) \\ &= R(X, Y + Z, X, Y + Z) - R(X, Y, X, Y) \\ &\quad - R(X, Z, X, Z). \end{aligned}$$

Thus the sectional curvatures $K(P)$ determine the values of $R(X, Y, X, Z)$ for all tangent vectors X, Y and Z at p. It follows from this that the sectional curvatures determine $R(X, Y, Z, X)$, $R(Y, X, X, Z)$ and $R(Y, X, Z, X)$. But

$$\begin{aligned} 3R(W, X, Y, Z) &= 2R(W, X, Y, Z) - R(W, Y, Z, X) - R(W, Z, X, Y) \\ &= (R(W, X, Y, Z) + R(W, Y, X, Z)) \\ &\quad + (R(W, X, Y, Z) + R(W, Z, Y, X)) \\ &= R(W, X + Y, X + Y, Z) - R(W, X, X, Z) \\ &\quad - R(W, Y, Y, Z) + R(W, X + Z, Y, X + Z) \\ &\quad - R(W, X, Y, X) - R(W, Z, Y, Z). \end{aligned}$$

We conclude that $R(W, X, Y, Z)$ is determined by the sectional curvatures of M, as required. ∎

10 Covariant Derivatives along Curves and Surfaces

10.1 Vector Fields along Smooth Maps

Definition Let Q and M be smooth manifolds, and let $\varphi: Q \to M$ be a smooth map. A *vector field V along* the map φ is a function $V: Q \to TM$ from Q to the total space TM of the tangent bundle $\pi_{TM}: TM \to M$ of M with the property that $\pi_{TM} \circ V = \varphi$.

Let Q and M be smooth manifolds. A smooth vector field $V: Q \to TM$ along a smooth map $\varphi: Q \to M$ is thus a smooth map from Q to TM which associates to each point q of Q a tangent vector $V(q)$ to the manifold M at the point $\varphi(q)$.

Let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system defined over some open set U in M. Given any smooth vector field $V: Q \to TM$ along the smooth map $\varphi: Q \to M$, there exist smooth real-valued functions $v^1, v^2, \ldots, v^n$ on Q such that

$$V(q) = \sum_{i=1}^{n} v^i(q) \left. \frac{\partial}{\partial x^i} \right|_{\varphi(q)}$$

for all $q \in \varphi^{-1}(U)$.

We say that a vector field V along the map φ is smooth if, given any smooth coordinate system $(x^1, x^2, \ldots, x^n)$ defined over an open subset U of M, the components $v^1, v^2, \ldots, v^n$ of V with respect to this coordinate system are smooth functions on $\varphi^{-1}(U)$. In particular, one can define in this way smooth vector fields along curves and surfaces in the smooth manifold M.

Example Let $\gamma: I \to M$ be a smooth curve in the smooth manifold M, defined on some open interval I in $\mathbb{R}$. Then a vector field V along the curve γ is a function which associates to each $t \in I$ a tangent vector $V(t)$ to M at $\gamma(t)$. The map that sends $t \in I$ to the velocity vector $\gamma'(t)$ of the curve at time t is a smooth vector field along γ.

10.2 Moving Frames

Definition Let M be a smooth manifold of dimension n, and let U be an open set in M. A *moving frame* over U is an n-tuple of smooth vector fields $E_1, E_2, \ldots, E_n$ over U such that, for each point p of U, the values

$$(E_1)_p, \; (E_2)_p, \ldots, \; (E_n)_p$$

of these vector fields at the point p constitute a basis for the tangent T_pM at that point.

Example Let M be a smooth manifold, and let U be the domain of a smooth coordinate system $(x^1, x^2, \ldots, x^n)$ for M. Then the vector fields

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots \frac{\partial}{\partial x^n}$$

constitute a moving frame over the open set U.

10.3 Covariant Differentiation of Vector Fields along Curves

Let M be a smooth manifold, and let $\gamma: I \to M$ be a smooth curve in the smooth manifold M, defined on some open interval I in $\mathbb{R}$. Let $t_0 \in I$, and let $E_1, E_2, \ldots, E_n$ be a moving frame for M defined around the point $\gamma(t_0)$ of M. Given any smooth vector field V along the curve γ, there exist smooth real-valued functions $v^1, v^2, \ldots, v^n$ on $\gamma^{-1}(U)$ such that

$$V(t) = \sum_{j=1}^{n} v^j(t)(E_j)_{\gamma(t)},$$

for all $t \in \gamma^{-1}(U)$.

Let ∇ be an affine connection on M. Given any smooth vector field V along the smooth curve γ we wish to define the *covariant derivative* $\dfrac{DV(t)}{dt}$ of the vector field V along the curve. Moreover this covariant derivative operator acting on vector fields along smooth curves should be determined in some natural fashion by the affine connection ∇.

Let us first consider the particular case where $V(t) = Y_{\gamma(t)}$ for all $t \in I$, where Y is some smooth vector field on M. Then there exist smooth real-valued functions $w^1, w^2, \ldots, w^n$ on U such that

$$Y(p) = \sum_{j=1}^{n} w^j(p)(E_j)_p$$

for all $p \in U$. Then

$$V(t) = \sum_{j=1}^{n} v^j(t)(E_j)_{\gamma(t)},$$

for all $t \in \gamma^{-1}(U)$, where $h^j(t) = w^j(\gamma(t))$ for $j = 1, 2, \ldots, n$. Now

$$\begin{aligned} \nabla_{\gamma'(t)} Y &= \sum_{j=1}^{n} \left(\frac{dw^j(\gamma(t))}{dt} (E_j)_{\gamma(t)} + w^j(\gamma(t)) \nabla_{\gamma'(t)} E_j \right) \\ &= \sum_{j=1}^{n} \left(\frac{dh^j(t)}{dt} (E_j)_{\gamma(t)} + h^j(t) \nabla_{\gamma'(t)} E_j \right) \end{aligned}$$

for all $t \in \gamma^{-1}(U)$.

This suggests that it might be reasonable to define the covariant derivative $\dfrac{dV(t)}{dt}$ of any smooth vector field V along the curve γ so that if

$$V(t) = \sum_{j=1}^{n} v^j(t) (E_j)_{\gamma(t)},$$

where $h^1, h^2, \ldots, h^n$ are smooth real-valued functions on $\gamma^{-1}(U)$, then

$$\frac{DV(t)}{dt} = \sum_{j=1}^{n} \left(\frac{dh^j(t)}{dt} (E_j)_{\gamma(t)} + h^j(t) \nabla_{\gamma'(t)} E_j \right)$$

for all $t \in \gamma^{-1}(U)$. However we need to verify that the value of the right hand side of the above equation is completely determined by the smooth vector field V along γ and the affine connection ∇ on M, and does not does not depend on the choice of the moving frame $E_1, E_2, \ldots, E_n$.

Let $\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n$ be another smooth moving frame for M defined over an open subset $\hat{U}$ of M, where $U \cap \hat{U} \cap \gamma(I)$ is non-empty. Then there exist smooth real-valued functions $A^j{}_k$ on $U \cap \hat{U}$ such that

$$\hat{E}_k = \sum_{j=1}^{n} A^j{}_k E_j$$

on $U \cap \hat{U}$. For all $p \in U \cap \hat{U}$, let $A(p)$ be the $n \times n$ matrix whose entry in the jth row and kth column is $A^j{}_k(p)$. Then this matrix $A(p)$ is a non-singular matrix. Now, given any smooth vector field V along the smooth curve $\gamma \colon I \to M$, there exist smooth real-valued functions $h^1, h^2, \ldots, h^n$ on $\gamma^{-1}(U)$ and smooth real-valued functions $\hat{h}^1, \hat{h}^2, \ldots, \hat{h}^n$ on $\gamma^{-1}(\hat{U})$ such that $V(t) = \sum_{j=1}^{n} h^j(t) (E_j)_{\gamma(t)}$ on $\gamma^{-1}(U)$ and $V(t) = \sum_{j=1}^{n} \hat{h}^k(t) (\hat{E}_k)_{\gamma(t)}$ on $\gamma^{-1}(\hat{U})$. Then

$$h^j(t) = \sum_{k=1}^{n} A^j{}_k(\gamma(t)) \hat{h}^k(t)$$

for all $q \in U \cap \hat{U}$. Then

$$\frac{dh^j(t)}{dt} = \sum_{k=1}^n \left(\frac{d(A^j{}_k(\gamma(t))}{dt} \hat{h}^k(t) + A^j{}_k(\gamma(t)) \frac{d\hat{h}^k(t)}{dt} \right).$$

It follows that

$$\begin{aligned}
&\sum_{j=1}^n \left(\frac{dh^j(t)}{dt} (E_j)_{\gamma(t)} + h^j(t) \nabla_{\gamma'(t)} E_j \right) \\
&= \sum_{j,k=1}^n \Big(\frac{d(A^j{}_k(\gamma(t))}{dt} \hat{h}^k(t) (E_j)_{\gamma(t)} + A^j{}_k(\gamma(t)) \frac{d\hat{h}^k(t)}{dt} (E_j)_{\gamma(t)} \\
&\qquad + A^j{}_k(\gamma(t)) \hat{h}^k(t) \nabla_{\gamma'(t)} E_j \Big) \\
&= \sum_{k=1}^n \frac{d\hat{h}^k(t)}{dt} (\hat{E}_k)_{\gamma(t)} \\
&\qquad + \sum_{j,k=1}^n \hat{h}^k(t) \Big(\frac{d(A^j{}_k(\gamma(t))}{dt} (E_j)_{\gamma(t)} + A^j{}_k(\gamma(t)) \nabla_{\gamma'(t)} E_j \Big) \\
&= \sum_{k=1}^n \frac{d\hat{h}^k(t)}{dt} (\hat{E}_k)_{\gamma(t)} + \sum_{j,k=1}^n \hat{h}^k(t) \nabla_{\gamma'(t)} (A^j{}_k(\gamma(t)) E_j) \\
&= \sum_{k=1}^n \left(\frac{d\hat{h}^k(t)}{dt} (\hat{E}_k)_{\gamma(t)} + \sum_{j,k=1}^n \hat{h}^k(t) \nabla_{\gamma'(t)} \hat{E}_k \right).
\end{aligned}$$

We may therefore employ the formula

$$\sum_{j=1}^n \left(\frac{dh^j(t)}{dt} (E_j)_{\gamma(t)} + h^j(t) \nabla_{\gamma'(t)} E_j \right)$$

on order to define the covariant derivative of the vector field on $\gamma^{-1}(U)$, since the tangent vector to M at $\gamma(t)$ determined by this expression does not depend on the choice of moving frame used when evaluating this expression.

Definition Let M be a smooth manifold, let ∇ be a smooth affine connection on M, let $\gamma: I \to M$ be a smooth curve in M defined on some open interval I in $\mathbb{R}$, and let $V: I \to TM$ be a smooth vector field along the curve γ. The *covariant derivative* $\dfrac{DV(t)}{dt}$ of the vector field V is the vector field determined, for values of t sufficiently close to some given value t_0, by the equation

$$\frac{DV(t)}{dt} = \sum_{j=1}^{n} \left(\frac{dh^j(t)}{dt} (E_j)_{\gamma(t)} + h^j(t) \nabla_{\gamma'(t)} E_j \right),$$

where $E_1, E_2, \ldots, E_n$ is a moving frame for the smooth manifold M defined on some open neighbourhood U of the point $\gamma(t_0)$ and $h^1, h^2, \ldots, h^n$ are the smooth functions on $\gamma^{-1}(U)$ determined such that

$$V(t) = \sum_{j=1}^{n} v^j(t) (E_j)_{\gamma(t)},$$

for all $t \in \gamma^{-1}(U)$.

Lemma 10.1 *Let M be a smooth manifold, let ∇ be an affine connection on M, and let $\gamma: I \to M$ be a smooth curve in M. Let V and W be smooth vector fields along γ and let $f: I \to \mathbb{R}$ be a smooth real-valued function. Then*

(i) $\dfrac{D(V(t) + W(t))}{dt} = \dfrac{DV(t)}{dt} + \dfrac{DW(t)}{dt}$,

(ii) $\dfrac{D(f(t)V(t))}{dt} = \dfrac{df(t)}{dt} V(t) + f(t) \dfrac{DV(t)}{dt}$,

(iii) *if $V(t) = X_{\gamma(t)}$ for all t, where X is some smooth vector field defined over an open set in M, then $\dfrac{DV(t)}{dt} = \nabla_{\gamma'(t)} X$.*

Moreover the differential operator D/dt is the unique operator on the space of smooth vector fields along the curve γ satisfying (i), (ii) and (iii).

Definition A smooth vector field V along a smooth curve γ is said to be *parallel* along γ if $\dfrac{DV(t)}{dt} = 0$ for all t.

10.4 Vector Fields along Parameterized Surfaces

Let M be a smooth manifold, let U be a connected open set in $\mathbb{R}^m$, and let $\varphi: U \to M$ be a smooth map from U to M. Given $(t^1, t^2, \ldots, t^m) \in U$, we define

$$\frac{\partial \varphi(t^1, t^2, \ldots, t^m)}{\partial t^i}$$

to be the velocity vector of the curve $t \mapsto \varphi(t^1, \ldots, t^{i-1}, t, t^{i+1}, \ldots, t^m)$ at $t = t^i$. Then $\partial \varphi / \partial t^i$ is a smooth vector field along the map φ for $i = 1, 2, \ldots, m$.

Let ∇ be an affine connection on M. Given any smooth vector field V along the map φ, and given $(t^1, t^2, \ldots, t^m) \in U$, we define

$$\frac{DV(t^1, t^2, \ldots, t^m)}{\partial t^i}$$

to be the covariant derivative of the vector field

$$t \mapsto V(t^1, \ldots, t^{i-1}, t, t^{i+1}, \ldots, t^m)$$

along the curve $t \mapsto \varphi(t^1, \ldots, t^{i-1}, t, t^{i+1}, \ldots, t^m)$ at $t = t^i$. Then the partial covariant derivative $DV/\partial t^i$ is a smooth vector field along the map φ of $i = 1, 2, \ldots, m$.

Let M be a smooth manifold of dimension n. A smooth *parameterized surface* in M is a smooth map $\varphi: U \to M$ defined on a connected open subset U on $\mathbb{R}^2$.

Lemma 10.2 *Let M be a smooth manifold and let ∇ be an affine connection on M. Let V be a smooth vector field along a smooth parameterized surface $\varphi: U \to M$ in M. Then*

$$\begin{aligned} \frac{D}{\partial s}\frac{\partial\varphi(s,t)}{\partial t} - \frac{D}{\partial t}\frac{\partial\varphi(s,t)}{\partial s} &= T\Big(\frac{\partial\varphi(s,t)}{\partial s}, \frac{\partial\varphi(s,t)}{\partial t}\Big), \\ \frac{D}{\partial s}\frac{DV(s,t)}{\partial t} - \frac{D}{\partial t}\frac{DV(s,t)}{\partial s} &= R\left(\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial t}\right) V(s,t), \end{aligned}$$

where T and R are the torsion and curvature tensors of the affine connection ∇.

Proof Without loss of generality, we may suppose that the image of the map $\varphi: U \to M$ is contained in the domain of some smooth coordinate system $(x^1, x^2, \ldots, x^n)$. Let $B_1, B_2, \ldots, B_n$ be the smooth vector fields over this coordinate patch defined by

$$B_i = \frac{\partial}{\partial x^i}$$

for $i = 1, 2, \ldots, n$. Then the vector fields $B_1, B_2, \ldots, B_n$ constitute a moving frame defined over some open set in M that contains $\varphi(U)$. Moreover

$$[B_j, B_k] = 0$$

for $j, k = 1, 2, \ldots, n$, and therefore

$$\nabla_{B_j} B_k - \nabla_{B_k} B_j = T(B_j, B_k), \qquad \nabla_{B_j}\nabla_{B_k} B_i - \nabla_{B_k}\nabla_{B_j} B_i = R(B_j, B_k) B_i,$$

for $i, j, k = 1, 2, \ldots, n$.

The map $\varphi: U \to M$ is specified, with respect to the coordinate system $(x^1, x^2, \ldots, x^n)$, by smooth real-valued functions $\varphi^1, \varphi^2, \ldots, \varphi^n$ on U, where $\varphi^i(s,t) = x^i(\varphi(s,t)$ for $i = 1, 2, \ldots, n$ and for all $s, t \in U$. It follows that

$$\frac{\partial \varphi}{\partial s} = \sum_{j=1}^n \frac{\partial \varphi^j}{\partial s} B_j, \qquad \frac{\partial \varphi}{\partial t} = \sum_{k=1}^n \frac{\partial \varphi^k}{\partial t} B_k.$$

Thus

$$\frac{DX}{\partial s} = \sum_{j=1}^n \frac{\partial \varphi^j}{\partial s} \nabla_{B_j} X, \qquad \frac{DX}{\partial t} = \sum_{k=1}^n \frac{\partial \varphi^k}{\partial t} \nabla_{B_k} X.$$

for all smooth vector fields X on M defined around points of $\varphi(U)$. Now

$$\begin{aligned} \frac{D}{\partial s}\frac{\partial \varphi}{\partial t} &= \sum_{k=1}^n \frac{D}{\partial s}\Bigl(\frac{\partial \varphi^k}{\partial t} B_k\Bigr) \\ &= \sum_{k=1}^n \frac{\partial^2 \varphi^k}{\partial s \partial t} B_k + \sum_{j,k=1}^n \frac{\partial \varphi^j}{\partial s}\frac{\partial \varphi^k}{\partial t} \nabla_{B_j} B_k. \end{aligned}$$

Thus

$$\begin{aligned} \frac{D}{\partial s}\frac{\partial \varphi}{\partial t} - \frac{D}{\partial t}\frac{\partial \varphi}{\partial s} &= \sum_{j,k=1}^n \frac{\partial \varphi^j}{\partial s}\frac{\partial \varphi^k}{\partial t} \left(\nabla_{B_j} B_k - \nabla_{B_k} B_j\right) \\ &= \sum_{j,k=1}^n \frac{\partial \varphi^j}{\partial s}\frac{\partial \varphi^k}{\partial t} T(B_j, B_k) = T\Bigl(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}\Bigr). \end{aligned}$$

Let $f: U \to \mathbb{R}$ be a smooth real-valued function on U, and let V be a smooth vector field along the map φ. Then

$$\begin{aligned} \frac{D}{\partial s}\frac{D(fV)}{\partial t} &= \frac{D}{\partial s}\Bigl(\frac{\partial f}{\partial t} V + f\frac{DV}{\partial t}\Bigr) \\ &= \frac{\partial^2 f}{\partial s \partial t} V + \frac{\partial f}{\partial t}\frac{DV}{\partial s} + \frac{\partial f}{\partial s}\frac{DV}{\partial t} + f\frac{D}{\partial s}\frac{DV}{\partial t}, \end{aligned}$$

and thus

$$\Bigl(\frac{D}{\partial s}\frac{D}{\partial t} - \frac{D}{\partial t}\frac{D}{\partial s}\Bigr)(fV) = f\Bigl(\frac{D}{\partial s}\frac{DV}{\partial t} - \frac{D}{\partial t}\frac{DV}{\partial s}\Bigr)$$

Now any smooth vector field V along the map φ can be expressed in the form

$$V(s,t) = \sum_{i=1}^n v^i(s,t) (B_i)_{\varphi(s,t)}$$

for some smooth real-valued functions $v^1, v^2, \ldots, v^n$ on U. It follows that

$$\frac{D}{\partial s}\frac{DV}{\partial t} - \frac{D}{\partial t}\frac{DV}{\partial s} = \sum_{i=1}^{n} v^i \left(\frac{D}{\partial s}\frac{D}{\partial t} - \frac{D}{\partial t}\frac{D}{\partial s}\right) B_i.$$

But

$$\begin{aligned}
\frac{D}{\partial s}\frac{D}{\partial t} B_i &= \sum_{k=1}^{n} \frac{D}{\partial s}\left(\frac{\partial \varphi^k}{\partial t} \nabla_{B_k} B_i\right) \\
&= \sum_{k=1}^{n} \frac{\partial^2 \varphi^k}{\partial s \partial t} \nabla_{B_k} B_i + \sum_{j,k=1}^{n} \frac{\partial \varphi^j}{\partial s}\frac{\partial \varphi^k}{\partial t} \nabla_{B_j} \nabla_{B_k} B_i,
\end{aligned}$$

and hence

$$\begin{aligned}
\left(\frac{D}{\partial s}\frac{D}{\partial t} - \frac{D}{\partial t}\frac{D}{\partial s}\right) B_i &= \sum_{j,k=1}^{n} \frac{\partial \varphi^j}{\partial s}\frac{\partial \varphi^k}{\partial t} \left(\nabla_{B_j} \nabla_{B_k} B_i - \nabla_{B_k} \nabla_{B_j} B_i\right) \\
&= \sum_{j,k=1}^{n} \frac{\partial \varphi^j}{\partial s}\frac{\partial \varphi^k}{\partial t} R(B_j, B_k) B_i \\
&= R\left(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}\right) B_i.
\end{aligned}$$

We deduce that

$$\frac{D}{\partial s}\frac{DV}{\partial t} - \frac{D}{\partial t}\frac{DV}{\partial s} = \sum_{i=1}^{n} v^i R\left(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}\right) B_i = R\left(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}\right) V,$$

as required. ∎

11 Geodesics and Jacobi Fields

11.1 Geodesics

Definition Let M be a Riemannian or pseudo-Riemannian manifold which is provided with a smooth affine connection ∇, and let $\gamma\colon I \to M$ be a smooth curve in M, defined over some interval I in $\mathbb{R}$. We say that γ is a *geodesic* (with respect to the connection ∇ if and only if

$$\frac{D}{dt}\left(\frac{d\gamma(t)}{dt}\right) = 0.$$

Thus γ is a geodesic if and only if the velocity vector field $t \mapsto \gamma'(t)$ is parallel along γ (with respect to the connection ∇ on M). The geodesic $\gamma\colon I \to M$ is said to be *maximal* if it cannot be extended to a geodesic defined over some interval J, where $I \subset J$ and $I \neq J$.

A smooth curve in a Riemannian or pseudo-Riemannian manifold is said to be a geodesic if it is a geodesic with respect to the Levi-Civita connection determined by the metric tensor on the manifold.

Covariant differentiation of smooth vector fields along curves and surfaces in a Riemannian or pseudo-Riemannian manifold is defined with respect to the Levi-Civita connection determined by the metric tensor on the manifold.

Lemma 11.1 *Let M be a Riemannian or pseudo-Riemannian manifold, and let g denote the metric tensor on M. Let $\gamma\colon I \to M$ be a smooth curve in M, defined on an open interval I in $\mathbb{R}$, and let $V\colon I \to TM$ and $W\colon I \to TM$ be smooth vector fields along the curve γ. Then*

$$\frac{d}{dt}\Big(g(V(t), W(t))\Big) = g\left(\frac{DV(t)}{dt}, W(t)\right) + g\left(V(t), \frac{DW(t)}{dt}\right).$$

Proof Let $E_1, E_2, \ldots, E_n$ be a moving frame on M defined over an open neighbourhood U of $\gamma(t_0)$ for some $t_0 \in I$. Then there are smooth functions $v^1, v^2, \ldots, v^n$ and $w^1, w^2, \ldots, w^n$ on $\gamma^{-1}(U)$ defined such that

$$V(t) = \sum_{j=1}^n v^j (E_j)_{\gamma(t)}, \quad W(t) = \sum_{k=1}^n w^k (E_k)_{\gamma(t)}$$

for all $t \in \gamma^{-1}(U)$. Then

$$\frac{DV(t)}{dt} = \sum_{j=1}^n \left(\frac{dv^j(t)}{dt}\,(E_j)_{\gamma(t)} + v^j(t)\,\nabla_{\gamma'(t)} E_j\right)$$

and

$$\frac{DW(t)}{dt} = \sum_{k=1}^{n} \left(\frac{dw^k(t)}{dt} (E_k)_{\gamma(t)} + w^k(t) \, \nabla_{\gamma'(t)} E_k \right).$$

Then

$$\begin{aligned}
g\left(\frac{DV(t)}{dt}, W(t)\right) &+ g\left(V(t), \frac{DW(t)}{dt}\right) \\
&= \sum_{j,k=1}^{n} \Big(\frac{dv^j(t)}{dt} \, w^k(t) \, g((E_j)_{\gamma(t)}, (E_k)_{\gamma(t)}) \\
&\qquad + v^j(t) \, w^k(t) \, g(\nabla_{\gamma'(t)} E_j, (E_k)_{\gamma(t)}) \\
&\qquad + v^j(t) \, \frac{dw^k(t)}{dt} \, g((E_j)_{\gamma(t)}, (E_k)_{\gamma(t)}) \\
&\qquad + v^j(t) \, w^k(t) \, g((E_j)_{\gamma(t)}, \nabla_{\gamma'(t)} E_k) \Big) \\
&= \sum_{j,k=1}^{n} \Big(\frac{d}{dt} \left(v^j(t) w^k(t) \right) g((E_j)_{\gamma(t)}, (E_k)_{\gamma(t)}) \\
&\qquad + v^j(t) \, w^k(t) \Big(g(\nabla_{\gamma'(t)} E_j, (E_k)_{\gamma(t)}) + g((E_j)_{\gamma(t)}, \nabla_{\gamma'(t)} E_k) \Big) \Big) \\
&= \sum_{j,k=1}^{n} \Big(\frac{d}{dt} \left(v^j(t) w^k(t) \right) g((E_j)_{\gamma(t)}, (E_k)_{\gamma(t)}) \\
&\qquad + v^j(t) \, w^k(t) \frac{d}{dt} \Big(g((E_j)_{\gamma(t)}, (E_k)_{\gamma(t)}) \Big) \Big) \\
&= \sum_{j,k=1}^{n} \frac{d}{dt} \Big(v^j(t) w^k(t) \, g((E_j)_{\gamma(t)}, (E_k)_{\gamma(t)}) \Big) \\
&= \frac{d}{dt} \Big(g(V(t), W(t)) \Big),
\end{aligned}$$

as required. ∎

Lemma 11.2 *Let M be a Riemannian or pseudo-Riemannian manifold, and let g denote the metric tensor on M. Let $\gamma\colon I \to M$ be a geodesic in M. Then*

$$\frac{d}{dt} g(\gamma'(t), \gamma'(t)) = 0,$$

and thus $g(\gamma'(t), \gamma'(t))$ is constant along the geodesic.

Proof

$$\frac{d}{dt} g(\gamma'(t), \gamma'(t)) = \frac{d}{dt} g\left(\frac{d\gamma(t)}{dt}, \frac{d\gamma(t)}{dt}\right)$$

$$
\begin{aligned}
&= g\left(\frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) + g\left(\frac{d\gamma}{dt}, \frac{D}{dt}\frac{d\gamma}{dt}\right) \\
&= 0,
\end{aligned}
$$

as required. ∎

Let us choose a smooth coordinate system $(x^1, x^2, \ldots, x^n)$ over some open set U in the smooth manifold M. Let the smooth functions Γ^i_{jk} on U be the *Christoffel symbols* of the Levi-Civita connection on the coordinate patch U, defined such that

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \sum_{i=1}^n \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

Then $\Gamma^i_{jk} = \Gamma^i_{kj}$ for all j and k, since

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} = \left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right] = 0.$$

Let $\gamma: I \to U$ be a smooth curve in U, and let $\gamma^i(t) = x^i \circ \gamma(t)$ for all $t \in \gamma^{-1}(U)$. Then

$$\frac{d\gamma(t)}{dt} = \sum_{k=1}^n \frac{d\gamma^k(t)}{dt} \frac{\partial}{\partial x^k},$$

so that

$$
\begin{aligned}
\frac{D}{dt}\frac{d\gamma(t)}{dt} &= \sum_{k=1}^n \left(\frac{d^2\gamma^k(t)}{dt^2} \frac{\partial}{\partial x^k} + \frac{d\gamma^k(t)}{dt} \sum_{j=1}^n \frac{d\gamma^j(t)}{dt} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \right) \\
&= \sum_{i=1}^n \left(\frac{d^2\gamma^i(t)}{dt^2} + \sum_{j=1}^n \sum_{k=1}^n \Gamma^i_{jk}(\gamma(t)) \frac{d\gamma^j(t)}{dt} \frac{d\gamma^k(t)}{dt} \right) \frac{\partial}{\partial x^i}.
\end{aligned}
$$

Thus $\gamma: I \to U$ is a geodesic if and only if

$$\frac{d^2\gamma^i(t)}{dt^2} + \sum_{j=1}^n \sum_{k=1}^n \Gamma^i_{jk}(\gamma(t)) \frac{d\gamma^j(t)}{dt} \frac{d\gamma^k(t)}{dt} = 0 \qquad (i = 1, 2, \ldots, n).$$

Standard existence and uniqueness theorems for solutions of ordinary differential systems of equations ensure that, given a tangent vector V at any point m of M, and given any real number t_0, there exists a unique maximal geodesic $\gamma: I \to M$, defined on some open interval I containing t_0, such that $\gamma(t_0) = m$ and $\gamma'(t_0) = V$.

11.2 The First Variations of Length and Energy

Definition Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, and let $\gamma\colon [a,b] \to M$ be a parameterized smooth curve in M. The *energy* $E(\gamma)$ of γ is is then defined to be the quantity

$$E(\gamma) = \tfrac{1}{2} \int_a^b g(\gamma'(t), \gamma'(t))\, dt.$$

Theorem 11.3 *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, let $\varphi\colon [a,b] \times (-\varepsilon, \varepsilon) \to M$ be a smooth map, and let $\gamma\colon [a,b] \to M$ and $\varphi_u\colon [a,b] \to M$ be the smooth curves in M defined by $\gamma(t) = \varphi(t, 0)$ and $\varphi_u(t) = \varphi(t, u)$ for all $t \in [a,b]$ and $u \in (-\varepsilon, \varepsilon)$ (so that $\gamma = \varphi_0$), and let $E(\varphi_u)$ denote the energy of φ_u. Then*

$$\begin{aligned} \left.\frac{dE(\varphi_u)}{du}\right|_{u=0} &= g(\gamma'(b), V(b)) - g(\gamma'(a), V(a)) \\ &\qquad - \int_a^b g\left(\frac{D}{dt}\left(\frac{d\gamma(t)}{dt}\right), V(t)\right) dt, \end{aligned}$$

where

$$V(t) = \left.\frac{\partial \varphi(t,u)}{\partial u}\right|_{u=0}.$$

Proof The Levi-Civita connection is torsion-free. It therefore follows from Lemma 10.2 that

$$\frac{D}{du}\frac{\partial \varphi(t,u)}{\partial t} = \frac{D}{dt}\frac{\partial \varphi(t,u)}{\partial u}.$$

It therefore follows from Lemma 11.1 that

$$\begin{aligned} \tfrac{1}{2}\frac{\partial}{\partial u} g\left(\frac{\partial \varphi(t,u)}{\partial t}, \frac{\partial \varphi(t,u)}{\partial t}\right) &= g\left(\frac{\partial \varphi(t,u)}{\partial t}, \frac{D}{du}\frac{\partial \varphi(t,u)}{\partial t}\right) \\ &= g\left(\frac{\partial \varphi(t,u)}{\partial t}, \frac{D}{dt}\frac{\partial \varphi(t,u)}{\partial u}\right) \end{aligned}$$

Thus

$$\frac{dE(\varphi_u)}{du} = \int_a^b g\left(\frac{\partial \varphi(t,u)}{\partial t}, \frac{D}{dt}\frac{\partial \varphi(t,u)}{\partial u}\right) dt,$$

and therefore

$$\left.\frac{dE(\varphi_u)}{du}\right|_{u=0} = \int_a^b g\left(\gamma'(t), \frac{DV(t)}{dt}\right) dt$$

$$
\begin{aligned}
&= \int_a^b \frac{d}{dt}\left(g(\gamma'(t),\, V(t))\right)\, dt \\
&\quad - \int_a^b g\left(\frac{D}{dt}\left(\frac{d\gamma(t)}{dt}\right),\, V(t)\right)\, dt \\
&= g(\gamma'(b),\, V(b)) - g(\gamma'(a),\, V(a)) \\
&\quad - \int_a^b g\left(\frac{D}{dt}\left(\frac{d\gamma(t)}{dt}\right),\, V(t)\right)\, dt,
\end{aligned}
$$

as required. ∎

Corollary 11.4 *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, let $\varphi\colon [a,b] \times (-\varepsilon, \varepsilon) \to M$ be a smooth map, and let $\gamma\colon [a,b] \to M$ and $\varphi_u\colon [a,b] \to M$ be the smooth curves in M defined by $\gamma(t) = \varphi(t, 0)$ and $\varphi_u(t) = \varphi(t, u)$ for all $t \in [a,b]$ and $u \in (-\varepsilon, \varepsilon)$ (so that $\gamma = \varphi_0$), and let $E(\varphi_u)$ denote the energy of φ_u. Suppose that $\varphi_u(a) = \gamma(a)$ and $\varphi_u(b) = \gamma(b)$ for all $u \in (-\varepsilon, \varepsilon)$. Suppose also that $\gamma\colon [a,b] \to M$ is a geodesic, and thus satisfies*

$$
\frac{D}{dt}\left(\frac{d\gamma(t)}{dt}\right) = 0.
$$

Then

$$
\left.\frac{dE(\varphi_u)}{du}\right|_{u=0} = 0.
$$

Definition Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, and let $\gamma\colon [a,b] \to M$ be a smooth curve in M which satisfies

$$
g(\gamma'(t), \gamma'(t)) > 0
$$

for all $t \in [a,b]$. The *length* $L(\gamma)$ of γ is is then defined to be the quantity

$$
L(\gamma) = \int_a^b |\gamma'(t)|\, dt,
$$

where $|\gamma'(t)|^2 = g(\gamma'(t), \gamma'(t))$ for all $t \in [a,b]$.

Theorem 11.5 *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, let $\varphi\colon [a,b] \times (-\varepsilon, \varepsilon) \to M$ be a smooth map with the property that*

$$
g\left(\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial t}\right) > 0
$$

for all $t \in [a,b]$ *and* $u \in (-\varepsilon, \varepsilon)$. *Let* $\gamma\colon [a,b] \to M$ *and* $\varphi_u\colon [a,b] \to M$ *be the smooth curves in* M *defined by* $\gamma(t) = \varphi(t,0)$ *and* $\varphi_u(t) = \varphi(t,u)$ *for all* $t \in [a,b]$ *and* $u \in (-\varepsilon, \varepsilon)$ *(so that* $\gamma = \varphi_0$*), and let* $L(\varphi_u)$ *denote the length of* φ_u. *Then*

$$\begin{aligned}\left.\frac{dL(\varphi_u)}{du}\right|_{u=0} &= \frac{1}{|\gamma'(b)|} g(\gamma'(b), V(b)) - \frac{1}{|\gamma'(a)|} g(\gamma'(a), V(a)) \\ &\quad - \int_a^b g\left(\frac{D}{dt}\left(\frac{1}{|\gamma'(t)|}\frac{d\gamma(t)}{dt}\right), V(t)\right) dt,\end{aligned}$$

where

$$V(t) = \left.\frac{\partial \varphi(t,u)}{\partial u}\right|_{u=0}.$$

In particular, if $\gamma\colon [a,b] \to M$ *is parameterized by arclength, then*

$$\left.\frac{dL(\varphi_u)}{du}\right|_{u=0} = g(\gamma'(b), V(b)) - g(\gamma'(a), V(a)) - \int_a^b g\left(\frac{D}{dt}\left(\frac{d\gamma(t)}{dt}\right), V(t)\right) dt.$$

Proof The Levi-Civita connection is torsion-free. It therefore follows from Lemma 10.2 that

$$\frac{D}{du}\frac{\partial \varphi(t,u)}{\partial t} = \frac{D}{dt}\frac{\partial \varphi(t,u)}{\partial u}.$$

It therefore follows from Lemma 11.1 that

$$\begin{aligned}\left|\frac{\partial \varphi(t,u)}{\partial t}\right| \frac{\partial}{\partial u}\left|\frac{\partial \varphi(t,u)}{\partial t}\right| &= \tfrac{1}{2}\frac{\partial}{\partial u}\left|\frac{\partial \varphi(t,u)}{\partial t}\right|^2 \\ &= \tfrac{1}{2}\frac{\partial}{\partial u} g\left(\frac{\partial \varphi(t,u)}{\partial t}, \frac{\partial \varphi(t,u)}{\partial t}\right) \\ &= g\left(\frac{\partial \varphi(t,u)}{\partial t}, \frac{D}{du}\frac{\partial \varphi(t,u)}{\partial t}\right) \\ &= g\left(\frac{\partial \varphi(t,u)}{\partial t}, \frac{D}{dt}\frac{\partial \varphi(t,u)}{\partial u}\right)\end{aligned}$$

Thus

$$\frac{dL(\varphi_u)}{du} = \int_a^b \left|\frac{\partial \varphi(t,u)}{\partial t}\right|^{-1} g\left(\frac{\partial \varphi(t,u)}{\partial t}, \frac{D}{dt}\frac{\partial \varphi(t,u)}{\partial u}\right) dt,$$

and therefore

$$\begin{aligned}\left.\frac{dL(\varphi_u)}{du}\right|_{u=0} &= \int_a^b \frac{1}{|\gamma'(t)|} g\left(\gamma'(t), \frac{DV(t)}{dt}\right) dt \\ &= \int_a^b \frac{d}{dt}\left(\frac{1}{|\gamma'(t)|} g(\gamma'(t), V(t))\right) dt\end{aligned}$$

$$
\begin{aligned}
& \quad - \int_a^b g\left(\frac{D}{dt}\left(\frac{1}{|\gamma'(t)|}\frac{d\gamma(t)}{dt}\right), V(t)\right) dt \\
= & \quad \frac{1}{|\gamma'(b)|} g(\gamma'(b), V(b)) - \frac{1}{|\gamma'(a)|} g(\gamma'(a), V(a)) \\
& \quad - \int_a^b g\left(\frac{D}{dt}\left(\frac{1}{|\gamma'(t)|}\frac{d\gamma(t)}{dt}\right), V(t)\right) dt,
\end{aligned}
$$

as required. ∎

Corollary 11.6 *Let M be a Riemannian or pseudo-Riemannian manifold with metric tensor g, let $\varphi\colon [a,b] \times (-\varepsilon, \varepsilon) \to M$ be a smooth map with the property that*

$$g\left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t},\right) > 0$$

for all $t \in [a,b]$ and $u \in (-\varepsilon, \varepsilon)$. Let $\gamma\colon [a,b] \to M$ and $\varphi_u\colon [a,b] \to M$ be the smooth curves in M defined by $\gamma(t) = \varphi(t,0)$ and $\varphi_u(t) = \varphi(t,u)$ for all $t \in [a,b]$ and $u \in (-\varepsilon, \varepsilon)$ (so that $\gamma = \varphi_0$), and let $L(\varphi_u)$ denote the length of φ_u. Suppose that $\varphi_u(a) = \gamma(a)$ and $\varphi_u(b) = \gamma(b)$ for all $u \in (-\varepsilon, \varepsilon)$. Suppose also that $\gamma\colon [a,b] \to M$ is a reparameterization of a geodesic, and thus satisfies

$$\frac{D}{dt}\left(\frac{1}{|\gamma'(t)|}\frac{d\gamma(t)}{dt}\right) = 0.$$

Then

$$\left.\frac{dL(\varphi_u)}{du}\right|_{u=0} = 0.$$

11.3 Jacobi Fields

Let (M,g) be a Riemannian manifold, and let $\gamma\colon [a,b] \to M$ be a geodesic in M. A *Jacobi field* along γ is a vector field V along γ which satisfies the *Jacobi equation*

$$\frac{D^2V(t)}{dt^2} = R(\gamma'(t), V(t))\gamma'(t),$$

where R denotes the curvature tensor of the Levi-Civita connection on M. First we show that Jacobi fields arise naturally from variations of the geodesic γ through neighbouring geodesics.

Lemma 11.7 *Let $\gamma\colon I \to M$ be a geodesic in a Riemannian manifold (M,g) and let*

$$\varphi\colon I \times (-\varepsilon, \varepsilon) \to M$$

be a smooth map satisfying $\varphi(t,0) = \gamma(t)$ for all $t \in I$. Let V be the vector field along the geodesic γ defined by

$$V(t) = \left.\frac{\partial \varphi(t,u)}{\partial u}\right|_{u=0}.$$

Suppose that, for each $u \in (-\varepsilon, \varepsilon)$, the curve $t \mapsto \varphi(t,u)$ is a geodesic in M. Then the vector field V satisfies the Jacobi equation

$$\frac{D^2 V(t)}{dt^2} = R(\gamma'(t), V(t))\gamma'(t).$$

Proof First we note that

$$\frac{D}{\partial t}\frac{\partial \varphi}{\partial t} = 0,$$

since each curve $t \mapsto \varphi(t,u)$ is a geodesic Now the Levi-Civita connection is torsion-free. It therefore follows from Lemma 10.2 that

$$\frac{D}{\partial t}\frac{D}{\partial u}\frac{\partial \varphi}{\partial t} - \frac{D}{\partial u}\frac{D}{\partial t}\frac{\partial \varphi}{\partial t} = R\Big(\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial u}\Big)\frac{\partial \varphi}{\partial t}.$$

and

$$\frac{D}{\partial t}\frac{\partial \varphi}{\partial u} = \frac{D}{\partial u}\frac{\partial \varphi}{\partial t}.$$

Therefore

$$\begin{aligned}
\frac{D^2}{\partial t^2}\frac{\partial \varphi}{\partial u} &= \frac{D}{\partial t}\frac{D}{\partial u}\frac{\partial \varphi}{\partial t} \\
&= R\Big(\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial u}\Big)\frac{\partial \varphi}{\partial t} + \frac{D}{\partial u}\frac{D}{\partial t}\frac{\partial \varphi}{\partial t} \\
&= R\Big(\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial u}\Big)\frac{\partial \varphi}{\partial t}.
\end{aligned}$$

Now

$$\left.\frac{\partial \varphi(t,u)}{\partial t}\right|_{u=0} = \gamma'(t), \qquad \left.\frac{\partial \varphi(t,u)}{\partial u}\right|_{u=0} = V(t).$$

Thus, on setting $u = 0$, we deduce that

$$\frac{D^2 V(t)}{dt^2} = R(\gamma'(t), V(t))\gamma'(t),$$

as required. ∎

11.4 The Second Variation of Energy

Let (M, g) be a Riemannian manifold and let $\gamma\colon [a, b] \to M$ be a geodesic in M. Let $\varphi\colon [a, b] \times (-\varepsilon, \varepsilon) \to M$ be a smooth map with the properties that

$$\begin{aligned} \varphi(t, 0) &= \gamma(t) \text{ for all } t \in [a, b], \\ \varphi(a, u) &= \gamma(a) \text{ for all } u \in (-\varepsilon, \varepsilon), \\ \varphi(b, u) &= \gamma(b) \text{ for all } u \in (-\varepsilon, \varepsilon). \end{aligned}$$

Thus if $\varphi_u\colon [a, b] \to M$ is the smooth curve defined by $\varphi_u(t) = \varphi(t, u)$ then each φ_u starts at $\gamma(a)$ and ends at $\gamma(b)$. We calculate

$$\left. \frac{d^2 E(\gamma(\varphi_u))}{du^2} \right|_{u=0},$$

where $E(\varphi_u)$ is the energy of φ_u. Let X and Y be the smooth vector fields along the map φ defined by

$$X(t, u) = \frac{\partial \varphi(t, u)}{\partial t}, \quad Y(t, u) = \frac{\partial \varphi(t, u)}{\partial u}.$$

Note that $Y(a, u) = 0$ and $Y(b, u) = 0$ for all $u \in (-\varepsilon, \varepsilon)$, on account of the fact that $\varphi(a, u) = \gamma(a)$ and $\varphi(b, u) = \gamma(b)$. The energy of the curve φ_u is given by

$$E(\varphi_u) = \tfrac{1}{2} \int_a^b g(X(t, u), X(t, u))\, dt.$$

Now

$$\frac{DX}{\partial u} = \frac{D}{\partial u} \frac{\partial \varphi}{\partial t} = \frac{D}{\partial t} \frac{\partial \varphi}{\partial u} = \frac{DY}{\partial t}$$

by Lemma 10.2. Thus

$$\frac{dE(\varphi_u)}{du} = \int_a^b g\left(X, \frac{DX}{\partial u}\right) dt = \int_a^b g\left(X, \frac{DY}{\partial t}\right) dt,$$

hence

$$\begin{aligned} \frac{d^2 E(\varphi_u)}{du^2} &= \int_a^b \left(g\left(\frac{DX}{\partial u}, \frac{DY}{\partial t}\right) + g\left(X, \frac{D}{\partial u} \frac{DY}{\partial t}\right)\right) dt \\ &= \int_a^b \left(g\left(\frac{DY}{\partial t}, \frac{DY}{\partial t}\right) + g\left(X, \frac{D}{\partial u} \frac{DY}{\partial t}\right)\right) dt. \end{aligned}$$

But

$$\frac{D}{\partial u} \frac{DY}{\partial t} = \frac{D}{\partial t} \frac{DY}{\partial u} + R(Y, X)Y$$

by Lemma 10.2. Therefore

$$
\begin{aligned}
\frac{d^2E(\varphi_u)}{du^2} &= \int_a^b g\left(\frac{DY}{\partial t}, \frac{DY}{\partial t}\right) dt \\
&\quad + \int_a^b g\left(X, \frac{D}{\partial t}\frac{DY}{\partial u} + R(Y,X)Y\right) dt.
\end{aligned}
$$

But

$$
\begin{aligned}
\int_a^b g\left(X, \frac{D}{\partial t}\frac{DY}{\partial u}\right) dt &= \int_a^b \frac{\partial}{\partial t}\left(g\left(X, \frac{DY}{\partial u}\right)\right) dt - \int_a^b g\left(\frac{DX}{\partial t}, \frac{DY}{\partial u}\right) dt \\
&= g\left(X(b,u), \frac{DY(b,u)}{\partial u}\right) - g\left(X(a,u), \frac{DY(a,u)}{\partial u}\right) \\
&\quad - \int_a^b g\left(\frac{DX}{\partial t}, \frac{DY}{\partial u}\right) dt \\
&= -\int_a^b g\left(\frac{DX}{\partial t}, \frac{DY}{\partial u}\right) dt,
\end{aligned}
$$

because $Y(a,u) = 0$ and $Y(b,u) = 0$ for all $u \in (-\varepsilon, \varepsilon)$. Thus

$$
\begin{aligned}
\frac{d^2E(\varphi_u)}{du^2} &= \int_a^b \left(g\left(\frac{DY}{\partial t}, \frac{DY}{\partial t}\right) + g(X, R(Y,X)Y)\right) dt \\
&\quad - \int_a^b g\left(\frac{DX}{\partial t}, \frac{DY}{\partial u}\right) dt.
\end{aligned}
$$

Now let us set $u = 0$. We define the vector field V along γ by

$$
V(t) = Y(t,0) = \left.\frac{\partial \varphi(t,u)}{\partial u}\right|_{u=0}.
$$

Note that $X(t,0) = \gamma'(t)$ and

$$
\frac{DX(t,0)}{dt} = \frac{D\gamma'(t)}{dt} = 0
$$

(since γ is a geodesic. Therefore

$$
\begin{aligned}
&\left.\frac{d^2E(\varphi_u)}{du^2}\right|_{u=0} \\
&\quad = \int_a^b \left(g\left(\frac{DV(t)}{\partial t}, \frac{DV(t)}{\partial t}\right) + g(\gamma'(t), R(V(t), \gamma'(t))V(t))\right) dt \\
&\quad = \int_a^b \left(g\left(\frac{DV(t)}{\partial t}, \frac{DV(t)}{\partial t}\right) + R(\gamma'(t), V(t), V(t), \gamma'(t))\right) dt.
\end{aligned}
$$

We can integrate the first term in this formula by parts. Using the fact that $V(a) = 0$ and $V(b) = 0$ we see that

$$\int_a^b g\left(\frac{DV(t)}{\partial t}, \frac{DV(t)}{\partial t}\right)\, dt = -\int_a^b g\left(V(t), \frac{D^2V(t)}{\partial t^2}\right)\, dt$$

Also the standard properties of the Riemann curvature tensor ensure that

$$\begin{aligned} R(\gamma'(t), V(t), V(t), \gamma'(t)) &= -R(V(t), \gamma'(t), V(t), \gamma'(t)) \\ &= R(V(t), \gamma'(t), \gamma'(t), V(t)) \\ &= g(V(t), R(\gamma'(t), V(t))\gamma'(t)) \end{aligned}$$

(see Proposition 9.7). We conclude that

$$\left.\frac{d^2E(\varphi_u)}{du^2}\right|_{u=0} = \int_a^b g\left(V(t), R(\gamma'(t), V(t))\gamma'(t) - \frac{D^2V(t)}{dt^2}\right)\, dt.$$

Thus if V is a Jacobi field along γ then

$$\left.\frac{d^2E(\varphi_u)}{du^2}\right|_{u=0} = 0.$$