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Part II: Sections 5 to 7

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Contents

5 Tensors and Multilinear Algebra

5.1 The Dual of a Finite-Dimensional Vector Space

Let V be a vector space over a field K. (In applications to differential geometry and theoretical physics, the field K is usually the field of real numbers, though sometimes it is appropriate to take the field K of scalars to be the field of complex numbers.) The *dual space* V^* of V is the vector space over the field K consisting of all linear functionals from V to K. We define

$$\langle \varphi, \mathbf{v} \rangle = \varphi(\mathbf{v})$$

for all $\varphi \in V^*$ and $\mathbf{v} \in V$.

Suppose that the vector space V is finite-dimensional. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be a basis for V, where n is the dimension of V. Then there is a corresponding *dual basis* $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n$ of the dual space V^*. The elements of this dual basis satisfy the identities

$$\langle \varepsilon^j, \mathbf{e}_k \rangle = \varepsilon^j(\mathbf{e}_k) = \delta^j_k$$

for $j, k = 1, 2, \ldots, n$, where δ^j_k is the *Kronecker delta*, defined such that

$$\delta^j_k = \begin{cases} 1 & \text{if } j = k; \\ 0 & \text{otherwise.} \end{cases}$$

The dual space $(V^*)^*$ of V^* can be identified with the vector space V itself. Indeed each element $\mathbf{v}$ of V determines a linear functional on V^* which sends ω to $\langle \omega, \mathbf{v} \rangle$ for all $\omega \in V^*$. Moreover every linear functional on V^* is determined in this fashion by some element of the vector space $\mathbf{v}$. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be a basis of V, and let $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n$ be the corresponding dual basis of V^*. Then the basis of V that is the dual basis of the basis $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n$ of V^* is the original basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of the vector space V.

5.2 Multilinear Forms on Finite-Dimensional Vector Spaces

Let $V_1, V_2, \ldots, V_r$ and W be vector spaces over some field K. A function

$$S\colon V_1 \times V_2 \times \cdots V_r \to W$$

is said to be *multilinear* (or *K-multilinear*) if

$$\begin{aligned} S(\alpha \mathbf{v}'_1 + \beta \mathbf{v}''_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_r) \\ = \quad \alpha S(\mathbf{v}'_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_r) + \beta S(\mathbf{v}''_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_r) \end{aligned}$$

$$\begin{aligned}
&S(\mathbf{v}_1, \alpha\mathbf{v}_2' + \beta\mathbf{v}_2'', \mathbf{v}_3, \ldots, \mathbf{v}_r) \\
&\qquad = \alpha S(\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3, \ldots, \mathbf{v}_r) + \beta S(\mathbf{v}_1, \mathbf{v}_2'', \mathbf{v}_3, \ldots, \mathbf{v}_r) \\
&\qquad \ldots \\
&S(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \alpha\mathbf{v}_r' + \beta\mathbf{v}_r'') \\
&\qquad = \alpha S(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_r') + \beta S(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_r'')
\end{aligned}$$

for all $\mathbf{v}_1, \mathbf{v}_1', \mathbf{v}_1'' \in V_1$, $\mathbf{v}_2, \mathbf{v}_2', \mathbf{v}_2'' \in V_2$, ..., $\mathbf{v}_r, \mathbf{v}_r', \mathbf{v}_r'' \in V_r$, and for all $\alpha, \beta \in K$. The collection of all K-multilinear maps (or functions) from $V_1, V_2, \ldots, V_r$ to W is a vector space over the field K, which we denote by

$$\mathcal{M}_K(V_1, V_2, \ldots, V_r; W)$$

In particular, we denote by $\mathcal{M}_K(V_1, V_2, \ldots, V_r; K)$ the vector space consisting of all multilinear maps from $V_1 \times V_2 \times \cdots \times V_r$ to the field K.

We define

$$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_r \in \mathcal{M}_K(V_1^*, V_2^*, \ldots, V_r^*; K)$$

and

$$\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_r \in \mathcal{M}_K(V_1, V_2, \ldots, V_r; K)$$

for all

$$(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r) \in V_1 \times V_2, \ldots, V_r$$

and

$$(\omega_1, \omega_2, \ldots, \omega_r) \in V_1^* \times V_2^* \times \cdots \times V_r^*$$

such that

$$\begin{aligned}
&(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_r)(\omega_1, \omega_2, \ldots, \omega_r) \\
&\qquad = (\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_r)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r) \\
&\qquad = \langle \omega_1, \mathbf{v}_1 \rangle \langle \omega_2, \mathbf{v}_2 \rangle \cdots \langle \omega_r, \mathbf{v}_r \rangle \\
&\qquad = \omega_1(\mathbf{v}_1)\omega_2(\mathbf{v}_2) \cdots \omega_r(\mathbf{v}_r).
\end{aligned}$$

The multilinear map

$$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_r : V_1^* \times V_2^* \cdots \times V_r^* \to K$$

represents the *tensor product* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, and the multilinear map

$$\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_r : V_1 \times V_2 \cdots \times V_r \to K$$

represents the *tensor product* of $\omega_1, \omega_2, \ldots, \omega_r$.

Proposition 5.1 *Let K be a field, let $V_1, V_2, \ldots, V_r$ be finite-dimensional vector spaces over K, let $V_1^*, V_2^*, \ldots, V_r^*$ be the corresponding dual spaces, and, for each integer q between 1 and r, let*

$$(\mathbf{e}_{(q),j} : j = 1, 2, \ldots, n_q)$$

be a basis of the vector space V_q, where $n_q = \dim_K V_q$, and let

$$(\varepsilon^j_{(q)} : j = 1, 2, \ldots, n_q)$$

be the corresponding dual basis of the dual space V_q^*. Let

$$S \in \mathcal{M}_K(V_1, V_2, \ldots, V_r; K),$$

be a multilinear map from $V_1 \times V_2 \times \cdots \times V_k$ to K, and let

$$S_{j_1,j_2,\ldots,j_r} = S(\mathbf{e}_{(1),j_1}, \mathbf{e}_{(2),j_2}, \ldots, \mathbf{e}_{(r),j_r})$$

for all $(j_1, j_2, \ldots, j_r) \in J$, where

$$J = \{(j_1, j_2, \ldots, j_r) \in \mathbb{Z}^r : 1 \leq j_q \leq n_q \text{ for } q = 1, 2, \ldots, r\}.$$

Then

$$S = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_r=1}^{n_r} S_{j_1,j_2,\ldots,j_r} \varepsilon^{j_1}_{(1)} \otimes \varepsilon^{j_2}_{(2)} \otimes \cdots \otimes \varepsilon^{j_r}_{(r)}.$$

Thus if $(\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots, \mathbf{v}_{(r)}) \in V_1 \times V_2 \times \cdots \times V_r$, and if

$$v^j_{(q)} = \langle \varepsilon^j_{(q)}, \mathbf{v}_q \rangle = \varepsilon^j_{(q)}(\mathbf{v}_q)$$

for $q = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n_q$, so that

$$\mathbf{v}_{(q)} = \sum_{j_q=1}^{n_q} v^{j_q}_{(q)} \ \mathbf{e}_{(q),j_q}$$

for $q = 1, 2, \ldots, r$ then

$$S(\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots, \mathbf{v}_{(r)}) = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_r=1}^{n_r} S_{j_1,j_2,\ldots,j_r} v^{j_1}_{(1)} v^{j_2}_{(2)} \cdots v^{j_r}_{(r)}.$$

Proof The definition of $\varepsilon^{j_1}_{(1)} \otimes \varepsilon^{j_2}_{(2)} \otimes \cdots \otimes \varepsilon^{j_r}_{(r)}$ ensures that

$$\left(\varepsilon^{j_1}_{(1)} \otimes \varepsilon^{j_2}_{(2)} \otimes \cdots \otimes \varepsilon^{j_r}_{(r)}\right) (\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots, \mathbf{v}_{(r)}) = v^{j_1}_{(1)} v^{j_2}_{(2)} \cdots v^{j_r}_{(r)}$$

for all $(j_1, j_2, \ldots, j_r) \in J$. It follows from the multilinearity of S that

$$\begin{aligned}
&S(\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots, \mathbf{v}_{(r)}) \\
&= \sum_{j_1=1}^{n_1} S(\mathbf{e}_{(1),j_1}, \mathbf{v}_{(2)}, \ldots, \mathbf{v}_{(r)}) v_{(1)}^{j_1} \\
&= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} S(\mathbf{e}_{(1),j_1}, \mathbf{e}_{(2),j_2}, \mathbf{v}_{(3)}, \ldots, \mathbf{v}_{(r)}) v_{(1)}^{j_1} v_{(2)}^{j_2} \\
&\cdots \\
&= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_r=1}^{n_r} S(\mathbf{e}_{(1),j_1}, \mathbf{e}_{(2),j_2}, \mathbf{e}_{(r),j_r}) v_{(1)}^{j_1} v_{(2)}^{j_2} \cdots v_{(r)}^{j_r} \\
&= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_r=1}^{n_r} S_{j_1,j_2,\ldots,j_r} v_{(1)}^{j_1} v_{(2)}^{j_2} \cdots v_{(r)}^{j_r} \\
&= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_r=1}^{n_r} S_{j_1,j_2,\ldots,j_r} (\varepsilon_{(1)}^{j_1} \otimes \varepsilon_{(2)}^{j_2} \otimes \cdots \otimes \varepsilon_{(r)}^{j_r}) (\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots, \mathbf{v}_{(r)}).
\end{aligned}$$

The result follows. ∎

Remark In order to simplify notation slightly, it is convenient to denote a summation such as

$$\sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_r=1}^{n_r} S_{j_1,j_2,\ldots,j_r} \varepsilon_{(1)}^{j_1} \otimes \varepsilon_{(2)}^{j_2} \otimes \cdots \otimes \varepsilon_{(r)}^{j_r}$$

simply as

$$\sum_{j_1,j_2,\ldots,j_r} S_{j_1,j_2,\ldots,j_r} \varepsilon_{(1)}^{j_1} \otimes \varepsilon_{(2)}^{j_2} \otimes \cdots \otimes \varepsilon_{(r)}^{j_r},$$

where it is understood that each index j_q ranges over the set of all integers between 1 and the dimension n_q of the corresponding vector space V_q.

Corollary 5.2 *Let $V_1, V_2, \ldots, V_r$ be finite-dimensional vector spaces over a field K, let $V_1^*, V_2^*, \ldots, V_r^*$ be the corresponding dual spaces, and, for each integer q between 1 and r, let $\varepsilon_{(q)}^1, \varepsilon_{(q)}^2, \ldots, \varepsilon_{(q)}^{n_q}$ be a basis of the dual space V_q^* of V_q. Then the collection of all tensor products of the form*

$$\varepsilon_{(1)}^{j_1} \otimes \varepsilon_{(2)}^{j_2} \otimes \cdots \otimes \varepsilon_{(r)}^{j_r},$$

with $(j_1, j_2, \ldots, j_r) \in J$ constitutes a basis for the vector space

$$\mathcal{M}_K(V_1, V_2, \ldots, V_r; K)$$

of multilinear maps from $V_1 \times V_2 \times \cdots \times V_r$ to K, where

$$J = \{(j_1, j_2, \ldots, j_r) \in \mathbb{Z}^r : 1 \leq j_q \leq n_q \text{ for } q = 1, 2, \ldots, r\}.$$

Thus $\mathcal{M}_K(V_1, V_2, \ldots, V_r; K)$ is vector space of dimension $n_1 n_2 \cdots n_r$ over the field K.

Proof It follows from Proposition 5.1 that the collection of all tensor products of the form

$$\varepsilon_{(1)}^{j_1} \otimes \varepsilon_{(2)}^{j_2} \otimes \cdots \otimes \varepsilon_{(r)}^{j_r}$$

spans the vector space $\mathcal{M}_K(V_1, V_2, \ldots, V_r; K)$. Suppose that

$$\sum_{j_1, j_2, \ldots, j_r} S_{j_1, j_2, \ldots, j_r} \varepsilon_{(1)}^{j_1} \otimes \varepsilon_{(2)}^{j_2} \otimes \cdots \otimes \varepsilon_{(r)}^{j_r} = 0.$$

where $S_{j_1, j_2, \ldots, j_r} \in K$ for each $(j_1, j_2, \ldots, j_r) \in J$. For each $q \in \{1, 2, \ldots, r\}$, let $(\mathbf{e}_{(q),j} : j = 1, 2, \ldots, n_q)$ be the basis of V_q that has as its dual basis the basis $(\varepsilon_{(q)}^{j} : j = 1, 2, \ldots, n_q)$ of V_q^*. Then

$$\begin{aligned} 0 &= \sum_{j_1, j_2, \ldots, j_r} S_{j_1, j_2, \ldots, j_r} (\varepsilon_{(1)}^{j_1} \otimes \varepsilon_{(2)}^{j_2} \otimes \cdots \otimes \varepsilon_{(r)}^{j_r})(\mathbf{e}_{(1),k_1}, \mathbf{e}_{(2),k_2}, \ldots, \mathbf{e}_{(r),k_r}) \\ &= S_{k_1, k_2, \ldots, k_r} \end{aligned}$$

for all $(k_1, k_2, \ldots, k_r) \in J$. We conclude from this that the elements

$$\varepsilon_{(1)}^{j_1} \otimes \varepsilon_{(2)}^{j_2} \otimes \cdots \otimes \varepsilon_{(r)}^{j_r}$$

of $\mathcal{M}_K(V_1, V_2, \ldots, V_r; K)$ are linearly independent, and therefore constitute a basis of this vector space. It follows immediately that this vector space is of dimension $n_1 n_2 \cdots n_r$. ∎

Corollary 5.3 *Let $V_1, V_2, \ldots, V_r$ be finite-dimensional vector spaces over a field K, let $V_1^*, V_2^*, \ldots, V_r^*$ be the corresponding dual spaces, and, for each integer q between 1 and r, let $\mathbf{e}_{(q),1}, \mathbf{e}_{(q),2}, \ldots, \mathbf{e}_{(q),n_q}$ be a basis of the vector space V_q. Then the collection of all tensor products of the form*

$$\mathbf{e}_{(1),j_1} \otimes \mathbf{e}_{(2),j_2} \otimes \cdots \otimes \mathbf{e}_{(r),j_r},$$

with $(j_1, j_2, \ldots, j_r) \in J$ constitutes a basis for the vector space

$$\mathcal{M}_K(V_1^*, V_2^*, \ldots, V_r^*; K)$$

of multilinear maps from $V_1^*, V_2^*, \ldots, V_r^*$ to the field K of scalars, where of multilinear maps from $V_1^* \times V_2^* \times \cdots \times V_r^*$ to K, where

$$J = \{(j_1, j_2, \ldots, j_r) \in \mathbb{Z}^r : 1 \leq j_q \leq n_q \text{ for } q = 1, 2, \ldots, r\}.$$

Thus $\mathcal{M}_K(V_1^*, V_2^*, \ldots, V_r^*; K)$ is vector space of dimension $n_1 n_2 \cdots n_r$ over the field K.

Proof The dual space of V_q^* is the vector space V_q, for $q = 1, 2, \ldots, r$. Moreover if $\varepsilon_{(q)}^1, \varepsilon_{(q)}^2, \ldots, \varepsilon_{(q)}^{n_q}$ is the basis of V_q^* that is dual to the basis $\mathbf{e}_{(q),1}, \mathbf{e}_{(q),2}, \ldots, \mathbf{e}_{(q),n_q}$ then the latter basis is also the dual of the former. The result therefore follows directly on applying Corollary 5.2. ∎

Corollary 5.4 *Let $V_1, V_2, \ldots, V_r$ and W be finite-dimensional vector spaces over a field K, and let $V_1^*, V_2^*, \ldots, V_r^*$ be the dual spaces of $V_1, V_2, \ldots, V_r$. Then every multilinear map*

$$\lambda: V_1 \times V_2 \times \cdots \times V_r \to W$$

from $V_1 \times V_2 \times \cdots \times V_r$ to W determines a unique linear transformation

$$\hat{\lambda}: \mathcal{M}_K(V_1^*, V_2^*, \ldots, V_r^*; K) \to W$$

from $\mathcal{M}_K(V_1^*, V_2^*, \ldots, V_r^*; K)$ to W which satisfies

$$\hat{\lambda}(\mathbf{v}_{(1)} \otimes \mathbf{v}_{(2)} \otimes \cdots \otimes \mathbf{v}_{(r)}) = \lambda(\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots, \mathbf{v}_{(r)})$$

for all $(\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots \mathbf{v}_{(r)}) \in V_1 \times V_2 \times \cdots \times V_r$.

Proof Let $\mathbf{e}_{(q),1}, \mathbf{e}_{(q),2}, \ldots, \mathbf{e}_{(q),n_q}$ be a basis of the vector space V_q. Then the collection of all tensor products of the form

$$\mathbf{e}_{(1),j_1} \otimes \mathbf{e}_{(2),j_2} \otimes \cdots \otimes \mathbf{e}_{(r),j_r},$$

where $j_q \in \{1, 2, \ldots, r\}$ for $q = 1, 2, \ldots, r$, constitutes a basis for the vector space $\mathcal{M}_K(V_1^*, V_2^*, \ldots, V_r^*; K)$. It follows that, given any multilinear map

$$\lambda: V_1 \times V_2 \times \cdots \times V_r \to W,$$

there exists a unique linear transformation

$$\hat{\lambda}: \mathcal{M}_K(V_1^*, V_2^*, \ldots, V_r^*; K) \to W$$

characterized by the property that

$$\hat{\lambda}(\mathbf{e}_{(1),j_1} \otimes \mathbf{e}_{(2),j_2} \otimes \cdots \otimes \mathbf{e}_{(r),j_r}) = \lambda(\mathbf{e}_{(1),j_1}, \mathbf{e}_{(2),j_2}, \ldots, \mathbf{e}_{(r),j_r})$$

for all $(j_1, j_2, \ldots, j_r) \in J$, where

$$J = \{(j_1, j_2, \ldots, j_r) \in \mathbb{Z}^r : 1 \leq j_q \leq n_q \text{ for } q = 1, 2, \ldots, r\}.$$

The multilinearity of λ and the linearity of $\hat{\lambda}$ then ensure that

$$\hat{\lambda}(\mathbf{v}_{(1)} \otimes \mathbf{v}_{(2)} \otimes \cdots \otimes \mathbf{v}_{(r)}) = \lambda(\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots, \mathbf{v}_{(r)})$$

for all $(\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots \mathbf{v}_{(r)}) \in V_1 \times V_2 \times \cdots \times V_r$, as required. ∎

5.3 Tensor Products of Finite-Dimensional Vector Spaces

Definition Let $V_1, V_2, \ldots, V_r$ be finite-dimensional vector spaces over some field K. We define the *tensor product* $V_1 \otimes V_2 \otimes \cdots \otimes V_r$ of $V_1, V_2, \ldots, V_r$ to be the vector space $\mathcal{M}_K(V_1^*, V_2^*, \ldots, V_r^*; K)$ whose elements are multilinear maps from $V_1^* \times V_2^* \times \cdots \times V_r^*$ to the field K of scalars.

Let $V_1, V_2, \ldots, V_r$ be finite-dimensional vector spaces over a field K. Then there is a well-defined multilinear map

$$\mu: V_1 \times V_2 \times \cdots \times V_r \to V_1 \otimes V_2 \otimes \cdots \otimes V_r$$

which is defined such that

$$\mu(\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots, \mathbf{v}_{(r)}) = \mathbf{v}_{(1)} \otimes \mathbf{v}_{(2)} \otimes \cdots \otimes \mathbf{v}_{(r)}$$

for all $(\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \ldots \mathbf{v}_{(r)}) \in V_1 \times V_2 \times \cdots \times V_r$. Moreover it follows from Corollary 5.4 that, given any finite-dimensional vector space W, and given any multilinear map

$$\lambda: V_1 \times V_2 \times \cdots \times V_r \to W,$$

there exists a unique linear transformation

$$\hat{\lambda}: V_1 \otimes V_2 \otimes \cdots \otimes V_r \to W$$

such that $\lambda = \hat{\lambda} \circ \mu$. This property is the *universal property* that characterizes tensor products of finite-dimensional vector spaces.

Proposition 5.5 *Let $T \in V_1 \otimes V_2 \otimes \cdots \otimes V_r$, where $V_1, V_2, \ldots, V_r$ are finite-dimensional vector spaces over a field K, let*

$$(\mathbf{e}_{(q),j} : j = 1, 2, \ldots, n_q) \quad \textit{and} \quad (\mathbf{f}_{(q),k} : k = 1, 2, \ldots, n_q)$$

be bases of the vector space V_q for $q = 1, 2, \ldots, r$, where $n_q = \dim_K V_q$, and let $T^{j_1, j_2, \ldots, j_r} \in K$ and $\hat{T}^{k_1, k_2, \ldots, k_r} \in K$ be defined for all $(j_1, j_2, \ldots, j_r) \in J$ and $(k_1, k_2, \ldots, k_r) \in J$, where

$$J = \{(j_1, j_2, \ldots, j_r) \in \mathbb{Z}^r : 1 \le j_q \le n_q \textit{ for } q = 1, 2, \ldots, r\},$$

so that

$$\begin{aligned} T &= \sum_{(j_1, j_2, \ldots, j_r) \in J} T^{j_1, j_2, \ldots, j_r} \mathbf{e}_{(1),j_1} \otimes \mathbf{e}_{(2),j_2} \otimes \cdots \otimes \mathbf{e}_{(r),j_r} \\ &= \sum_{(k_1, k_2, \ldots, k_r) \in J} \hat{T}^{k_1, k_2, \ldots, k_r} \mathbf{f}_{(1),k_1} \otimes \mathbf{f}_{(2),k_2} \otimes \cdots \otimes \mathbf{f}_{(r),k_r}. \end{aligned}$$

Suppose that

$$\mathbf{f}_{(q),k} = \sum_{j=1}^{n_q} (A_{(q)})^j_k \mathbf{e}_{(q),j}$$

for $q = 1, 2, \ldots, r$, *where* $(A_{(q)})^j_k \in K$ *for* $j, k = 1, 2, \ldots, n_q$. *Then*

$$T^{j_1,j_2,\ldots,j_r} = \sum_{(k_1,k_2,\ldots,k_r)\in J} (A_{(1)})^{j_1}_{k_1} (A_{(2)})^{j_2}_{k_2} \cdots (A_{(r)})^{j_r}_{k_r} \hat{T}^{k_1,k_2,\ldots,k_r}$$

for all $(j_1, j_2, \ldots, j_r) \in J$.

Proof It follows from the multilinearity of the tensor product that

$$\begin{aligned} &\mathbf{f}_{(1),k_1} \otimes \mathbf{f}_{(2),k_2} \otimes \cdots \otimes \mathbf{f}_{(r),k_r}. \\ &= \sum_{(j_1,j_2,\ldots,j_k)\in J} (A_{(1)})^{j_1}_{k_1} (A_{(2)})^{j_2}_{k_2} \cdots (A_{(r)})^{j_r}_{k_r} \mathbf{e}_{(1),j_1} \otimes \mathbf{e}_{(2),j_2} \otimes \cdots \otimes \mathbf{e}_{(r),j_r} \end{aligned}$$

for all $(k_1, k_2, \ldots, k_r) \in J$. The required result follows directly, by substituting in the above equation into the equation

$$T = \sum_{(k_1,k_2,\ldots,k_r)\in J} \hat{T}^{k_1,k_2,\ldots,k_r} \mathbf{f}_{(1),k_1} \otimes \mathbf{f}_{(2),k_2} \otimes \cdots \otimes \mathbf{f}_{(r),k_r}$$

and then equating coefficients of

$$\mathbf{e}_{(1),j_1} \otimes \mathbf{e}_{(2),j_2} \otimes \cdots \otimes \mathbf{e}_{(r),j_r}$$

in the resulting formula for T. ∎

5.4 Tensors

Definition Let V be a finite-dimensional vector space over a field K. A *tensor* of type (r, s) on V is an element of the vector space $V^{\otimes r} \otimes V^{*\otimes s}$ that is the tensor product of r copies of the vector space V and s copies of the dual vector space V^*.

Let V be a vector space of dimension n over a field K, let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be a basis of V, and let $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n$ be the dual basis of V^*, which is defined so that $\langle \varepsilon^k, \mathbf{e}_j \rangle = \delta^k_j$ for $j, k = 1, 2, \ldots, n$, where δ^k_j is the Kronecker delta. Let T be a tensor of type (r, s) on V. Then there exist scalars $T^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} \in K$ such that

$$T = \sum_{j_1,j_2,\ldots,j_r} \sum_{k_1,k_2,\ldots,k_s} T^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} \mathbf{e}_{j_1} \otimes \mathbf{e}_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_r} \otimes \varepsilon^{k_1} \otimes \varepsilon^{k_2} \otimes \cdots \otimes \varepsilon^{k_s}.$$

Let $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ be another basis for V, and let $\eta^1, \eta^2, \ldots, \eta^n$ be the corresponding dual basis for V^*, so that $\langle \eta^q, \mathbf{f}_p \rangle = \delta_p^q$ for $p, q = 1, 2, \ldots, n$. Then there exist non-singular matrices A and B, with coefficients A_p^j and B_k^q in K, such that

$$\mathbf{f}_p = \sum_{j=1}^n A_p^j \mathbf{e}_j \quad \text{and} \quad \eta^q = \sum_{k=1}^n B_k^q \varepsilon^k$$

for $p, q = 1, 2, \ldots, n$. Then

$$\delta_q^p = \langle \eta^q, \mathbf{f}_p \rangle = \sum_{j=1}^n \sum_{k=1}^n B_k^q A_p^j \langle \varepsilon^k, \mathbf{e}_j \rangle = \sum_{j=1}^n \sum_{k=1}^n B_k^q A_p^j \delta_j^k = \sum_{j=1}^n B_j^q A_p^j$$

for $p, q = 1, 2, \ldots, n$. It follows that the matrix product BA is the identity matrix, and thus $B = A^{-1}$. We may therefore write $B_k^q = (A^{-1})_k^q$.

Proposition 5.6 *Let V be a vector space of dimension n over a field K, let*

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \quad \textit{and} \quad \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$$

be bases of V, let

$$\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n \quad \textit{and} \quad \eta^1, \eta^2, \ldots, \eta^n$$

be the corresponding dual bases of V^*, and let A be the $n \times n$ matrix with coefficients A_k^j in K such that $\mathbf{f}_p = \sum_{j=1}^n A_p^j \mathbf{e}_j$ for $p = 1, 2, \ldots, n$. Let $T \in V^{r,s}$ be a tensor of type (r, s) on V, and let

$$\begin{aligned} T &= \sum_{j_1, j_2, \ldots, j_r} \sum_{k_1, k_2, \ldots, k_s} T^{j_1, j_2, \ldots, j_r}_{k_1, k_2, \ldots, k_s} \mathbf{e}_{j_1} \otimes \mathbf{e}_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_r} \otimes \varepsilon^{k_1} \otimes \varepsilon^{k_2} \otimes \cdots \otimes \varepsilon^{k_s} \\ &= \sum_{p_1, p_2, \ldots, p_r} \sum_{q_1, q_2, \ldots, q_s} \hat{T}^{p_1, p_2, \ldots, p_r}_{q_1, q_2, \ldots, q_s} \mathbf{f}_{p_1} \otimes \mathbf{f}_{p_2} \otimes \cdots \otimes \mathbf{f}_{p_r} \otimes \eta^{q_1} \otimes \eta^{q_2} \otimes \cdots \otimes \eta^{q_s}, \end{aligned}$$

where the coefficients $T^{j_1, j_2, \ldots, j_r}_{k_1, k_2, \ldots, k_s}$ and $\hat{T}^{p_1, p_2, \ldots, p_r}_{q_1, q_2, \ldots, q_s}$ of T with respect to the relevant bases are scalars belonging to the field K. Then

$$\begin{aligned} &T^{j_1, j_2, \ldots, j_r}_{k_1, k_2, \ldots, k_s} \\ &\quad = \sum_{p_1, p_2, \ldots, p_r} \sum_{q_1, q_2, \ldots, q_s} A^{j_1}_{p_1} A^{j_2}_{p_2} \cdots A^{j_r}_{p_r} (A^{-1})^{q_1}_{k_1} (A^{-1})^{q_2}_{k_2} \cdots (A^{-1})^{q_s}_{k_s} \hat{T}^{p_1, p_2, \ldots, p_r}_{q_1, q_2, \ldots, q_s} \end{aligned}$$

for all values of the indices $j_1, j_2, \ldots, j_r$ and $k_1, k_2, \ldots, k_s$ as these indices range from 1 to n.

Proof This result follows directly on applying Proposition 5.5. ∎

Let V be a vector space over a field K. A *covariant tensor* of rank s on V is a tensor of type $(0, s)$. A covariant tensor of rank s corresponds to a multilinear map from V^s to K. Covariant tensors thus represent multilinear forms on the vector space V.

A *contravariant tensor* of rank r on V is a tensor of type $(r, 0)$. A contravariant tensor of rank r corresponds to a multilinear map from V^{*r} to K. Contravariant tensors thus represent multilinear forms on the dual V^* of the vector space V.

The space $V^{(1,0)}$ of tensors of rank $(1, 0)$ on the vector space V is isomorphic to the vector space V itself.

The space $V^{(0,1)}$ of tensors of rank $(0, 1)$ on the vector space V is isomorphic to the dual V^* of the vector space V.

Tensors of type $(1, 1)$ on the vector space V represent linear operators on V.

A tensor of type $(1, s)$ represents a multilinear map from V^s to V.

Example Let V be a vector space of dimension n over a field K, and let R be a tensor of type $(1, 3)$ on V. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be a basis of V, and let $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n$ be the dual basis of V^*, which is defined so that $\langle \varepsilon^k, \mathbf{e}_j \rangle = \delta^k_j$ for $j, k = 1, 2, \ldots, n$, where δ^k_j is the Kronecker delta. Then

$$R = \sum_{h,i,j,k} R^h{}_{ijk}\, \mathbf{e}_h \otimes \varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^k,$$

where the above summation is taken over all values of the indices h, i, j and k between 1 and n. This tensor determines a trilinear map from $V \times V \times V$ to V. This trilinear map sends $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ to $R(\mathbf{u}, \mathbf{v}, \mathbf{w})$ for all $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in V \times V \times V$, where

$$R(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{h,i,j,k} R^h{}_{ijk}\, \langle \varepsilon^i, \mathbf{u} \rangle\, \langle \varepsilon^j, \mathbf{v} \rangle\, \langle \varepsilon^k, \mathbf{w} \rangle\, \mathbf{e}_h.$$

Let u^p, v^p and w_p denote the pth components of the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ for $p = 1, 2, \ldots, n$, where these components are taken with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, so that

$$\mathbf{u} = \sum_{p=1}^n u^p \mathbf{e}_p, \quad \mathbf{v} = \sum_{p=1}^n v^p \mathbf{e}_p, \quad \mathbf{w} = \sum_{p=1}^n w^p \mathbf{e}_p.$$

Then

$$\langle \varepsilon^i, \mathbf{u} \rangle = u^i, \quad \langle \varepsilon^j, \mathbf{v} \rangle = v^j, \quad \langle \varepsilon^k, \mathbf{w} \rangle = w^k,$$

and therefore

$$R(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{h,i,j,k} R^h{}_{ijk} u^i v^j w^k \, \mathbf{e}_h.$$

Remark The *Riemann curvature tensor* of Riemannian geometry and General Relativity is a tensor of type $(1, 3)$ on each tangent space of a Riemannian or pseudo-Riemannian manifold. The Riemann curvature tensor on the tangent space at any point of a Riemannian manifold thus determines a trilinear map sending a triple of tangent vectors at that point to a single tangent vector.

6 Vector Bundles

6.1 Smooth Vector Bundles

Definition Let E and M be a smooth manifolds, let k be a non-negative integer, and let $\pi_E: E \to M$ be a smooth surjective map. Suppose that, for each point p of M, the subset E_p of E consisting of those elements e of E that satisfy $\pi_E(e) = p$ has operations of addition and scalar multiplication defined on it, with respect to which it is a real vector space of dimension k. Suppose also that, given any point p_0 of M, there exists an open set U containing p_0 and a smooth map $\psi: U \times \mathbb{R}^k \to E$ which satisfies the following conditions:

(i) the function ψ maps $U \times \mathbb{R}^k$ diffeomorphically onto $\pi_E^{-1}(U)$;

(ii) $\pi_E(\psi(p, \mathbf{v})) = p$ for all $p \in U$ and $\mathbf{v} \in \mathbb{R}^k$;

(iii) for each $p \in U$, the map $\psi_p: \mathbb{R}^k \to E_p$ is an isomorphism of real vector spaces, where

$$\psi_p(\mathbf{v}) = \psi(p, \mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R}^k$.

The smooth manifold E and the smooth map $\pi_E: E \to M$ then constitute a *smooth real vector bundle* over M of *rank* k with *total space* E, *base space* M and *projection map* $\pi_E: E \to M$.

Definition Let $\pi_E: E \to M$ be a smooth vector bundle over a smooth manifold M. Given any point p of M, the *fibre* of the vector bundle $\pi_E: E \to M$ over the point p of M is the real vector space E_p, where

$$E_p = \pi_E^{-1}(\{p\}) = \{e \in E : \pi_E(e) = p\}.$$

Definition Let $\pi_E: E \to M$ and $\pi_{\hat{E}}: \hat{E} \to M$ be smooth vector bundles over a smooth manifold M. A function $\varphi E \to \hat{E}$ between the total spaces of these vector bundles is said to be an *isomorphism of vector bundles* over M provided that it satisfies the following conditions:

(i) $\varphi: E \to \hat{E}$ is a diffeomorphism;

(ii) given any point p of M, the restriction of φ to the fibre E_p of the vector bundle $\pi_E: E \to M$ over the point p yields an isomorphism $\varphi_p: E_p \to \hat{E}_p$ of vector spaces between E_p and the corresponding fibre $\hat{E}_p$ of the vector bundle $\pi_{\hat{E}}: \hat{E} \to M$ over the point p.

Vector bundles $\pi_E: E \to M$ and $\pi_{\hat{E}}: \hat{E} \to M$ are said to be *isomorphic* as vector bundles over M if there exists an isomorphism between them.

Let $\pi_E: E \to M$ and $\pi_{\hat{E}}: \hat{E} \to M$ be smooth vector bundles over M, and let the smooth map $\varphi: E \to \hat{E}$ be an isomorphism of vector bundles over M. Then $\pi_{\hat{E}} \circ \varphi = \pi_E$.

Definition Let M be a smooth manifold. The *product bundle* of rank k over M is the smooth vector bundle $\pi_E: E \to M$, where $E = M \times \mathbb{R}^k$, $\pi_E(p, \mathbf{v}) = p$ for all $p \in M$ and $\mathbf{v} \in \mathbb{R}^k$, and where, for each point p of M, the vector space structure on the fibre $\pi_E^{-1}(\{p\})$ is defined such that

$$\lambda \mathbf{v}_p + \mu \mathbf{w}_p = (\lambda \mathbf{v} + \mu \mathbf{w})_p$$

for all $p \in M$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^k$ and $\lambda, \mu \in \mathbb{R}$, where $\mathbf{v}_p = (p, \mathbf{v})$ for all $p \in M$ and $\mathbf{v} \in \mathbb{R}_k$.

Definition A smooth vector bundle $\pi_E: E \to M$ of rank k over a smooth manifold M is said to be (topologically) *trivial* if it is isomorphic (as a smooth vector bundle) to the product bundle of rank k over M.

Lemma 6.1 *A smooth vector bundle $\pi_E: E \to M$ over a smooth manifold M is trivial if and only if there exists a diffeomorphism $\psi: M \times \mathbb{R}^k \to E$ such that $\pi_E(\psi(p, \mathbf{v})) = p$ and $\psi_p: \mathbb{R}^k \to E_p$ is an isomorphism of real vector spaces for all $p \in M$, where $E_p = \pi_E^{-1}(\{p\})$ and $\psi_p(\mathbf{v}) = \psi(p, \mathbf{v})$ for all $p \in M$ and $\mathbf{v}$.*

Proof This result follows immediately from the relevant definitions. ∎

Definition Let $\pi_E: E \to M$ be a smooth vector bundle over a smooth manifold M, and let U be an open subset of M. Then $\pi_{E|U}: E|U \to U$ is a smooth vector bundle over U, where $E|U = \pi_E^{-1}(U)$ and $\pi_{E|U} = \pi_E|\pi_E^{-1}(U)$ (so that $E|U$ is the union of the fibres of $\pi_E: E \to M$ that project to points of U, and $\pi_{E|U}$ is the restriction of the projection map π_E to $E|U$). We refer to this smooth vector bundle $\pi_{E|U}: E|U \to U$ as the *restriction* of the vector bundle $\pi_E: E \to M$ to the open set U.

Lemma 6.2 *Let $\pi_E: E \to M$ be a smooth vector bundle over a smooth manifold M, and let $p \in M$. Then there exists an open set U in M, where $p \in U$, such that the restriction $\pi_{E|U}: E|U \to U$ of this vector bundle to the open set U is isomorphic to a product bundle over U, and is thus a trivial bundle over U.*

Proof The result follows immediately from the relevant definitions and from Lemma 6.1. ∎

Definition Let $\pi_E: E \to M$ be a smooth vector bundle over a smooth manifold M, and let U be an open set in M. A continous map $s: U \to E$ is said to be a continuous *section* of the vector bundle over U if $\pi_E(s(p)) = p$ for all $p \in U$.

Lemma 6.3 *A smooth vector bundle $\pi_E: E \to M$ of rank r over a smooth manifold M is trivial if and only if there exist smooth sections $s_1, s_2, \ldots, s_r$ of $\pi_E: E \to M$ such that, for each point p of M, the elements*

$$s_1(p), s_2(p), \ldots, s_r(p)$$

of the fibre E_p of the bundle over p constitute a basis of the real vector space E_p.

Proof Suppose that there exist smooth sections $s_1, s_2, \ldots, s_r$ of $\pi_E: E \to M$ such that, for all $p \in M$, the elements $s_1(p), s_2(p), \ldots, s_r(p)$ of the fibre E_p of the vector bundle over the point p constitute a basis of the real vector space E_p. Define $\psi: M \times \mathbb{R}^r \to E$ so that

$$\psi(p, (v_1, v_2, \ldots, v_k)) = v_1 s_1(p) + v_2 s_2(p) + \cdots + v_r s_r(p)$$

for all $p \in M$ and $(v_1, v_2, \ldots, v_r) \in \mathbb{R}^r$. Then $\psi: M \times \mathbb{R}^r \to E$ is a diffeomorphism. Moreover this diffeomorphism is an isomorphism of smooth vector bundles over M, where we regard $M \times \mathbb{R}^r$ as a product bundle over M with fibre $\mathbb{R}^r$. Thus the smooth vector bundle $\pi_E: E \to M$ is trivial.

Conversely if the smooth vector bundle $\pi_E: E \to M$ is trivial then there exists a diffeomorphism $\psi: M \times \mathbb{R}^r \to E$ which is an isomorphism of smooth vector bundles over M. Let Let $s_1, s_2, \ldots, s_r$ be the smooth sections of $\pi_E: E \to M$ defined such that

$$\begin{aligned} s_1(p) &= \psi(p, (1, 0, \ldots, 0)), \\ s_2(p) &= \psi(p, (0, 1, \ldots, 0)), \\ &\vdots \\ s_r(p) &= \psi(p, (0, 1, \ldots, r)) \end{aligned}$$

for all $p \in M$. Then the elements $s_1(p), s_2(p), \ldots, s_r(p)$ of E_p constitute a basis of the real vector space E_p at each point of M, as required. ∎

6.2 Patching Constructions

Let k be a non-negative integer. We denote by $\mathrm{GL}(k,\mathbb{R})$ the group of all non-singular $k \times k$ matrices with real coefficients. This group is an open subset of the real vector space consisting of all $k \times k$ matrices with real coefficients. The operation of matrix multiplication determines a smooth function from $\mathrm{GL}(k,\mathbb{R}) \times \mathrm{GL}(k,\mathbb{R})$ to $\mathrm{GL}(k,\mathbb{R})$, and the operation of matrix inversion is a smooth function from $\mathrm{GL}(k,\mathbb{R})$ to itself. Each element B of the group $\mathrm{GL}(k,\mathbb{R})$ determines an isomorphism of real vector spaces from $\mathbb{R}^k$ to itself that sends $\mathbf{v} \in \mathbb{R}^k$ to $B\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^k$. Moreover every vector space isomorphism from $\mathbb{R}^k$ to itself is determined in this fashion by some element of the group $\mathrm{GL}(k,\mathbb{R})$.

Proposition 6.4 *Let $\pi_E: E \to M$ be a smooth vector bundle of rank k over a smooth manifold M. Then there exists an open cover $(U_\alpha : \alpha \in A)$ of M, indexed by some indexing set A, and smooth maps*

$$\psi_\alpha: U_\alpha \times \mathbb{R}^k \to E,$$

and

$$g_{\alpha,\beta}: U_\alpha \cap U_\beta \to GL(k,\mathbb{R})$$

for all $\alpha, \beta \in A$, where these smooth maps satisfy the following properties:—

(i) $\pi_E(\psi_\alpha(p,\mathbf{v})) = p$ *for all* $p \in U_\alpha$ *and* $\mathbf{v} \in \mathbb{R}^k$*;*

(ii) ψ_α *maps* $U_\alpha \times \mathbb{R}^k$ *diffeomorphically onto* $\pi_E^{-1}(U_\alpha)$

(iii) *for each* $p \in U_\alpha$*, the map* $(\psi_\alpha)_p: \mathbb{R}^k \to E_p$ *is an isomorphism of real vector spaces, where*

$$(\psi_\alpha)_p(\mathbf{v}) = \psi_\alpha(p,\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R}^k$*;*

(iv) $\psi_\beta(p,\mathbf{v}) = \psi_\alpha(p, g_{\alpha,\beta}(p)\mathbf{v})$ *for all* $\alpha, \beta \in A$*,* $p \in U_\alpha \cap U_\beta$ *and* $\mathbf{v} \in \mathbb{R}^k$*;*

(v) $g_{\alpha,\beta}(p) = (\psi_\alpha)_p^{-1}(\psi_\beta)_p$ *for all* $\alpha, \beta \in A$ *and* $p \in U_\alpha \cap U_\beta$*;*

(vi) $g_{\alpha,\alpha}(p)$ *is the identity matrix for all* $\alpha \in A$ *and* $p \in U_\alpha$*;*

(vii) $g_{\beta,\alpha}(p) = g_{\alpha,\beta}(p)^{-1}$ *for all* $\alpha, \beta \in A$ *and* $p \in U_\alpha \cap U_\beta$*;*

(viii) $g_{\alpha,\beta}(p) g_{\beta,\gamma}(p) = g_{\alpha,\gamma}(p)$ *for all* $\alpha, \beta, \gamma \in A$ *and* $p \in U_\alpha \cap U_\beta \cap U_\gamma$*.*

Proof The existence of the open cover $(U_\alpha : \alpha \in A)$ and the smooth functions ψ_α satisfying conditions (i), (ii), (ii) follows immediately from the definition of a smooth vector bundle, and is a mere restatement of that definition. Then functions $g_{\alpha,\beta}$ can be defined by the equation given in (v), and these functions will satisfy propertites (iv), (vi), (vii) and (viii). ∎

Proposition 6.5 *Let M be a smooth manifold, let E be a set, let $\pi_E: E \to M$ be a surjective function, let $(U_\alpha : \alpha \in A)$ be collection of open sets in M indexed by a set A, let k be a non-negative integer, and, for all $\alpha, \beta \in A$, let $\psi_\alpha: U_\alpha \times \mathbb{R}^k \to E$ and $g_{\alpha,\beta}: U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ be functions that satisfy the following conditions:—*

(i) $\bigcup_{\alpha \in A} U_\alpha = M$,

(ii) $\pi_E(\psi_\alpha(p, \mathbf{v})) = p$ *for all* $\alpha \in A$, $p \in U_\alpha$ *and* $\mathbf{v} \in \mathbb{R}^k$*;*

(iii) *the function* $\psi_\alpha: U_\alpha \times \mathbb{R}^k \to E$ *maps* $U \times \mathbb{R}^k$ *bijectively onto* $\pi_E^{-1}(U_\alpha)$ *for all* $\alpha \in A$*;*

(iv) $\psi_\beta(p, \mathbf{v}) = \psi_\alpha(p, g_{\alpha,\beta}(p)\mathbf{v})$ *for all* $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$ *and* $\mathbf{v} \in \mathbb{R}^k$*;*

(v) *the function* $g_{\alpha,\beta}: U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ *is smooth for all* $\alpha, \beta \in A$.

Then there exists a topology and smooth structure on the set E with respect to which E is a smooth manifold, $\pi_E: E \to M$ is a smooth map, and the function $\psi_\alpha: U_\alpha \times \mathbb{R}^k \to E$ maps $U_\alpha \times \mathbb{R}^k$ diffeomorphically onto $\pi_E^{-1}(U_\alpha)$ for all $\alpha \in A$. The smooth manifold E and the smooth map $\pi_E: E \to M$ then constitute a smooth vector bundle of rank k over the smooth manifold M.

Proof Let $\tau_{\alpha,\beta}: (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to \mathbb{R}^k$ be defined for all $\alpha, \beta \in A$ such that

$$\tau_{\alpha,\beta}(p, \mathbf{v}) = g_{\alpha,\beta}(p)\mathbf{v}$$

for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$ and $\mathbf{v} \in \mathbb{R}^k$. Then these functions $\tau_{\alpha,\beta}$ are smooth functions, and $\psi_\beta(p, \mathbf{v}) = \psi_\alpha(p, \tau_{\alpha,\beta}(p, \mathbf{v}))$ for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$ and $\mathbf{v} \in \mathbb{R}^k$. The result therefore follows on applying Proposition 4.6. ∎

Corollary 6.6 *Let M be a smooth manifold, let $(U_\alpha : \alpha \in A)$ be an open cover of M indexed by a set A, let k be a non-negative integer, and, for all $\alpha, \beta \in A$, let $g_{\alpha,\beta}: U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ be a smooth map that satisfy the following conditions:—*

(i) $g_{\alpha,\alpha}(p)$ *is the identity matrix for all* $\alpha \in A$ *and* $p \in U_\alpha$*;*

(ii) $g_{\beta,\alpha}(p) = g_{\alpha,\beta}(p)^{-1}$ *for all* $\alpha, \beta \in A$ *and* $p \in U_\alpha \cap U_\beta$*;*

(iii) $g_{\alpha,\beta}(p)g_{\beta,\gamma}(p) = g_{\alpha,\gamma}(p)$ *for all* $\alpha, \beta, \gamma \in A$ *and* $p \in U_\alpha \cap U_\beta \cap U_\gamma$.

Then there exists a smooth vector bundle $\pi_E: E \to M$ *over* M *and smooth maps*

$$\psi_\alpha: U_\alpha \times \mathbb{R}^k \to E$$

which satisfy the following properties:

(iv) $\pi_E(\psi_\alpha(p, \mathbf{v})) = p$ *for all* $p \in U_\alpha$ *and* $\mathbf{v} \in \mathbb{R}^k$*;*

(v) ψ_α *maps* $U_\alpha \times \mathbb{R}^k$ *diffeomorphically onto* $\pi_E^{-1}(U_\alpha)$*;*

(vi) *for each* $p \in U_\alpha$*, the map* $(\psi_\alpha)_p: \mathbb{R}^k \to E_p$ *is an isomorphism of real vector spaces, where*

$$(\psi_\alpha)_p(\mathbf{v}) = \psi_\alpha(p, \mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R}^r$*;*

(vii) $\psi_\beta(p, \mathbf{v}) = \psi_\alpha(p, g_{\alpha,\beta}(p)\mathbf{v})$ *for all* $\alpha, \beta \in A$*,* $p \in U_\alpha \cap U_\beta$ *and* $\mathbf{v} \in \mathbb{R}^k$*.*

Proof Let

$$X = \{(\alpha, p, \mathbf{v}) \in A \times M \times \mathbb{R}^k : p \in U_\alpha\}.$$

We define a relation $\sim$ on X, where elements $(\alpha, p, \mathbf{v})$ and $(\beta, q, \mathbf{w})$ of X satisfy $(\alpha, p, \mathbf{v}) \sim (\beta, q, \mathbf{w})$ if and only if

$$p = q \text{ and } \mathbf{w} = g_{\beta,\alpha}(p)\mathbf{v}.$$

Conditions (i), (ii) and (iii) ensure that the relation $\sim$ on X is reflexive, symmetric and transitive, and is thus an equivalence relation. Let E be the set of equivalence classes of elements of X under the equivalence relation $\sim$. We denote by $[\alpha, p, \mathbf{v}]$ the equivalence class of an element $(\alpha, p, \mathbf{v})$ of X. The definition of the equivalence relation $\sim$ ensures that there is a well-defined function $\pi_E: E \to M$, where $\pi_E([\alpha, p, \mathbf{v}]) = p$ for all $(\alpha, p, \mathbf{v}) in X$.

Let $\psi_\alpha(p, \mathbf{v}) = [\alpha, p, \mathbf{v}]$ for all $\alpha \in A$, $p \in U_\alpha$ and $\mathbf{v} \in \mathbb{R}^k$. Then

$$\begin{aligned} \psi_\beta(p, \mathbf{w}) &= [\beta, p, \mathbf{w}] = [\beta, p, g_{\beta,\alpha}(p)(g_{\alpha,\beta}(p)(\mathbf{w}))] = [\alpha, p, g_{\alpha,\beta}(p)\mathbf{w}] \\ &= \psi_\alpha(p, g_{\alpha,\beta}(p)\mathbf{w}) \end{aligned}$$

for all $p \in U_\alpha \cap U_\beta$ and $\mathbf{w} \in \mathbb{R}^k$. Let $E_p = \pi_E^{-1}(\{p\})$ for all $p \in M$. Then

$$E_p = \{[\alpha, p, \mathbf{v}] : \mathbf{v} \in \mathbb{R}^k\}.$$

for all $p \in U_\alpha$. Now if elements $(\alpha, p, \mathbf{v}_1)$, $(\alpha, p, \mathbf{v}_2)$, $(\beta, p, \mathbf{w}_1)$ and $(\beta, p, \mathbf{w}_2)$ are elements of X, and if $[\alpha, p, \mathbf{v}_1] = [\beta, p, \mathbf{w}_1]$ and $[\alpha, p, \mathbf{v}_2] = [\beta, p, \mathbf{w}_2]$ then

$$[\alpha, p, \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2] = [\beta, p, \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2]$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$, because

$$\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 = \lambda_1 g_{\beta,\alpha}(p)(\mathbf{v}_1) + \lambda_2 g_{\beta,\alpha}(p)(\mathbf{v}_2) = g_{\beta,\alpha}(p)(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2).$$

It follows that, for all $p \in M$, the fibre E_p of $\pi_E: E \to M$ over p can be given the structure of a real vector space, where

$$\lambda_1[\alpha, p, \mathbf{v}_1] + \lambda_2[\alpha, p, \mathbf{v}_2] = [\alpha, p, \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2]$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^k$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. The function that sends $\mathbf{v} \in \mathbb{R}$ to $\psi_\alpha(p, \mathbf{v})$ is then an isomorphism of vector spaces for all $\alpha \in A$ and $p \in U_\alpha$. The conditions of Proposition 6.5 are then satisfied by the smooth manifold M, the set E, the surjective function $\pi_E: E \to M$, the open cover $(U_\alpha : \alpha \in A)$ and the functions ψ_α and $g_{\alpha,\beta}$. The result therefore follows immediately from that proposition. ∎

6.3 The Tangent Bundle of a Smooth Manifold

Proposition 6.7 *Let M be a smooth manifold of dimension n, let TM be the set whose elements are the tangent vectors to M, and let $\pi_{TM}: TM \to M$ be the function that satisfies $\pi_{TM}(X_p) = p$ for all points p of M and for all tangent vectors X_p belonging to the tangent space T_pM to M at the point p. Then the set TM can be given a topology and smooth structure so that it becomes a smooth manifold. The surjective function $\pi_{TM}: TM \to M$ is then a smooth map, and TM and $\pi_{TM}: TM \to M$ are the total space and projection function of a smooth vector bundle over M. Moreover, given any smooth coordinate system $(x^1, x^2, \ldots, x^n)$ for M, defined over some open subset U of M, there is a smooth map $\psi: U \times \mathbb{R}^n \to TM$ which maps $U \times \mathbb{R}^n$ diffeomorphically onto $\pi_{TM}^{-1}(U)$ and sends $(p, (v^1, v^2, \ldots, v^n)) \in U \times \mathbb{R}^n$ to the tangent vector $\psi(p, (v^1, v^2, \ldots, v^n))$ determined by the following equation:*

$$\psi(p, (v^1, v^2, \ldots, v^n)) = \sum_{j=1}^n v^j \left. \frac{\partial}{\partial x^j} \right|_p .$$

The inverse of the diffeomorphism from $U \times \mathbb{R}^n$ to $\pi_{TM}^{-1}(U)$ determined by ψ is thus a smooth chart for TM which sends each tangent vector $\displaystyle\sum_{j=1}^n v^i \left. \frac{\partial}{\partial x^j} \right|_p$ in $\pi_{TM}^{-1}(U)$ to the element

$$(x^1(p), x^2(p), \ldots, x^n(p), v^1, v^2, \ldots, v^n)$$

of $\mathbb{R}^{2n}$. These requirements uniquely determine the topology and smooth structure on the smooth manifold TM.

Proof Let $(x^1, x^2, \ldots, x^n)$ and $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^n)$ be a smooth coordinate systems for M, defined over open subsets U and $\hat{U}$ respectively of M, where $U \cap \hat{U}$ is non-empty. Let $\psi: U \times \mathbb{R}^n \to TM$ and $\hat{\psi}: \hat{U} \times \mathbb{R}^n \to TM$ be defined such that

$$\psi(p, (v^1, v^2, \ldots, v^n)) = \sum_{j=1}^{n} v^i \left. \frac{\partial}{\partial x^j} \right|_p .$$

and

$$\hat{\psi}(p, (w^1, w^2, \ldots, w^n)) = \sum_{j=1}^{n} w^i \left. \frac{\partial}{\partial \hat{x}^j} \right|_p .$$

Then

$$\begin{aligned} \psi(p, (v^1, v^2, \ldots, v^n)) &= \sum_{j=1}^{n} v^i \left. \frac{\partial}{\partial x^j} \right|_p \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} v^i \left. \frac{\partial \hat{x}^k}{\partial x^j} \right|_p \left. \frac{\partial}{\partial \hat{x}^k} \right|_p \\ &= \hat{\psi}(p, (w^1, w^2, \ldots, w^n)), \end{aligned}$$

where

$$w^k = \sum_{j=1}^{n} \left. \frac{\partial \hat{x}^k}{\partial x^j} \right|_p v^j = \sum_{j=1}^{n} (J(p))^k_j v^j,$$

where $(J(p))^k_j = \left. \dfrac{\partial \hat{x}^k}{\partial x^j} \right|_p$ for all $p \in U \cap \hat{U}$. Now the entries of the Jacobian matrix $J(p)$ depend smoothly on P. The result therefore follows on applying Proposition 6.5. ∎

Definition Let M be a smooth manifold. A smooth *vector field* defined over an open subset V of M is a smooth section $X: V \to TM$ of the tangent bundle $\pi_{TM}: TM \to M$ of M defined over the open set V.

6.4 Examples of Vector Bundles

Example Let S^n be the unit sphere centred on the origin in $\mathbb{R}^{n+1}$, so that

$$S^n = \{\mathbf{p} \in \mathbb{R}^{n+1} : |\mathbf{p}| = 1\},$$

where $|\mathbf{p}|^2 = \mathbf{p}.\mathbf{p}$, and let

$$E = \{(\mathbf{p}, \mathbf{v}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |\mathbf{p}| = 1 \text{ and } \mathbf{p}.\mathbf{v} = 0\}.$$

Now $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a $(2n+2)$-dimensional Euclidean space, and the subset E of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a $2n$-dimensional submanifold of this Euclidean space. Indeed let $(\mathbf{p}, \mathbf{v})$ be an element of E. Then at last one of the components $p_1, p_2, \ldots, p_{n+1}$ is non-zero. We may suppose, without loss of generality, that $p_{n+1} \neq 0$. Then $\mathbf{p} \in E \setminus (H_{n+1} \times \mathbb{R}^{n+1})$, where

$$H_{n+1} = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0 \right\},$$

and

$$\begin{aligned} E \setminus (H_{n+1} \times \mathbb{R}^{n+1}) \\ = \Bigl\{ & (x_1, x_2, \ldots, x_{n+1}, z_1, z_2, \ldots, z_{n+1}) \in \mathbb{R}^{2n+2} : \\ & x_{n+1} = \sqrt{1 - x_1^2 - \cdots - x_n^2} \text{ and} \\ & z_{n+1} = -\frac{x_1 z_1 + x_2 z_2 + \cdots + x_n z_n}{\sqrt{1 - x_1^2 - \cdots - x_n^2}} \Bigr\}. \end{aligned}$$

There is a smooth surjective map $\pi_E \colon E \to S^n$, where $\pi_E(\mathbf{p}, \mathbf{v}) = \mathbf{p}$ for all $(\mathbf{p}, \mathbf{v}) \in E$. Then $\pi_E \colon E \to S^n$ is the projection map of a fibre bundle over S^n with total space S^n.

Let $E_{\mathbf{p}} = \pi_E^{-1}(\{\mathbf{p}\})$ for all $\mathbf{p} \in S^n$, and, for each element $\mathbf{v}$ of $\mathbb{R}^{n+1}$ for which $(\mathbf{p}, \mathbf{v}) \in E$, let us denote by $\mathbf{v}_{\mathbf{p}}$ the element of $E_{\mathbf{p}}$ represented by the ordered pair $(\mathbf{p}, \mathbf{v})$. We give the fibre $E_{\mathbf{p}}$ of $\pi_E \colon E \to S^n$ over the point $\mathbf{p}$ of S^n the structure of a vector space, where

$$\lambda \mathbf{v}_{\mathbf{p}} + \mu \mathbf{w}_{\mathbf{p}} = (\lambda \mathbf{v} + \mu \mathbf{w})_{\mathbf{p}}$$

for all $\mathbf{v}_{\mathbf{p}}, \mathbf{w}_{\mathbf{p}} \in E_{\mathbf{p}}$ and for all real numbers λ and μ. Then the fibre bundle $\pi_E \colon E \to S^n$ acquires thereby the structure of a smooth vector bundle over S^n.

Now an element $\mathbf{v}$ of $\mathbb{R}^{n+1}$ represents the Cartesian components of a vector in $\mathbb{R}^{n+1}$. This vector is tangent to the submanifold S^n of $\mathbb{R}^{n+1}$ at some point $\mathbf{p}$ of S^n if and only if $\mathbf{p} . \mathbf{v} = 0$, in which case $(\mathbf{p}, \mathbf{v}) \in E$, and thus $\mathbf{v}_{\mathbf{p}} \in \mathbf{E}_{\mathbf{p}}$. It follows that the tangent space $T_{\mathbf{p}} S^n$ to S^n at the point $\mathbf{p}$ is naturally isomorphic to the fibre $E_{\mathbf{p}}$ of the smooth vector bundle $\pi_E \colon E \to S^n$ over the point $\mathbf{p}$. These natural isomorphisms between the fibres of the respective bundles over S^n give rise to a smooth map $\varphi \colon TS^n \to E$ that is an isomorphism of vector bundles over S^n. Thus the tangent bundle $\pi_{TS^n} \colon TS^n \to S^n$ of the n-dimensional sphere S^n is naturally isomorphic to the smooth vector bundle $\pi_E \colon E \to S^n$ constructed above.

Example Consider the tangent bundle $\pi_{TS^1}: TS^1 \to S^1$ of the unit circle S^1 in $\mathbb{R}^2$, where

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Let $\pi_0: S^1 \times \mathbb{R} \to S^1$ be the projection map of the product bundle with total space $S^1 \times \mathbb{R}$, fibre $\mathbb{R}$ and base space S^1, where $\pi_0((x, y), t) = (x, y)$ for all $(x, y) \in S^1$ and $\mathbb{R}$, and where the vector space structure on each fibre of π_0 is defined such that, for all $(x, y) \in S^1$, the function from $\mathbb{R}$ to $S^1 \times \mathbb{R}$ that sends $t \in \mathbb{R}$ to $(x, y), t)$ is an isomorphism of one-dimensional real vector spaces. Then there is an isomorphism $\psi: S^1 \times \mathbb{R} \to TS^1$ of smooth vector bundles over S^1 that sends each element $((x, y), t)$ of $S^1 \times \mathbb{R}$ to the tangent vector to S^1 at the point (x, y) whose Cartesian components are $(-yt, xt)$. (Thus, for each point $\mathbf{p}$ of S^1, and for each real number t, the smooth map ψ sends $(\mathbf{p}, t) \in S^1 \times \mathbb{R}$ to the tangent vector at $\mathbf{p}$ obtained on rotating the displacement vector $\mathbf{p}$ of the point anticlockwise through a right angle, and then multiplying the resulting vector by the real number t so as to obtain a tangent vector to S^1 at the point $\mathbf{p}$.) We have thus shown that the tangent bundle $\pi_{TS^1}: TS^1 \to S^1$ of the circle S^1 is isomorphic to a product bundle over S^1, and is therefore (topologically) trivial.

Example We now construct a non-trivial smooth vector bundle of rank 1 over the circle S^1. Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

let

$$M = \{(x, y, u, v) \in \mathbb{R}^4 : x^2 + y^2 = 1 \text{ and } y(u^2 - v^2) = 2xuv\},$$

and let $\pi_M: M \to S^1$ be defined so that

$$\pi_M(x, y, u, v) = (x, y)$$

for all $(x, y, u, v) \in M$.

Let $(x, y, u, v) \in M$. Then there exist real numbers θ, φ and z such that $x = \cos\theta$, $y = \sin\theta$, $u = z\cos\varphi$ and $v = z\sin\varphi$. Then

$$y(u^2 - v^2) = z^2 \sin\theta(\cos^2\varphi - \sin^2\varphi) = z^2 \sin\theta \cos 2\varphi$$

and

$$2xuv = 2z^2 \cos\theta \cos\varphi \sin\varphi = z^2 \cos\theta \sin 2\varphi,$$

and therefore

$$\sin\theta \cos 2\varphi = \cos\theta \sin 2\varphi.$$

It follows that $2\varphi - \theta$ is an integer multiple of 2π. Thus, given any point $\mathbf{q}$ of M, there exist real numbers θ and z such that

$$\mathbf{q} = \left(\cos\theta, \sin\theta, z\cos\frac{\theta}{2}, z\sin\frac{\theta}{2}\right).$$

It follows easily from this that $\pi_M\colon M \to S^1$ is a smooth fibre bundle over the circle. We can give each fibre of this map the structure of a real vector space of dimension 1. This vector space structure is determined by the requirement that, for all $(x, y, u, v) \in M$, the function from $\mathbb{R}$ to M that sends $z \in \mathbb{R}$ to (x, y, zu, zv) is a linear transformation from $\mathbb{R}$ to the fibre of $\pi_M\colon M \to S^1$ over the point (x, y) of S^1. Then $\pi_M\colon M \to S^1$ carries the structure of a smooth vector bundle of rank 1 over S^1.

Let $s\colon S^1 \to M$ be a continuous section of $\pi_M\colon M \to S^1$. Then s determines a continuous real-valued function $f\colon \mathbb{R} \to \mathbb{R}$ characterized by the property that

$$s(\cos\theta, \sin\theta) = \left(\cos\theta,\ \sin\theta,\ f(\theta)\cos\frac{\theta}{2},\ f(\theta)\sin\frac{\theta}{2}\right)$$

for all $\theta \in \mathbb{R}$. But then

$$\begin{aligned} & s(\cos\theta, \sin\theta) \\ &\quad = s(\cos(\theta+2\pi), \sin(\theta+2\pi)) \\ &\quad = \left(\cos\theta,\ \sin\theta,\ f(\theta+2\pi)\cos\frac{\theta+2\pi}{2},\ f(\theta+2\pi)\sin\frac{\theta+2\pi}{2}\right) \\ &\quad = \left(\cos\theta,\ \sin\theta,\ -f(\theta+2\pi)\cos\frac{\theta}{2},\ -f(\theta+2\pi)\sin\frac{\theta}{2}\right) \end{aligned}$$

and therefore $f(\theta + 2\pi) = -f(\theta)$ for all $\theta \in \mathbb{R}$. Thus if the function f on $\mathbb{R}$ is not identically equal to zero then it assumes both positive and negative values. It follows from the Intermediate Value Theorem that there exists $\theta_0 \in \mathbb{R}$ for which $f(\theta_0) = 0$. But then $s(\mathbf{p}_0)$ is the zero element of the fibre $M_{\mathbf{p}_0}$ of $\pi_M\colon M \to S^1$ over the point $\mathbf{p}_0$ of S^1, where $\mathbf{p}_0 = (\cos\theta_0, \sin\theta_0)$. We have thus shown that the smooth vector bundle $\pi_M\colon M \to S^1$ has no continuous sections that are non-zero throughout the circle S^1. It follows from this that the vector bundle $\pi_M\colon M \to S^1$ is not isomorphic to a product bundle, and is therefore (topologically) non-trivial.

Example The tangent bundle of a two-dimensional sphere is not isomorphic to a product bundle.

Let S^2 be the unit sphere in $\mathbb{R}^3$, where

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

The *Hairy Ball Theorem* (or *Hairy Dog Theorem*) in two dimensions states that there is no continuous vector field on S^2 that is everywhere tangential to S^2 and that is non-zero everywhere on S^2. We now give a somewhat informal proof of this theorem.

Given any point $\mathbf{p}$ of the unit sphere S^2, let $\mathbf{n_p}$ denote the outward normal at the point $\mathbf{p}$. If the point $\mathbf{p}$ has cartesian components (x, y, z) then the vector $\mathbf{n_p}$ also has components (x, y, z).

Also, given any point $\mathbf{p}$ of the unit sphere S^2, there exist angles $\theta \in [0, \pi]$ and $\varphi \in (-\pi, \pi]$ such that

$$\mathbf{p} = (\sin\theta\, \cos\varphi,\ \sin\theta\, \sin\varphi,\ \cos\theta).$$

Let $\mathbf{Q_p}$ denote the tangent vector at $\mathbf{p}$ that is the velocity vector $\gamma_\theta'(\varphi)$ of the smooth curve $\gamma_\theta\colon \mathbb{R} \to S^2$ at time φ, where

$$\gamma_\theta(t) = (\sin\theta\, \cos t,\ \sin\theta\, \sin t,\ \cos\theta)$$

for all $t \in \mathbb{R}$. Then $\mathbf{Q_p}$ is a tangent vector to the sphere at the point $\mathbf{p}$, and

$$\mathbf{Q_p} = (-\sin\theta\, \sin\varphi,\ \sin\theta\, \cos\varphi,\ 0).$$

It follows that the map $\mathbf{p} \mapsto \mathbf{Q_p}$ is a smooth vector field, defined over S^2, which is everywhere tangential to S^2. It is zero at the points $(0, 0, 1)$ and $(0, 0, -1)$ and is non-zero everywhere else. If one imagines the unit sphere being rotated at constant speed about the z-axis, where the angle of rotation (measured in radians) increases at unit speed, then $\mathbf{Q_p}$ will represent the velocity vector of a particle currently at the point $\mathbf{p}$ of the sphere.

Let $\mathbf{p} \mapsto \mathbf{V_p}$ be a continuous vector field on the sphere S^2 which is everywhere tangential to the sphere. Let θ be an angle satisfying $0 < \theta < \pi$, and let

$$C_\theta = (x, y, z) \in S^2 : z = \cos\theta\}.$$

Then $\gamma_\theta(t) \in C_\theta$ for all $t \in \mathbb{R}$. Suppose that $\mathbf{V_p} \neq \mathbf{0}$ for all $\mathbf{p} \in C_\theta$. Then there exists a continous strictly positive function $f_\theta\colon \mathbb{R} \to (0, +\infty)$ and a smooth function $\psi_\theta\colon \mathbb{R} \to \mathbb{R}$ with the property that

$$\mathbf{V}_{\gamma_\theta(t)} = f_\theta(t) \left(\cos\psi_\theta(t)\, \mathbf{Q}_{\gamma_\theta(t)} + \sin\psi_\theta(t)\, \mathbf{n}_{\gamma_\theta(t)} \times \mathbf{Q}_{\gamma_\theta(t)}\right)$$

for all $t \in \mathbb{R}$. Then, for all $t \in \mathbb{R}$, the quantity $\psi(t)$ represents the angle between the tangent vectors $\mathbf{Q}_{\gamma_\theta(t)}$ and $\mathbf{V}_{\gamma_\theta(t)}$ at the point $\gamma(t)$. Now the function ψ_θ is not necessarily periodic. But

$$\frac{\psi_\theta(t + 2\pi) - \psi_\theta(t)}{2\pi}$$

is an integer for all real numbers t (because $\gamma_\theta(t+2\pi) = \gamma_\theta(t)$ and therefore $\psi_\theta(t+2\pi)$ and $\psi_\theta(t)$ both represent the angle between the vectors $\mathbf{Q}_{\gamma_\theta(t)}$ and $\mathbf{V}_{\gamma_\theta(t)}$ at the point $\gamma_\theta(t)$). Moreover the function mapping real number t to the integer $(2\pi)^{-1}(\psi_\theta(t+2\pi) - \psi_\theta(t))$ is a continuous function of t. It is therefore a constant function of t. We conclude therefore that there is an integer n_θ with the property that

$$\psi_\theta(t+2\pi) = \psi_\theta(t) + 2\pi n_\theta.$$

It is not difficult to see that if $\mathbf{V}_{(0,0,1)} \neq \mathbf{0}$ then $n_\theta = -1$ for values of the angle θ that are sufficiently close to 0. (Think of the unit sphere as representing the surface of the earth, where the point $(0,0,1)$ represents the north pole. If we have a continuous tangential vector field $\mathbf{V}$ which is non-zero at the north pole then the angle between this vector field $\mathbf{V}$ and the velocity vector in the direction of motion would increase through an angle of 2π in the clockwise direction as one traverses a sufficiently small circle of latitude in the anticlockwise direction around the north pole.) Similarly if $\mathbf{V}_{(0,0,-1)} \neq \mathbf{0}$ then $n_\theta = 1$ for values of the angle θ that are sufficiently close to π.

Now if the tangential vector field $\mathbf{V}$ were non-zero over the entire sphere then the function sending θ to n_θ for all $\theta \in (0,\pi)$ would be a continuous integer-valued function of θ on the open interval $(0,\pi)$. It would therefore be a constant function of θ on this open interval. But this constant function would have the value -1 when θ was sufficiently close to 0, and it would have the value 1 when θ was sufficiently close to π. This however is clearly impossible. We conclude therefore that there cannot exist any continuous vector field on the two-dimensional sphere S^2 that is everywhere tangential to the sphere, and that is non-zero at every point of the sphere. This proves the *Hairy Ball Theorem* for vector fields on a two-dimensional sphere.

It follows immediately from the Hairy Ball Theorem that the tangent bundle of the two-dimensional sphere S^2 is not isomorphic to a product bundle over the sphere, and is therefore a non-trivial vector bundle.

Example The tangent bundle of a three-dimensional sphere is isomorphic to a product bundle.

Let S^3 be the unit sphere in $\mathbb{R}^4$, defined such that

$$S^3 = \{(w,x,y,z) \in \mathbb{R}^4 : w^2 + x^2 + y^2 + z^2 = 1\}.$$

We note that S^3 is diffeomorphic to the group $\mathrm{SU}(2)$ of 2×2 unitary matrices A satisfying $\det A = 1$. A 2×2 matrix A with complex coefficients belongs to the group SU(2) if and only if $A^{-1} = A^\dagger$ and $\det A = 1$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{C}$. Then $A \in \mathrm{SU}(2)$ if and only if $ad - bc = 1$ and

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}.$$

It follows that $A \in \mathrm{SU}(2)$ if and only if $ad - bc = 1$, $d = \overline{a}$ and $c = -\overline{b}$. Moreover if $d = \overline{a}$ and $c = -\overline{b}$ then $ad - bc = |a|^2 + |b|^2$. We conclude therefore that

$$\begin{aligned} \mathrm{SU}(2) &= \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\} \\ &= \left\{ \begin{pmatrix} w - iz & -ix - y \\ -ix + y & w + iz \end{pmatrix} : (w, x, y, z) \in S^3 \right\} \\ &= \left\{ wI - ix\sigma_x - iy\sigma_y - iz\sigma_z : (w, x, y, z) \in S^3 \right\}, \end{aligned}$$

where the identity matrix I and the *Pauli matrices* σ_x, σ_y and σ_z are defined as follows:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The group SU(2) is a smooth submanifold of the algebra $M_2(\mathbb{C})$ of 2×2 matrices with complex coefficients, and the tangent space to SU(2) at the identity matrix I is the 3-dimensional real subspace of $M_2(\mathbb{C})$ spanned by the matrices $-i\sigma_x$, $-i\sigma_y$ and $-i\sigma_z$. The elements of this subspace are the 2×2 skew-Hermitian matrices whose trace is zero. Let

$$X_A = -iA\sigma_x, \quad Y_A = -iA\sigma_y, \quad Z_A = -iA\sigma_z$$

for all $A \in \mathrm{SU}(2)$. Then the matrices X_A, Y_A, Z_A consitute a basis of the tangent space $T_A\mathrm{SU}(2)$ to SU(2) at each $A \in \mathrm{SU}(2)$. Indeed let $\gamma\colon (-\varepsilon, \varepsilon) \to \mathrm{SU}(2)$ be a smooth curve in SU(2) satisfying $\gamma(0) = I$, and let

$$\gamma'(0) = \left.\frac{d(\gamma(t))}{dt}\right|_{t=0} = -iu\sigma_x - iv\sigma_y - iw\sigma_z = uX_I + vY_I + wZ_I,$$

where $u, v, w \in \mathbb{R}$. Then the map sending $t \in (-\varepsilon, \varepsilon)$ to $A\gamma(t)$ is a smooth curve in SU(2), and the velocity vector to this smooth curve at time $t = 0$ is

$uX_A+vY_A+wZ_A$. Thus the function that sends $(A,(u,v,w)) \in \mathrm{SU}(2)\times\mathbb{R}^3$ to the tangent vector $uX_A+vY_A+wZ_A$ at A is an isomorphism of smooth vector bundles over SU(2), and thus the tangent bundle of SU(2) is isomorphic to a product bundle.

We have thus shown that the tangent bundle of the three-dimensional sphere S^3 is isomorphic to a product bundle.

6.5 Dual Bundles

Proposition 6.8 *Let $\pi_E\colon E \to M$ be a smooth vector bundle over a smooth manifold M. For each point p of M let E_p^* be the real vector space that is the dual space of the fibre E_p of $\pi_E\colon E \to M$ over the point p. Let E^* be the disjoint union of the vector spaces E_p^*, and let $\pi_{E_p^*}\colon E^* \to M$ be the surjective function on E^* that maps elements of E_p^* to p for all points p of M. Then the set E^* can be given the structure of a smooth manifold so as to ensure that $\pi_{E^*}\colon E^* \to M$ is a smooth vector bundle over M satisfying the following condition:*

> *if $s\colon V \to E$ and $\tau\colon V \to E^*$ are smooth sections of the vector bundles $\pi_E\colon E \to M$ and $\pi_{E^*}\colon E^* \to M$ defined over some open subset V of M, then the function on V that sends $p \in V$ to $\langle \tau(p), s(p)\rangle$ is a smooth real-valued function on M.*

Proof It follows from Proposition 6.4 that there exists an open cover $(U_\alpha : \alpha \in A)$ of M, indexed by some indexing set A, and smooth maps

$$\psi_\alpha\colon U_\alpha \times \mathbb{R}^k \to E,$$

and

$$g_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to GL(k,\mathbb{R})$$

for all $\alpha, \beta \in A$, where these smooth maps satisfy the properties listed in the statement of Proposition 6.4. In particular the maps ψ_α satisfy the following properties:

(i) $\pi_E(\psi_\alpha(p,\mathbf{v})) = p$ for all $p \in U_\alpha$ and $\mathbf{v} \in \mathbb{R}^k$;

(ii) ψ_α maps $U_\alpha \times \mathbb{R}^k$ diffeomorphically onto $\pi_E^{-1}(U_\alpha)$

(iii) for each $p \in U_\alpha$, the map $(\psi_\alpha)_p\colon \mathbb{R}^k \to E_p$ is an isomorphism of real vector spaces, where

$$(\psi_\alpha)_p(\mathbf{v}) = \psi_\alpha(p,\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R}^k$.

The smooth maps $g_{\alpha,\beta}: U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ are then defined so that $g_{\alpha,\beta}(p) = (\psi_\alpha)_p^{-1}(\psi_\beta)_p$, and therefore satisfy the identity

$$\psi_\beta(p, \mathbf{v}) = \psi_\alpha(p, g_{\alpha,\beta}(p)\mathbf{v})$$

for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$ and $\mathbf{v} \in \mathbb{R}^k$.

Let p be a point of the open set U_α for some $\alpha \in A$. The isomorphism $(\psi_\alpha)_p: \mathbb{R}^k \to E_p$ determines an isomorphism $(\chi_\alpha)_p: \mathbb{R}^{k*} \to E_p^*$ of the corresponding dual spaces, where $(\chi_\alpha)_p(\lambda) = \lambda \circ (\psi_\alpha)_p^{-1}$ for all linear functionals $\lambda: \mathbb{R}^k \to \mathbb{R}$ on $\mathbb{R}^k$. These isomorphisms of dual spaces then determine functions $\chi_\alpha: U_\alpha \times \mathbb{R}^{k*} \to E^*$, where $\chi_\alpha(p, \lambda) = (\chi_\alpha)_p(\lambda) = \lambda \circ (\psi_\alpha)_p^{-1}$ for all $p \in U_\alpha$ and $\lambda \in \mathbb{R}^{k*}$. Clearly $\pi_{E^*}(\chi_\alpha(p, \lambda)) = p$ for all $p \in U_\alpha$. Moreover the function $\chi_\alpha: U_\alpha \times \mathbb{R}^{k*} \to E^*$ maps $U \times \mathbb{R}^{k*}$ bijectively onto $\pi_{E^*}^{-1}(U_\alpha)$ for all $\alpha \in A$. Then

$$\begin{aligned} \chi_\beta(p, \lambda) &= (\chi_\beta)_p(\lambda) = \lambda \circ (\psi_\beta)_p^{-1} \\ &= \lambda \circ (\psi_\beta)_p^{-1} \circ (\psi_\alpha)_p \circ (\psi_\alpha)_p^{-1} = \lambda \circ g_{\alpha,\beta}(p)^{-1} \circ (\psi_\alpha)_p^{-1} \\ &= \chi_\alpha(p, \lambda \circ g_{\alpha,\beta}(p)^{-1}). \end{aligned}$$

for all $p \in U_\alpha \cap U_\beta$ and $\lambda \in \mathbb{R}^{k*}$.

Let $GL(\mathbb{R}^{k*})$ be the group of invertible linear operators on the dual space $\mathbb{R}^{k*}$ of $\mathbb{R}^k$. We define the standard basis on $\mathbb{R}^{k*}$ to be the dual basis determined by the standard basis on $\mathbb{R}^k$. The jth element of this standard basis on $\mathbb{R}^{k*}$ is then the linear functional $(x_1, x_2, \ldots, x_k) \mapsto x_j$ on $\mathbb{R}^k$. Let $h_{\alpha,\beta}: U_\alpha \cap U_\beta \to GL(\mathbb{R}^{k*})$ be defined so that $h_{\alpha,\beta}(p)\lambda = \lambda \circ g_{\alpha,\beta}(p)^{-1}$ for all $\lambda \in \mathbb{R}^{k*}$ and $p \in U_\alpha \cap U_\beta$. Then $h_{\alpha,\beta}$ is a smooth function on $U_\alpha: U_\beta$. Indeed the matrix that represents $h_{\alpha,\beta}(p)$ with respect to the standard basis on $\mathbb{R}^{k*}$ is the inverse of the transpose of the matrix $g_{\alpha,\beta}(p)$. The open cover $(U_\alpha : \alpha \in A)$ and the smooth maps χ_α and $h_{\alpha,\beta}$ then satisfy conditions (i)–(v) of Proposition 6.5, (with ψ_α and $g_{\alpha,\beta}$ replaced by χ_α and $h_{\alpha,\beta}$ in the statements of those conditions), and therefore there exists a topology and smooth structure on $\pi_{E^*}: E^* \to M$. Each function $\chi_\alpha: U_\alpha \times \mathbb{R}^{k*} \to E^*$ is then a smooth map which maps is domain diffeomorphically onto $\pi_{E^*}^{-1}(U_\alpha)$.

Now

$$\langle (\chi_\alpha)_p\lambda, (\psi_\alpha)_p\mathbf{v}\rangle = (\lambda \circ (\psi_\alpha)_p^{-1})((\psi_\alpha)_p\mathbf{v}) = \lambda(\mathbf{v}) = \langle \lambda, \mathbf{v}\rangle$$

for all $p \in U_\alpha$, $\lambda \in \mathbb{R}^{k*}$ and $\mathbf{v} \in \mathbb{R}^k$. If V is an open set in M and if $s: V \to E$ and $\tau: V \to E^*$ are smooth sections of the vector bundles $\pi_E: E \to M$ and $\pi_{E^*}: E^* \to M$ defined over V then, for each $\alpha \in A$ there exist smooth functions $\mathbf{u}: V \cap U_\alpha \to \mathbb{R}^k$ and $\omega: V \cap U_\alpha \to \mathbb{R}^{k*}$ such that $s(p) = \psi_p(\mathbf{u}(p))$ and $\tau(p) = \chi_p(\omega(p))$ for all $p \in V \cap U_\alpha$. Then

$$\langle \tau(p), s(p)\rangle = \langle \chi_p(\omega(p)), \psi_p(\mathbf{u}(p))\rangle = \langle \omega(p), \mathbf{u}(p)\rangle$$

It follows that the real-valued function on V which sends $p \in V$ to $\langle \tau(p), s(p) \rangle$ restricts to a smooth function on $V \cap U_\alpha$ for all $\alpha \in A$, and is thus itself smooth, as required. ∎

6.6 Some Results concerning Local Trivializations

Lemma 6.9 *Let M be a smooth manifold, and, for each integer q between 1 and k, let $\pi_{E_q}: E_q \to M$ be a smooth vector bundle over M. Then, given any point p_0 of M, there exists an open set V such that $p_0 \in V$ and the vector bundle $\pi_{E_q}: E_q \to M$ is trivial over V for $q = 1, 2, \ldots, k$. There then exists smooth functions $\psi_q: V \times \mathbb{R}^{r_q} \to E_q$, where r_q is the rank of the vector bundle $\pi_{E_q}: E_q \to M$, which satisfy the following properties:—*

(i) $\pi_{E_q}(\psi_q(p, \mathbf{v}_q)) = p$ *for all* $p \in V$ *and* $\mathbf{v}_q \in \mathbb{R}^{r_q}$;

(ii) ψ_q *maps* $V \times \mathbb{R}^{r_q}$ *diffeomorphically onto* $\pi_{E_q}^{-1}(U_\alpha)$

(iii) *for each* $p \in V$ *the map* $(\psi_q)_p: \mathbb{R}^{r_q} \to (E_q)_p$ *is an isomorphism of real vector spaces, where*

$$(\psi_q)_p(\mathbf{v}_q) = \psi_q(p, \mathbf{v}_q)$$

for all $\mathbf{v}_q \in \mathbb{R}^{r_q}$.

Proof Let $(E_q)_p = \pi_{E_q}^{-1}(\{p\})$ for $q = 1, 2, \ldots, k$ and for all $p \in M$. Then each fibre $(E_q)_p$ is a finite dimensional real vector space of dimension r_q, where r_q denotes the rank of the corresponding vector bundle E_q. Now given any point p of M, there exist open sets V_q in M for $q = 1, 2, \ldots, k$, where $p_0 \in V_q$ for all q, such that the smooth vector bundle $\pi_{E_q}: E_q \to M$ is trivial over V_q. Let $V = V_1 \cap V_2 \cap \cdots \cap V_k$. Then V is an open set in M, $p_0 \in V$, and each vector bundle $\pi_{E_q}: E_q \to M$ is trivial over V. The restriction of each vector bundle $\pi_{E_q}: E_q \to M$ to this open set V is then isomorphic to a product bundle, and therefore there exist smooth maps $\psi_q: V \times \mathbb{R}^{r_q} \to E_q$ satisfying the required properties. ∎

Lemma 6.10 *Let M be a smooth manifold, and, for each integer q between 1 and k, let $\pi_{E_q}: E_q \to M$ be a smooth vector bundle over M. Then there exists an open cover $(U_\alpha : \alpha \in A)$ of M, indexed by some indexing set A, and smooth maps*

$$\psi_{q,\alpha}: U_\alpha \times \mathbb{R}^{r_q} \to E_q,$$

and

$$g_{q,\alpha,\beta}: U_\alpha \cap U_\beta \to GL(r_q, \mathbb{R})$$

for $q = 1, 2, \ldots, k$ and for all $\alpha, \beta \in A$, where these smooth maps satisfy the following properties:—

(i) $\pi_{E_q}(\psi_{q,\alpha}(p, \mathbf{v}_q)) = p$ *for all* $p \in U_\alpha$ *and* $\mathbf{v}_q \in \mathbb{R}^{r_q}$*;*

(ii) $\psi_{q,\alpha}$ *maps* $U_\alpha \times \mathbb{R}^{r_q}$ *diffeomorphically onto* $\pi_{E_q}^{-1}(U_\alpha)$

(iii) *for each* $p \in U_\alpha$*, the map* $(\psi_{q,\alpha})_p : \mathbb{R}^{r_q} \to (E_q)_p$ *is an isomorphism of real vector spaces, where*

$$(\psi_{q,\alpha})_p(\mathbf{v}_q) = \psi_{q,\alpha}(p, \mathbf{v}_q)$$

for all $\mathbf{v}_q \in \mathbb{R}^{r_q}$*;*

(iv) $\psi_{q,\beta}(p, \mathbf{v}_q) = \psi_{q,\alpha}(p, g_{q,\alpha,\beta}(p)\mathbf{v}_q)$ *for* $q = 1, 2, \ldots, k$ *and for all* $\alpha, \beta \in A$*,* $p \in U_\alpha \cap U_\beta$ *and* $\mathbf{v}_q \in \mathbb{R}^{r_q}$*;*

(v) $g_{q,\alpha,\beta}(p) = (\psi_{q,\alpha})_p^{-1}(\psi_{q,\beta})_p$ *for* $q = 1, 2, \ldots, k$ *and for all* $\alpha, \beta \in A$ *and* $p \in U_\alpha \cap U_\beta$*.*

Proof It follows from Lemma 6.9 that there exists an open cover $(U_\alpha : \alpha \in A)$ of M, indexed by some indexing set A, such that the smooth vector bundle $\pi_{E_q} : E_q \to M$ is trivial over U_α for $q = 1, 2, \ldots, k$ and for all $\alpha \in A$. There then exist smooth maps

$$\psi_{q,\alpha} : U_\alpha \times \mathbb{R}^{r_q} \to E_q,$$

where r_q is the rank of the vector bundle $\pi_{E_q} : E_q \to M$, which satisfy properties (i), (ii), (iii). These functions $\psi_{q,\alpha}$ then determine smooth maps $g_{q,\alpha,\beta} : U_\alpha \cap U_\beta \to GL(r_q, \mathbb{R})$ that satisfy property (iv). Property (v) then follows directly from properties (iii) and (iv). ∎

6.7 Direct Sums of Vector Bundles

Let $\bigoplus_{q=1}^{k} V_q$ denote the direct sum

$$V_1 \oplus V_2 \oplus \cdots V_k$$

of real vector spaces $V_1, V_2, \ldots, V_k$. The elements of $\bigoplus_{q=1}^{k} V_q$ may be represented as ordered k-tuples $(\xi_1, \xi_2, \ldots, \xi_k)$, where $\xi_q \in V_q$ for $q = 1, 2, \ldots, k$. Given elements ξ_q of V_q for $q = 1, 2, \ldots, k$, we shall denote by $\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_k$ the element of $\bigoplus_{q=1}^{k} V_q$ that is also represented by the ordered k-tuple $(\xi_1, \xi_2, \ldots, \xi_k)$. Then

$$(\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_k) + (\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k) = (\xi_1 + \eta_1) \oplus (\xi_2 + \eta_2) \oplus \cdots \oplus (\xi_k + \eta_k)$$

and

$$\lambda(\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_k) = (\lambda\xi_1) \oplus (\lambda\xi_2) \oplus \cdots \oplus (\lambda\xi_k)$$

for all elements $\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_k$ and $\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k$ of $\bigoplus_{q=1}^{k} V_q$ and for all $\lambda \in \mathbb{R}$.

Proposition 6.11 *Let M be a smooth manifold, and, for each integer q between 1 and k, let $\pi_{E_q}: E_q \to M$ be a smooth vector bundle over M. For each point p of M let $E_p^\oplus$ be the real vector space that is the direct sum of the fibres of the given vector bundles over the point p, so that*

$$E_p^\oplus = \bigoplus_{q=1}^{k} (E_q)_p = (E_1)_p \oplus (E_2)_p \oplus \cdots (E_k)_p,$$

where $(E_q)_p = \pi_{E_q}^{-1}(\{p\})$ for $q = 1, 2, \ldots, k$. Also let $E^\oplus$ be the disjoint union of the vector spaces $E_p^\oplus$, and let $\pi_{E^\oplus}: E^\oplus \to M$ be the surjective function, defined on the disjoint union $E^\oplus$ of all these vector spaces $E_p^\oplus$, that sends elements of $E_p^\oplus$ to p for all points p of M. Then $E^\oplus$ can be given the structure of a smooth manifold so as to ensure that $\pi_{E^\oplus}: E^\oplus \to M$ is a smooth vector bundle over M satisfying the following condition:

> *if $s: V \to E^\oplus$ is a function mapping some open subset V of M into $E^\oplus$, and if*
>
> $$s(p) = s_1(p) \oplus s_2(p) \oplus \cdots \oplus s_k(p)$$
>
> *for all $p \in V$, where $s_q: V \to E_q$ is a smooth section of $\pi_{E_q}: E_q \to M$ defined over V for $q = 1, 2, \ldots, k$, then $s: V \to E^\oplus$ is a smooth section of $\pi_{E^\oplus}: E^\oplus \to M$ defined over V.*

Proof Let $(U_\alpha: \alpha \in A)$ be an open cover of M, where the smooth vector bundles $\pi_{E_q}: E_q \to M$ are all trivial over each open set U_α, and, for $q = 1, 2, \ldots, k$ and for all $\alpha, \beta \in A$, let $\psi_{q,\alpha}: U_\alpha \times \mathbb{R}^{r_q} \to E_q$ and $g_{q,\alpha,\beta}: U_\alpha \cap U_\beta \to GL(r_q, \mathbb{R})$ be smooth maps with the properties (i)–(v) listed in the statement of Corollary 6.9.

Now the real vector space $\bigoplus_{q=1}^{k} \mathbb{R}^{r_q}$ is isomorphic to $\mathbb{R}^m$, where

$$m = r_1 + r_2 + \cdots + r_k.$$

Let $\nu: \bigoplus_{q=1}^{k} \mathbb{R}^{r_q} \to \mathbb{R}^m$ be an isomorphism between these vector spaces. Then the functions $\psi_\alpha: U_\alpha \times \mathbb{R}^{r_q} \to E^\oplus$ and $g_{q,\alpha,\beta}: U_\alpha \cap U_\beta \to GL(r_q, \mathbb{R})$ and the isomorphism ν determine functions $\psi^{\oplus}_{\alpha}: U_\alpha \times \mathbb{R}^m \to E^{\oplus}$, and $g^{\oplus}_{\alpha,\beta}: U_\alpha \cap U_\beta \to \mathrm{GL}(m, \mathbb{R})$ such that

$$\psi^{\oplus}_{\alpha}(p, \nu(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)) = \psi_{1,\alpha}(p, \mathbf{v_1}) \oplus \psi_{2,\alpha}(p, \mathbf{v_2}) \oplus \cdots \oplus \psi_{s,\alpha}(p, \mathbf{v_k})$$

and

$$g^{\oplus}_{\alpha,\beta}(p)(\nu(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)) = \nu(g_{1,\alpha,\beta}(p)(\mathbf{v}_1), g_{2,\alpha,\beta}(p)(\mathbf{v}_2), \ldots, g_{k,\alpha,\beta}(p)(\mathbf{v}_k))$$

for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$ and $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) \in \bigoplus_{q=1}^{k} \mathbb{R}^{r_q}$. Now, given any $\mathbf{v} \in \mathbb{R}^m$, the function that sends $p \in U_\alpha \times U_\beta$ to $g^{\oplus}_{\alpha,\beta}(p)(\mathbf{v})$ is a smooth function from $U_\alpha \times U_\beta$ to $\mathbb{R}^m$. It follows that $g^{\oplus}_{\alpha,\beta}: U_\alpha \cap U_\beta \to \mathrm{GL}(m, \mathbb{R})$ is a smooth map on $U_\alpha \cap U_\beta$. Moreover

$$\begin{aligned}
&\psi^{\oplus}_{\alpha}(p, g^{\oplus}_{\alpha,\beta}(p)(\nu(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k))) \\
&\quad = \psi^{\oplus}_{\alpha}(p, \nu(g_{1,\alpha,\beta}(p)(\mathbf{v}_1), g_{2,\alpha,\beta}(p)(\mathbf{v}_2), \ldots, g_{k,\alpha,\beta}(p)(\mathbf{v}_k))) \\
&\quad = \psi_{1,\alpha}(p, g_{1,\alpha,\beta}(\mathbf{v_1})) \oplus \psi_{2,\alpha}(p, g_{1,\alpha,\beta}(\mathbf{v_2})) \oplus \cdots \oplus \psi_{s,\alpha}(p, g_{1,\alpha,\beta}(\mathbf{v_k})) \\
&\quad = \psi_{1,\beta}(p, \mathbf{v_1}) \oplus \psi_{2,\beta}(p, \mathbf{v_2}) \oplus \cdots \oplus \psi_{s,\beta}(p, \mathbf{v_k}) \\
&\quad = \psi^{\oplus}_{\beta}(p, \nu(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)),
\end{aligned}$$

for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$ and $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) \in \bigoplus_{q=1}^{k} \mathbb{R}^{r_q}$, and thus

$$\psi^{\oplus}_{\beta}(p, \mathbf{v}) = \psi^{\oplus}_{\alpha}(p, g^{\oplus}_{\alpha,\beta}(p)(\mathbf{v}))$$

for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$ and $\mathbf{v} \in \mathbb{R}^m$. Thus the open cover $(U_\alpha : \alpha \in A)$ of M and the functions $\psi^{\oplus}_{\alpha}: U_\alpha \times \mathbb{R}^m \to E^{\oplus}$ and $g^{\oplus}_{\alpha,\beta}: U_\alpha \cap U_\beta \to \mathrm{GL}(m, \mathbb{R})$ satisfy the conditions (i)–(v) in the statement of Proposition 6.5, and therefore there is a topology and smooth structure on $E^{\oplus}$ with respect to which $E^{\oplus}$ is a smooth manifold, $\pi_{E^{\oplus}}: E^{\oplus} \to M$ is a smooth vector bundle, and the function $\psi^{\oplus}_{\alpha}: U_\alpha \times \bigoplus_{q=1}^{k} \mathbb{R}^{r_q} \to E^{\oplus}$ maps its domain diffeomorphically onto $\pi^{-1}_{E^{\oplus}}(U_\alpha)$ for all $\alpha \in A$.

Let V be an open set in M, let $s_q: V \to E_q$ be a smooth section of $\pi_{E_q}: E_q \to M$ for $q = 1, 2, \ldots, k$, and let

$$s(p) = s_1(p) \oplus s_2(p) \oplus \cdots \oplus s_k(p)$$

for all $p \in V$. Then there are smooth functions $f_{q,\alpha}: V \cap U_\alpha \to \mathbb{R}^{r_q}$ such that $s_q(p) = \psi_{q,\alpha}(p, f_{q,\alpha}(p))$ for all $p \in V \cap U_\alpha$. Let

$$f_\alpha(p) = \nu(f_{1,\alpha}(p), f_{2,\alpha}(p), \ldots, f_{k,\alpha}(p))$$

for all $p \in V \cap U_\alpha$. Then $f_\alpha: V \cap U_\alpha \to \mathbb{R}^m$ is a smooth function on $V \cap U_\alpha$, and

$$\begin{aligned}
\psi_\alpha^\oplus(p, f_\alpha(p)) &= \psi_\alpha^\oplus(p, \nu(f_{1,\alpha}(p), f_{2,\alpha}(p), \ldots, f_{k,\alpha}(p))) \\
&= \psi_{1,\alpha}(p, f_{1,\alpha}(p)) \oplus \psi_{2,\alpha}(p, f_{2,\alpha}(p)) \oplus \cdots \oplus \psi_{k,\alpha}(p, f_{k,\alpha}(p)) \\
&= s_1(p) \oplus s_2(p) \oplus \cdots \oplus s_k(p) \\
&= s(p)
\end{aligned}$$

for all $p \in V \cap U_\alpha$. Therefore the restriction of the section $s: V \to E^\oplus$ to $V \cap U_\alpha$ is smooth on $V \cap U_\alpha$. It follows that $s: V \to E^\oplus$ is a smooth section of the smooth vector bundle $\pi_{E^\oplus}: E^\oplus \to M$, as required. ∎

Definition Let M be a smooth manifold, and, for each integer q between 1 and k, let $\pi_{E_q}: E_q \to M$ be a smooth vector bundle over M. The *direct sum* of the vector bundles $E_1, E_2, \ldots, E_k$ is denoted by $\bigoplus_{q=1}^{k} E_q$, or by

$$E_1 \oplus E_2 \oplus \cdots \oplus E_k,$$

and it is the smooth vector bundle $\pi_{E^\oplus}: E^\oplus \to M$ whose fibre over each point p is the direct sum of the fibres of the given vector bundles over M, where the topology and smooth structure on $E^\oplus$ are as described in the statement of Proposition 6.11

6.8 Tensor Products of Vector Bundles

Proposition 6.12 *Let M be a smooth manifold, and, for each integer q between 1 and k, let $\pi_{E_q}: E_q \to M$ be a smooth vector bundle over M. For each point p of M let $E_p^\otimes$ be the real vector space that is the tensor product of the fibres of the given vector bundles over the point p, so that*

$$E_p^\otimes = \bigotimes_{q=1}^{k} (E_q)_p = (E_1)_p \otimes (E_2)_p \otimes \cdots (E_k)_p,$$

where $(E_q)_p = \pi_{E_q}^{-1}(\{p\})$ for $q = 1, 2, \ldots, k$. Also let $E^\otimes$ be the disjoint union of the vector spaces $E_p^\otimes$, and let $\pi_{E^\otimes}: E^\otimes \to M$ be the surjective function, defined on the disjoint union $E^\otimes$ of all these vector spaces $E_p^\otimes$, that sends elements of $E_p^\otimes$ to p for all points p of M. Then $E^\otimes$ can be given the structure of a smooth manifold so as to ensure that $\pi_{E^\otimes}: E^\otimes \to M$ is a smooth vector bundle over M satisfying the following condition:

if $s: V \to E^{\otimes}$ is a function mapping some open subset V of M into $E^{\otimes}$, and if

$$s(p) = s_1(p) \otimes s_2(p) \otimes \cdots \otimes s_k(p)$$

for all $p \in V$, where $s_q: V \to E_q$ is a smooth section of $\pi_{E_q}: E_q \to M$ defined over V for $q = 1, 2, \ldots, k$, then $s: V \to E^{\otimes}$ is a smooth section of $\pi_{E^{\otimes}}: E^{\otimes} \to M$ defined over V.

Proof Let $(U_\alpha: \alpha \in A)$ be an open cover of M, where the smooth vector bundles $\pi_{E_q}: E_q \to M$ are all trivial over each open set U_α, and, for $q = 1, 2, \ldots, k$ and for all $\alpha, \beta \in A$, let $\psi_{q,\alpha}: U_\alpha \times \mathbb{R}^{r_q} \to E_q$ and $g_{q,\alpha,\beta}: U_\alpha \cap U_\beta \to GL(r_q, \mathbb{R})$ be smooth maps with the properties (i)–(v) listed in the statement of Corollary 6.9. These smooth maps determine smooth maps

$$\psi_\alpha^{\otimes}: U_\alpha \times \bigotimes_{q=1}^{k} \mathbb{R}^{r_q} \to E^{\otimes}$$

and

$$g_{\alpha,\beta}^{\otimes}: U_\alpha \cap U_\beta \to \mathrm{GL}\left(\bigotimes_{q=1}^{k} \mathbb{R}^{r_q}\right)$$

for all $\alpha, \beta \in A$, where $\mathrm{GL}\left(\bigotimes_{q=1}^{k} \mathbb{R}^{r_q}\right)$ denotes the group of invertible linear operators on the tensor product $\bigotimes_{q=1}^{k} \mathbb{R}^{r_q}$. These smooth maps $\psi_\alpha^{\otimes}$ and $g_{\alpha,\beta}^{\otimes}(p)$ are defined so that

$$\psi_\alpha^{\otimes}(p, \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_k) = \psi_{1,\alpha}(p, \mathbf{v_1}) \otimes \psi_{2,\alpha}(p, \mathbf{v_2}) \otimes \cdots \otimes \psi_{s,\alpha}(p, \mathbf{v_k})$$

and

$$g_{\alpha,\beta}^{\otimes}(p)(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_k) = g_{1,\alpha,\beta}(p)(\mathbf{v}_1) \otimes g_{2,\alpha,\beta}(p)(\mathbf{v}_2) \otimes \cdots \otimes g_{k,\alpha,\beta}(p)(\mathbf{v}_k)$$

for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$ and for all $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, where $\mathbf{v}_q \in \mathbb{R}^{r_q}$ for $q = 1, 2, \ldots, k$. Moreover

$$\psi_\beta^{\otimes}(p, \mathbf{v}) = \psi_\alpha^{\otimes}(p, g_{\alpha,\beta}^{\otimes}(p)(\mathbf{v}))$$

for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$ and $\mathbf{v} \in \bigotimes_{q=1}^{k} \mathbb{R}^{r_q}$.

Let $\mathbf{e}_{q,1}, \mathbf{e}_{q,2}, \ldots, \mathbf{e}_{q,r_q}$ be a basis of the real vector space $\mathbb{R}^{r_q}$ for $q = 1, 2, \ldots, k$, and let

$$J = (j_1, j_2, \ldots, j_k) : j_q \in \mathbb{Z} \text{ and } 1 \leq j_q \leq r_q \text{ for } q = 1, 2, \ldots, k\}.$$

Then these bases of vector spaces $\mathbb{R}^{r_q}$ together determine a basis for the tensor product $\bigotimes_{q=1}^{k} \mathbb{R}^{r_q}$ consisting of all elements of this tensor product that are of the form

$$\mathbf{e}_{1,j_1} \otimes \mathbf{e}_{2,j_2} \otimes \cdots \mathbf{e}_{k,j_k},$$

for some $(j_1, j_2, \ldots, j_k) \in J$.

Let $\alpha, \beta \in A$. Then each function $g_{q,\alpha,\beta}: U_\alpha \cap U_\beta \to GL(r_q, \mathbb{R})$ is smooth, and therefore there exist smooth real-valued functions $(A_q)^{i_q}{}_{j_q}$, where $q \in \{1, 2, \ldots, k\}$ and $1 \leq i_q, j_q \leq r_q$, such that

$$g_{q,\alpha,\beta}(p)(\mathbf{e}_{q,j_q}) = \sum_{i_q=1}^{r_q} (A_q)^{i_q}{}_{j_q}(p) \mathbf{e}_{q,i_q}$$

for $q = 1, 2, \ldots, k$. Then

$$\begin{aligned} & g^{\otimes}_{\alpha,\beta}(p)(\mathbf{e}_{1,j_1} \otimes \mathbf{e}_{2,j_2} \otimes \cdots \mathbf{e}_{k,j_k}) \\ & \qquad = \sum_{(i_1,i_2,\ldots,i_k) \in J} \left(\left(\prod_{q=1}^{k} (A_q)^{i_q}{}_{j_q}(p) \right) \mathbf{e}_{1,i_1} \otimes \mathbf{e}_{2,i_2} \otimes \cdots \mathbf{e}_{k,i_k} \right). \end{aligned}$$

Thus the matrix that represents the linear operator $g^{\otimes}_{\alpha,\beta}(p)$ on $\bigotimes_{q=1}^{k} \mathbb{R}^{r_q}$ with respect to the basis

$$\left(\mathbf{e}_{1,j_1} \otimes \mathbf{e}_{2,j_2} \otimes \cdots \mathbf{e}_{k,j_k} : (j_1, j_2, \ldots, j_k) \in J \right)$$

has entries that are products of the form $\prod_{q=1}^{k} (A_q)^{i_q}{}_{j_q}(p)$, where

$$(i_1, i_2, \ldots, i_k), \;\; (j_1, j_2, \ldots, j_k) \in J.$$

It follows that the entries of this matrix are smoooth functions of p as the point p ranges over the open set $U_\alpha \cap U_\beta$ in M. Thus the open cover $(U_\alpha : \alpha \in A)$ of M and the functions

$$\psi^{\otimes}_{\alpha}: U_\alpha \times \bigotimes_{q=1}^{k} \mathbb{R}^{r_q} \to E^{\otimes}$$

and

$$g^{\otimes}_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}\left(\bigotimes_{q=1}^{k} \mathbb{R}^{r_q}\right)$$

satisfy the conditions (i)–(v) in the statement of Proposition 6.5, and therefore there is a topology and smooth structure on $E^{\otimes}$ with respect to which $E^{\otimes}$ is a smooth manifold, $\pi_{E^{\otimes}}\colon E^{\otimes} \to M$ is a smooth vector bundle, and the function $\psi^{\otimes}_{\alpha}\colon U_\alpha \times \bigotimes_{q=1}^{k} \mathbb{R}^{r_q} \to E^{\otimes}$ maps its domain diffeomorphically onto $\pi_{E^{\otimes}}^{-1}(U_\alpha)$ for all $\alpha \in A$.

Let V be an open set in M, let $s_q\colon V \to E_q$ be a smooth section of $\pi_{E_q}\colon E_q \to M$ for $q = 1, 2, \ldots, k$, and let

$$s(p) = s_1(p) \otimes s_2(p) \otimes \cdots \otimes s_k(p)$$

for all $p \in V$. Then there are smooth real-valued functions $f^{j_q}_{q,\alpha}$ on $V \cap U_\alpha$ for $j_q = 1, 2, \ldots, r_q$ such that such that

$$s_q(p) = \sum_{j_q=1}^{r_q} f^{j_q}_{q,\alpha}(p)\psi_q(p, \mathbf{e}_{q,j_q})$$

for all $p \in V \cap U_\alpha$. Then

$$s(p) = \sum_{(j_1,j_2,\ldots,j_k)\in J} \left(\left(\prod_{q=1}^{k} f^{j_q}_{q,\alpha}(p)\right) b_{j_1,j_2,\ldots,j_k}(p)\right),$$

where

$$b_{j_1,j_2,\ldots,j_k}(p) = \psi^{\otimes}_{\alpha}(p, \mathbf{e}_{1,j_1} \otimes \mathbf{e}_{2,j_2} \otimes \cdots \mathbf{e}_{k,j_k})$$

for all $p \in U_\alpha$ and $(j_1, j_2, \ldots, j_k) \in J$. Now $b_{j_1,j_2,\ldots,j_k}$ is a the smooth section of $\pi_{E^{\otimes}}\colon E^{\otimes} \to M$ defined over U_α for all $(j_1, j_2, \ldots, j_k) \in J$. It follows that the section $s\colon V \to E^{\otimes}$ of $\pi_{E^{\otimes}}\colon E^{\otimes} \to M$ is smooth, as required. ∎

Definition Let M be a smooth manifold, and, for each integer q between 1 and k, let $\pi_{E_q}\colon E_q \to M$ be a smooth vector bundle over M. The *tensor product* of the vector bundles $E_1, E_2, \ldots, E_k$ is denoted by $\bigotimes_{q=1}^{k} E_q$, or by

$$E_1 \otimes E_2 \otimes \cdots \otimes E_k,$$

and it is the smooth vector bundle $\pi_{E^{\otimes}}\colon E^{\otimes} \to M$ whose fibre over each point p is the tensor product of the fibres of the given vector bundles over M, where the topology and smooth structure on $E^{\otimes}$ are as described in the statement of Proposition 6.12

6.9 Pullbacks of Smooth Vector Bundles

Proposition 6.13 *Let M and N be a smooth manifolds of dimensions m and n respectively, let $\pi_E: E \to N$ be a smooth vector bundle of rank k over N, and let $\varphi: M \to N$ be a smooth map. Let*

$$\varphi^* E = \{(p, e) \in M \times E : \varphi(p) = \pi_E(e)\}$$

and let $\pi_{\varphi^* E}: \varphi^* E \to M$ and $\varphi_*: \varphi^* E \to E$ be defined such that

$$\pi_{\varphi^* E}(p, e) = p \quad \textit{and} \quad \varphi_*(p, e) = e$$

for all $(p, e) \in \varphi^* E$. Then $\varphi^* E$ is a smooth submanifold of $M \times E$, the maps $\pi_{\varphi^* E}: \varphi^* E \to M$ and $\varphi_*: \varphi^* E \to E$ are smooth, and $\pi_{\varphi^* E}: \varphi^* E \to M$ is the projection map of a smooth vector bundle of rank k over the manifold M whose total space is $\varphi^* E$.

Proof Let $p_0 \in M$. Then there exists an open set U in N, where $\varphi(p_0) \in U$, and a smooth map $\psi: U \times \mathbb{R}^k \to E$ which satisfies the following properties: $\pi_E(\psi(q, \mathbf{v})) = q$ for all $q \in U$; the smooth map ψ maps its domain $U \times \mathbb{R}^k$ diffeomorphically onto $\pi_E^{-1}(U)$; for each $q \in U$, the function mapping $\mathbf{v} \in \mathbb{R}^k$ to $\psi(q, \mathbf{v})$ maps $\mathbb{R}^k$ isomorphically onto the real vector space E_q, where $E_q = \pi_E^{-1}(\{q\})$.

We can choose the open set U so that it is the domain of a smooth coordinate system $(y^1, y^2, \ldots, y^n)$ for the smooth manifold N. Now $\varphi^{-1}(U)$ is an open set in M, because $\varphi: M \to N$ is continuous. Let $(x^1, x^2, \ldots, x^m)$ be a smooth coordinate system on M whose domain W satisfies $p_0 \in W$ and $W \subset \varphi^{-1}(U)$.

Let $\xi^1, \xi^2, \ldots, \xi^k$ be the smooth functions from $\pi_E^{-1}(U) \to \mathbb{R}$ defined such that

$$\xi^l(\psi(q, (v_1, v_2, \ldots, v_k))) = v_l$$

for $l = 1, 2, \ldots, k$ and for all $q \in U$ and $(v_1, v_2, \ldots, v_k)$ in $\mathbb{R}^k$. Also let $\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^k$ be the smooth functions on $\pi_E^{-1}(U)$ defined such that $\tilde{y}^j = y^j \circ \pi_E$ for $j = 1, 2, \ldots, n$. Then the smooth functions

$$\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^n, \xi^1, \xi^2, \ldots, \xi^k$$

represent a smooth coordinate system throughout the open subset $\pi_E^{-1}(U)$ of the $(n+k)$-dimensional manifold E. Given $(p, e) \in W \times \pi_E^{-1}(U)$, let $\overline{x}^i(p, e) = x^i(p)$ for $i = 1, 2, \ldots, m$, $\overline{y}^j(p, e) = \tilde{y}^j(e) = y^j(\pi_E(e))$ for $j = 1, 2, \ldots, n$ and $\overline{\xi}^l(p, e) = \xi^l(e)$ for $l = 1, 2, \ldots, k$ Then

$$\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^m, \overline{y}^1, \overline{y}^2, \ldots, \overline{y}^n, \overline{\xi}^1, \overline{\xi}^2, \ldots, \overline{\xi}^k$$

are smooth functions on $M \times E$ which represent a smooth coordinate system throughout the open subset $W \times \pi_E^{-1}(U)$ of $M \times E$. Let

$$z^j(p, e) = \overline{y}^j(p, e) - y^j(\varphi(p)) = y^j(\pi_E(e)) - y^j(\varphi(p)).$$

Then the smooth functions

$$\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^m, z^1, z^2, \ldots, z^n, \overline{\xi}^1, \overline{\xi}^2, \ldots, \overline{\xi}^k$$

also represent a smooth coordinate system throughout $W \times \pi_E^{-1}(U)$, and

$$\begin{aligned}(\varphi^* E) \cap (W \times \pi_E^{-1}(U)) \\ &= \{(p, e) \in W \times \pi_E^{-1}(U) : z^j(p, e) = 0 \text{ for } j = 1, 2, \ldots, n\}.\end{aligned}$$

We conclude from this that $\varphi^*(E)$ is a smooth submanifold of $M \times E$.

Now each fibre $(\varphi^* E)_p$ of the surjective map $\pi_{\varphi^* E}: \varphi^* E \to M$ may be given the structure of a vector space so as to ensure that the smooth map $\varphi_*: \varphi^* E \to E$ maps $(\varphi^* E)_p$ isomorphically onto that fibre isomorphically onto the corresponding fibre $E_{\varphi(p)}$ of $\pi_E: E \to N$.

Let $\overline{\psi}: W \times \mathbb{R}^k \to \varphi^* E$ be the smooth map defined such that

$$\overline{\psi}(p, \mathbf{v}) = (p, \psi(\varphi(p), \mathbf{v})$$

for all $p \in W$ and $\mathbf{v} \in \mathbb{R}^k$. Then $\pi_{\varphi^* E}(\overline{\psi}(p, \mathbf{v}) = p$. Morover $\overline{\psi}$ maps $W \times \mathbb{R}^k$ diffemorphically onto $\pi_{\varphi^* E}^{-1}(W)$, and, for each $p \in W$, the map sending $\mathbf{v} \in \mathbb{R}^k$ to $\overline{\psi}(p, \mathbf{v}$ is an isomorphism of real vector spaces. It follows that $\pi_{\varphi^* E}: \varphi^* E \to M$ is a smooth vector bundle over M, as required. ∎

Definition Let M and N be a smooth manifolds of dimension m and n respectively, let $\pi_E: E \to N$ be a smooth vector bundle of rank k over N, and let $\varphi: M \to N$ be a smooth map. The *pullback* of the smooth vector bundle $\pi_E: E \to N$ along the smooth map $\varphi: M \to N$ is the smooth vector bundle $\pi_{\varphi^* E}: \varphi^* E \to M$ over M with total space $\varphi^* E$, where

$$\pi_{\varphi^* E}(p, e) = p$$

for all $(p, e) \in \varphi^* E$. The smooth map $\varphi_*: \varphi^* E \to E$ defined such that $\varphi_*(p, e) = e$ for all $(p, e) \in \varphi^* E$ is then a *morphism of smooth vector bundles* which induces vector space isomorphisms between corresponding fibres and which covers the smooth map $\varphi: M \to N$.

6.10 The Cotangent Bundle of a Smooth Manifold

Let M be a smooth manifold, and let $\pi_{TM}: TM \to M$ be the tangent bundle of M. This tangent bundle is a smooth vector bundle. There is a corresponding dual bundle $\pi_{T^*M}: T^*M \to M$ whose fibre over a point p of M is the dual space T_p^*M of the tangent space T_pM at the point p. This dual space T_p^*M is referred to as the *cotangent space* at the point p: its elements are linear functionals on the tangent space T_pM.

Definition The *cotangent bundle* of a smooth manifold is the smooth vector bundle $\pi_{T^*M}: T^*M \to M$ that is the dual of the tangent bundle $\pi_{TM}: TM \to M$ of M.

Let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system defined over an open set U in M. Then the differentials

$$dx_p^1,\ dx_p^2, \ldots,\ dx_p^n$$

of these coordinate functions constitute a basis of the cotangent space T_p^*M at each point p of U, where

$$\left\langle dx_p^j, \sum_{k=1}^n v^k \left.\frac{\partial}{\partial x^k}\right|_p \right\rangle = v^j$$

for $j = 1, 2, \ldots, n$. Then there are diffeomorphisms

$$\psi: U \times \mathbb{R}^n \to \pi_{TM}^{-1}(U) \quad \text{and} \quad \chi: U \times \mathbb{R}^n \to \pi_{T^*M}^{-1}(U)$$

that are isomorphisms of vector bundles over U, where

$$\psi(p, (v^1, v^2, \ldots, v^n) = \sum_{j=1}^n v^j \left.\frac{\partial}{\partial x^j}\right|_p$$

and

$$\chi(p, (b_1, b_2, \ldots, b_n)) = \sum_{j=1}^n b_j\, dx_p^j$$

for all $p \in U_\alpha$ and for all elements $(v^1, v^2, \ldots, v^n)$ and $(b_1, b_2, \ldots, b_n)$ of $\mathbb{R}^n$. Moreover

$$\langle \chi_p(b_1, b_2, \ldots, b_n), \psi_p(v^1, v^2, \ldots, v^n) \rangle = \sum_{j=1}^n b_j v^j$$

for all $p \in U_\alpha$ and for all elements $(v^1, v^2, \ldots, v^n)$ and $(b_1, b_2, \ldots, b_n)$ of $\mathbb{R}^n$, where

$$\psi_p(v^1, v^2, \ldots, v^n) = \psi(p, (v^1, v^2, \ldots, v^n))$$

and

$$\chi_p(b_1, b_2, \ldots, b_n) = \chi(p, (b_1, b_2, \ldots, b_n)).$$

The map that sends each point p of U to the differential dx^j_p is a smooth section of the cotangent bundle $\pi_{T^*M}: T^*M \to M$ over U. We denote this section by dx^j. Then, given any section $\tau: V \to T^*M$ of the cotangent bundle over an open subset V of U, there exist real-valued functions $b_1, b_2, \ldots, b_n$ on V such that $\tau = \sum_{j=1}^{n} b_j\, dx^j$. These functions $b_1, b_2, \ldots, b_n$ are uniquely determined, because the values of $dx^1, dx^2, \ldots, dx^n$ at any point p of U constitute a basis of the cotangent space $T^*_p M$. The section τ is smooth on V if and only if its components $b_1, b_2, \ldots, b_n$ are smooth real-valued functions on V.

Definition Let M be a smooth manifold. A *differential form* of degree 1 on an open subset V of M is a section $\tau: V \to T^*M$ of the cotangent bundle $\pi_{T^*M}: T^*M \to M$ defined over the open set V. Differential forms of degree 1 are also known as 1-*forms*.

Lemma 6.14 *Let M be a smooth manifold, let $X: V \to TM$ be a smooth vector field defined over an open subset V of M and let $\tau: V \to T^*M$ be a smooth differential form of degree 1 on V, and let X_p and τ_p denote the values of X and τ at each point p of V. Then the real-valued function on V that sends $p \in V$ to $\langle \tau_p, X_p \rangle$ is a smooth real-valued function on M.*

Proof Let U be the domain of a smooth coordinate system on M. Then there are uniquely-determined smooth real-valued functions $v^1, v^2, \ldots, v^n$ and $b_1, b_2, \ldots, b_n$ on $V \cap U$ such that

$$X = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j} \quad \text{and} \quad \tau = \sum_{k=1}^{n} b_k\, dx^k.$$

Then

$$\langle \tau, X \rangle = \tau(X) = \sum_{j=1}^{n} \sum_{k=1}^{n} b_k v^j \left\langle dx^k, \frac{\partial}{\partial x^j} \right\rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} b_k v^j \delta^k_j = \sum_{j=1}^{n} b_j v^j$$

on $U \cap V$, where δ^k_j is the Kronecker delta that is equal to 1 when $k = j$, but is equal to 0 otherwise. Thus the function on $V \cap U$ that sends points p of $V \cap U$ to the corresponding value $\langle \tau_p, X_p \rangle$ of $\tau(X)$ is a sum of products of smooth real-valued functions, and is thus itself smooth. ∎

6.11 Tensor Fields on Smooth Manifolds

Definition Let M be a smooth manifold. A *tensor field* of type (r,s) on M is a section of the smooth vector bundle $T^{\otimes r}M \otimes T^{*\otimes s}M$ that is the tensor product

$$TM \otimes TM \otimes \cdots \otimes TM \otimes T^*M \otimes T^*M \otimes \cdots \otimes T^*M$$

of r copies of the tangent bundle TM and s copies of the cotangent bundle T^*M of M.

Let M be a smooth manifold of dimension n, let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system defined over an open subset U of M, and let S be a smooth tensor field of type (r,s) defined over U. Then there are smooth real-valued functions $S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s}$ defined on U such that

$$S = \sum_{j_1,j_2,\ldots,j_r=1}^{n} \sum_{k_1,k_2,\ldots,k_s=1}^{n} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \otimes dx^{k_1} \otimes \cdots \otimes dx^{k_s}.$$

Suppose that

$$S = \sum_{l_1,l_2,\ldots,l_r=1}^{n} \sum_{m_1,m_2,\ldots,m_s=1}^{n} \hat{S}^{l_1,l_2,\ldots,l_r}_{m_1,m_2,\ldots,m_s} \frac{\partial}{\partial \hat{x}^{l_1}} \otimes \cdots \otimes \frac{\partial}{\partial \hat{x}^{l_r}} \otimes d\hat{x}^{m_1} \otimes \cdots \otimes d\hat{x}^{m_s}$$

where $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^n)$ is another smooth coordinate system that is also defined over the open set U. Then

$$\frac{\partial}{\partial x^{j_i}} = \sum_{l=1}^{n} \frac{\partial \hat{x}^l}{\partial x^{j_i}} \frac{\partial}{\partial \hat{x}^l}$$

for $i = 1, 2, \ldots, r$, and

$$dx^{k_i} = \sum_{m=1}^{n} \frac{\partial x^{k_i}}{\partial \hat{x}^m} d\hat{x}^m$$

for $i = 1, 2, \ldots, s$. It follows that

$$\hat{S}^{l_1,l_2,\ldots,l_r}_{m_1,m_2,\ldots,m_s} = \sum_{j_1,j_2,\ldots,j_r=1}^{n} \sum_{k_1,k_2,\ldots,k_s=1}^{n} S^{j_1,j_2,\ldots,j_r}_{k_1,k_2,\ldots,k_s} \left(\prod_{i=1}^{r} \frac{\partial \hat{x}^{l_i}}{\partial x^{j_i}} \right) \left(\prod_{i=1}^{s} \frac{\partial x^{k_i}}{\partial \hat{x}^{m_i}} \right).$$

Example Let M be a smooth manifold of dimension n and let g be a smooth tensor field of type $(0,2)$ on M. Let $(x^1, x^2, \ldots, x^n)$ be a smooth local coordinate system defined over some open subset U of M. Then there are smooth functions g_{ij} on U for $i, j = 1, 2, \ldots, n$ such that

$$g = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}\, dx^i \otimes dx^j$$

over U. Let $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^n)$ be another smooth local coordinate system defined over U, and let $\hat{g}^{kl}$ be smooth real-valued functions on U such that

$$g = \sum_{k=1}^{n} \sum_{l=1}^{n} \hat{g}_{kl} \, d\hat{x}^k \otimes d\hat{x}^l.$$

Then

$$\hat{g}_{kl} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} \, \frac{\partial x^i}{\partial \hat{x}^k} \, \frac{\partial x^j}{\partial \hat{x}^l}.$$

Example Let M be a smooth manifold of dimension n and let R be a smooth tensor field of type $(1, 3)$ on M. Let $(x^1, x^2, \ldots, x^n)$ be a smooth local coordinate system defined over some open subset U of M. Then there are smooth functions ${R^i}_{jkl}$ on U for $i, j, k, l = 1, 2, \ldots, n$ such that

$$R = \sum_{i,j,k,l=1}^{n} R^i_{jkl} \, \frac{\partial}{\partial x^i} \otimes \, dx^j \otimes dx^k \otimes dx^l$$

over U. Let $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^n)$ be another smooth local coordinate system defined over U, and let $\hat{R}^a_{bcd}$ be smooth real-valued functions on U such that

$$R = \sum_{a,b,c,d=1}^{n} \hat{R}^a_{bcd} \, \frac{\partial}{\partial \hat{x}^a} \otimes \, d\hat{x}^b \otimes d\hat{x}^c \otimes d\hat{x}^d$$

Then

$$\hat{R}^a_{bcd} = \sum_{i,j,k,l=1}^{n} R^i_{jkl} \, \frac{\partial \hat{x}^a}{\partial x^i} \, \frac{\partial x^j}{\partial \hat{x}^b} \, \frac{\partial x^k}{\partial \hat{x}^c} \, \frac{\partial x^l}{\partial \hat{x}^d}.$$

6.12 Sections of Tensor Product Bundles Determined by Multilinear Operators

Definition Let $\tilde{E}, E_1, E_2, \ldots, E_k$ be smooth vector bundles over a smooth manifold M, and let $\mathcal{Q}$ be an operator that, over each open set U on M, assigns to smooth sections $s_1, s_2, \ldots s_k$ of the respective vector bundles $E_1, E_2, \ldots, E_k$ defined over U a smooth section $\mathcal{Q}(s_1, s_2, \ldots, s_k)$ of the vector bundle $\tilde{E}$ defined over this open set U. The operator $\mathcal{Q}$ on sections is said to be $\mathbb{R}$-*multilinear* if

$$\begin{aligned} \mathcal{Q}(as_1 + bt_1, s_2, \ldots, s_k) &= a\mathcal{Q}(s_1, s_2, \ldots, s_k) + b\mathcal{Q}(t_1, s_2, \ldots, s_k), \\ \mathcal{Q}(s_1, as_2 + bt_2, \ldots, s_k) &= a\mathcal{Q}(s_1, s_2, \ldots, s_k) + b\mathcal{Q}(s_1, t_2, \ldots, s_k), \\ &\quad \text{etc.} \end{aligned}$$

for all real numbers a and b, and for all $s_1, s_2, \ldots, s_k$ and $t_1, t_2, \ldots, t_k$, where s_j and t_j are smooth sections of the vector bundle E_j defined over U for $j = 1, 2, \ldots, k$.

Proposition 6.15 *Let $\tilde{E}, E_1, E_2, \ldots, E_k$ be smooth vector bundles over a smooth manifold M, and let $\mathcal{Q}$ be an operator that, over each open set U on M, assigns to smooth sections $s_1, s_2, \ldots s_k$ of the respective vector bundles $E_1, E_2, \ldots, E_k$ defined over U a smooth section $\mathcal{Q}(s_1, s_2, \ldots, s_k)$ of the vector bundle $\tilde{E}$ defined over this open set U. Suppose that this operator $\mathcal{Q}$ on sections is $\mathbb{R}$-multilinear, so that*

$$\begin{aligned} \mathcal{Q}(as_1 + bt_1, s_2, \ldots, s_k) &= a\mathcal{Q}(s_1, s_2, \ldots, s_k) + b\mathcal{Q}(t_1, s_2, \ldots, s_k), \\ \mathcal{Q}(s_1, as_2 + bt_2, \ldots, s_k) &= a\mathcal{Q}(s_1, s_2, \ldots, s_k) + b\mathcal{Q}(s_1, t_2, \ldots, s_k), \\ & \quad \textit{etc.} \end{aligned}$$

for all real numbers a and b, and for all $s_1, s_2, \ldots, s_k$ and $t_1, t_2, \ldots, t_k$, where s_j and t_j are smooth sections of the vector bundle E_j defined over U for $j = 1, 2, \ldots, k$. Suppose also that

$$\mathcal{Q}(f_1 s_1, f_2 s_2, \ldots, f_k s_k) = f_1 \cdot f_2 \cdots f_k \, \mathcal{Q}(s_1, s_2, \ldots, s_k)$$

for all smooth functions $f_1, f_2 \cdots f_k$ on U, and for all $s_1, s_2, \ldots, s_k$, where s_j is a smooth section of the vector bundle E_j defined over U for $j = 1, 2, \ldots, k$. Then there exists a smooth section Q of the vector bundle

$$\tilde{E} \otimes E_1^* \otimes E_2^* \otimes \cdots \otimes E_k^*$$

such that

$$\mathcal{Q}(s_1, s_2, \ldots, s_k) = Q(s_1, s_2, \ldots, s_k)$$

for all $s_1, s_2, \ldots, s_k$, where s_j is a smooth section of the vector bundle E_j over U for $j = 1, 2, \ldots, k$.

Proof Let U be an open set in M over which each of the vector bundles $\pi_{E_j}: E_j \to M$ and $\pi_{\tilde{E}}: \tilde{E} \to M$ is trivial, let $e_{j,1}, e_{j,2}, \ldots, e_{j,r_j}$ be smooth sections of $\pi_{E_j}: E_j \to M$ for $j = 1, 2, \ldots, r_j$, where r_j the rank of this vector bundle, and where the values $e_{j,1}(p), e_{j,2}(p), \ldots, e_{j,r_j}(p)$ of these sections at each point p of U constitute a basis for the fibre $(E_j)_p$ of the vector bundle E_j over p, and let $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_r$ be smooth sections of $\pi_{\tilde{E}}: \tilde{E} \to M$ for $j = 1, 2, \ldots, r_j$, where r the rank of this vector bundle, and where the values $\tilde{e}_1(p), \tilde{e}_2(p), \ldots, \tilde{e}_2(p)$ of these sections at each point p of U constitute a basis for the fibre $\tilde{E}_p$ of the vector bundle $\tilde{E}$ over p.

Let $s_j\colon U \to E_j$ be a smooth section of $\pi_j\colon E_j \to M$ defined over U for $j = 1, 2, \ldots, k$. Then there exist smooth real-valued functions $f_{(j)}^{\alpha_j}$ on U for $j = 1, 2, \ldots, k$ and $\alpha_j = 1, 2, \ldots, r_j$ such that

$$s_j = \sum_{\alpha_j=1}^{r_j} f_{(j)}^{\alpha_j} e_{j,\alpha_j}.$$

Then

$$\begin{aligned} &\mathcal{Q}(s_1, s_2, \ldots, s_k) \\ &\quad= \mathcal{Q}_p\left(\sum_{\alpha_1=1}^{r_1} f_{(1)}^{\alpha_1} e_{1,\alpha_1}, \ldots, \sum_{\alpha_k=1}^{r_k} f_{(k)}^{\alpha_k} e_{k,\alpha_k}\right) \\ &\quad= \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_k=1}^{r_k} \mathcal{Q}_p(f_{(1)}^{\alpha_1} e_{1,\alpha_1}, f_{(2)}^{\alpha_2} e_{2,\alpha_2}, \ldots, f_{(k)}^{\alpha_k} e_{k,\alpha_k}) \\ &\quad= \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_k=1}^{r_k} f_{(1)}^{\alpha_1} f_{(2)}^{\alpha_2} \cdots f_{(k)}^{\alpha_k} \mathcal{Q}_p(e_{1,\alpha_1}, e_{2,\alpha_2}, \ldots, e_{k,\alpha_k}) \end{aligned}$$

Now there exist smooth functions $Q^\beta_{\alpha_1,\alpha_2,\ldots,\alpha_k}$ on U such that

$$\mathcal{Q}_p(e_{1,\alpha_1}, e_{2,\alpha_2}, \ldots, e_{k,\alpha_k}) = \sum_{\beta=1}^{r} Q^\beta_{\alpha_1,\alpha_2,\ldots,\alpha_k} \tilde{e}_\beta$$

for all k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of integers that satisfy $1 \leq \alpha_j \leq r_j$ for $j = 1, 2, \ldots, k$. Then

$$\mathcal{Q}(s_1, s_2, \ldots, s_k) = \sum_{\beta=1}^{r} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_k=1}^{r_k} Q^\beta_{\alpha_1,\alpha_2,\ldots,\alpha_k} f_{(1)}^{\alpha_1} f_{(2)}^{\alpha_2} \cdots f_{(k)}^{\alpha_k} \tilde{e}_\beta.$$

Let Q be the smooth section of the vector bundle

$$\tilde{E} \otimes E_1^* \otimes E_2^* \otimes \cdots \otimes E_k^*$$

over U defined by the equation

$$Q = \sum_{\beta=1}^{r} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_k=1}^{r_k} Q^\beta_{\alpha_1,\alpha_2,\ldots,\alpha_k} \tilde{e}_\beta \otimes \varepsilon_{(1)}^{\alpha_1} \otimes \varepsilon_{(2)}^{\alpha_2} \otimes \cdots \otimes \varepsilon_{(k)}^{\alpha_k},$$

where $\varepsilon_{(j)}^1, \varepsilon_{(j)}^2, \ldots, \varepsilon_{(j)}^{r_j}$ are the smooth sections of $\pi_{E_j^*}\colon E_j^* \to M$ over U whose values at each point p of U constitute the basis of the fibre $(E_j^*)_p$ over p that is the dual basis to the basis of $(E_j)_p$ determined by the values of the sections $e_{j,1}, e_{j,2}, \ldots, e_{j,r}$ of E_j at p. Then $\langle \varepsilon^{\alpha_j}_{(j)}, e_{j,\beta_j} \rangle = \delta^{\alpha_j}_{\beta_j}$, where $\delta^{\alpha_j}_{\beta_j}$ denotes the Kronecker delta, and thus

$$\langle \varepsilon^{\alpha_j}_{(j)}, s_j \rangle = \sum_{\beta_j=1}^{r_j} f^{\beta_j}_{(j)} \langle \varepsilon^{\alpha_j}_{(j)}, e_{j,\beta_j} \rangle = f^{\alpha_j}_{(j)}$$

for $j = 1, 2, \ldots, k$ and $\alpha_j = 1, 2, \ldots, r_j$. It follows that

$$\begin{aligned} & Q(s_1, \ldots, s_k) \\ = & \sum_{\beta=1}^{r} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_k=1}^{r_k} Q^{\beta}_{\alpha_1,\alpha_2,\ldots,\alpha_k} \langle \varepsilon^{\alpha_1}_{(1)}, s_1 \rangle \langle \varepsilon^{\alpha_2}_{(2)}, s_2 \rangle \cdots \langle \varepsilon^{\alpha_k}_{(k)}, s_k \rangle \, \tilde{e}_\beta \\ = & \sum_{\beta=1}^{r} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_k=1}^{r_k} Q^{\beta}_{\alpha_1,\alpha_2,\ldots,\alpha_k} f^{\alpha_1}_{(1)} f^{\alpha_2}_{(2)} \cdots f^{\alpha_k}_{(k)} \, \tilde{e}_\beta \\ = & \mathcal{Q}(s_1, \ldots, s_k), \end{aligned}$$

as required. ∎

Corollary 6.16 *Let M be a smooth manifold, and let Q be an operator on M which, given smooth vector fields $X_1, X_2, \ldots, X_k$ defined over an open subset U of M, determines a smooth real-valued function $\mathcal{Q}(X_1, X_2, \ldots, X_k)$ on U. Suppose that this operator is $\mathbb{R}$-multilinear, and that*

$$\mathcal{Q}(f_1 X_1, f_2 X_2, \ldots, f_k X_k) = f_1 \cdot f_2 \cdots f_k \, \mathcal{Q}(X_1, X_2, \ldots, X_k)$$

for all smooth real-valued functions $f_1, f_2, \ldots, f_k$ and smooth vector fields $X_1, X_2, \ldots, X_k$ defined over the open set U. Then there is a smooth tensor field Q of type $(0, s)$ on M such that

$$\mathcal{Q}(X_1, X_2, \ldots, X_k) = Q(X_1, X_2, \ldots, X_k)$$

for all open subsets U of M and for all smooth vector fields $X_1, X_2, \ldots, X_k$ defined over U.

Corollary 6.17 *Let M be a smooth manifold, and let S be an operator on M which, given smooth vector fields $X_1, X_2, \ldots, X_k$ defined over an open subset U of M, determines a smooth vector field $\mathcal{S}(X_1, X_2, \ldots, X_k)$ on U. Suppose that this operator is $\mathbb{R}$-multilinear, and that*

$$\mathcal{S}(f_1 X_1, f_2 X_2, \ldots, f_k X_k) = f_1 \cdot f_2 \cdots f_k \, \mathcal{S}(X_1, X_2, \ldots, X_k)$$

for all smooth real-valued functions $f_1, f_2, \ldots, f_k$ and smooth vector fields $X_1, X_2, \ldots, X_k$ defined over the open set U. Then there is a smooth tensor field S of type $(1, s)$ on M such that

$$\mathcal{S}(X_1, X_2, \ldots, X_k) = S(X_1, X_2, \ldots, X_k)$$

for all open subsets U of M and for all smooth vector fields $X_1, X_2, \ldots, X_k$ defined over U.

6.13 Subbundles of Vector Bundles

Definition Let $\pi_E: E \to M$ and $\pi_F: F \to M$ be smooth vector bundles over a smooth manifold M. We say that $\pi_F: F \to M$ is a *subbundle* of $\pi_E: E \to M$ if F is a smooth submanifold of E, $\pi_F = \pi_E|F$ and, for each $p \in M$, the fibre F_p of $\pi_F: F \to M$ over p is a vector subspace of the fibre E_p of $\pi_E: E \to M$ over p.

Proposition 6.18 *Let $\pi_E: E \to M$ be a smooth vector bundle over a smooth manifold M, and, for all $p \in M$, let F_p be a vector subspace of the fibre E_p of $\pi_E: E \to M$ over the point p, and let $F = \bigcup_{p \in M} F_p$ and π_F be the restriction $\pi_E|F$ of the projection map π_E to the submanifold F of E. Suppose that, given any point p_0 of M there exists some open set U in M, where $p_0 \in U$, and smooth sections $s_1, s_2, \ldots, s_k$ of $\pi_E: E \to M$ defined over U such that the values $s_1(p), s_2(p), \ldots, s_k(p)$ at each point p of U constitute a basis for the subspace F_p of E_p. Then F is a smooth submanifold of E, and $\pi_F: F \to M$ is a smooth vector bundle which is a subbundle of $\pi_E: E \to M$.*

Proof Let $p_0 \in M$. Then there exists an open set U in M, where $p_0 \in U$, and smooth sections $s_1, s_2, \ldots, s_k$ of $\pi_E: E \to M$ defined over U_1 such that the values $s_1(p), s_2(p), \ldots, s_k(p)$ at each point p of U_1 constitute a basis for the subspace F_p of E_p. It then follows from basic linear algebra and the definition of smooth vector bundles that there exist smooth sections $s_{k+1}, \ldots, s_r$, of $\pi_E \to M$ defined over some open subset V_2 of U_1, where r is the rank of the vector bundle E, and where $p_0 \in U_2$, such that the values

$$s_1(p_0), s_2(p_0), \ldots, s_k(p_0), s_{k+1}(p_0), \ldots, s_r(p_0)$$

of the smooth sections $s_1, s_2, \ldots, s_r$ at the point p_0 constitute a basis of E_{p_0}. The continuity of these smooth sections then ensures that there exists some open subset U of U, where $p_0 \in U$, such that the values of the sections $s_1, s_2, \ldots, s_p$ at each point p of U constitute a basis of the fibre E_p of the vector bundle E over the point p. Let $\psi: U \times \mathbb{R}^r \to E$ be the smooth map defined such that

$$\psi(p, (v_1, v_2, \ldots, v_r)) = \sum_{\alpha=1}^{r} v^\alpha s_\alpha(p)$$

for all $p \in U$ and $(v_1, v_2, \ldots, v_r) \in \mathbb{R}^r$. Then ψ maps $U \times \mathbb{R}^r$ diffeomorphically onto $\pi_E^{-1}(U)$. Moreover

$$\begin{aligned} \psi^{-1}(F) &= \psi^{-1}\left\{\sum_{\alpha=1}^{k} v^\alpha s_\alpha(p) : v_1, v_2, \ldots, v_k \in \mathbb{R}\right\} \\ &= \{(p, (v_1, v_2, \ldots, v_k, 0, \ldots, 0) : v_1, v_2, \ldots, v_k \in \mathbb{R}\}. \\ &= U \times K, \end{aligned}$$

where K is the k-dimensional vector subspace of $\mathbb{R}^r$ defined such that

$$K = \{(v_1, v_2, \ldots, v_r) \in \mathbb{R}^r : v_{k+1} = \cdots = v_r = 0\}.$$

Clearly $\psi^{-1}(F)$ is a smooth submanifold of the domain $U \times \mathbb{R}^r$ that is a smooth product bundle over U. It follows from this that F is a smooth submanifold of the total space $\pi_E: E \to M$ of the smooth vector bundle E, and that if $\pi_F: F \to M$ is the restriction of the projection map $\pi_E: E \to M$ to the submanifold F of E, then $\pi_F: F \to M$ is itself a smooth vector bundle over M, as required. ∎

7 Vector Fields, Lie Brackets and Flows

7.1 Smooth Vector Fields

Let M be a smooth manifold, and let $\pi_{TM}: TM \to M$ be the tangent bundle of M. The total space TM of this tangent bundle is a smooth manifold, and the fibre $\pi_{TM}^{-1}(\{p\})$ of this bundle over any point p on M is the tangent space T_pM to M at the point p. A *vector field* on M associates to each point p of M a corresponding tangent vector X_p to M at the point p. It is therefore represented by a function $X: M \to TM$ from M to TM. Moreover the composition function $\pi_{TM} \circ X: M \to M$ is the identity map of the manifold M. A vector field on M is thus represented by a section $X: M \to TM$ of the tangent bundle $\pi_{TM}: TM \to M$ of M.

Definition Let M be a smooth manifold. A *continuous vector field* X on M is a continuous section $X: M \to TM$ of the tangent bundle $\pi_{TM}: TM \to M$ of M.

Definition Let M be a smooth manifold. A *smooth vector field* X on M is a smooth section $X: M \to TM$ of the tangent bundle $\pi_{TM}: TM \to M$ of M.

A subset U of a smooth manifold M is itself a smooth manifold, and moreover the tangent bundle $\pi_{TU}: TU \to U$ satisfies $TU = \pi_{TM}^{-1}(U)$ and $\pi_{TU} = \pi_{TM}|TU$. A vector field X on U is thus represented by a function $X: U \to TM$ that satisfies $\pi_{TM}(X_p) = p$ for all $p \in U$, where X_p denotes the value of the function X at p. The vector field X on U is *continuous* if and only if $X: U \to TM$ is a continuous map. This vector field is *smooth* if and only if $X: U \to TM$ is a smooth map.

We now show that a vector field X on a smooth manifold is smooth if and only if its components with respect to any smooth coordinate system are smooth functions on the domain of that coordinate system.

Proposition 7.1 *Let M be a smooth manifold of dimension n, let U be an open set in M, and let $x^1, x^2, \ldots, x^n$ be a smooth coordinate system defined over U. Let X be a vector field on U, and let $v^1, v^2, \ldots, v^n$ be real-valued functions such that*

$$X = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}.$$

Then the vector field X is continuous on U if and only if the component functions $v^1, v^2, \ldots, v^n$ are continuous. Also the vector field X is smooth on U if and only if these component functions are smooth.

Proof The smooth chart (U, φ) for M whose components are the smooth coordinate functions $x^1, x^2, \ldots, x^n$ determines a corresponding smooth chart $(\pi_{TM}^{-1}(U), \tilde{\varphi})$ for the smooth manifold TM, where $\tilde{\varphi}: \pi_{TM}^{-1}(U) \to \mathbb{R}^{2n}$ is defined such that

$$\tilde{\varphi}\left(\sum_{j=1}^{n} a^j \left.\frac{\partial}{\partial x^j}\right|_p\right) = (x^1(p), x^2(p), \ldots, x^n(p), a^1, a^2, \ldots, a^n)$$

for all $p \in U$ and $a^1, a^2, \ldots, a^n \in \mathbb{R}$ (see Proposition 6.7). Now $X: U \to TM$ is continuous if and only if $\tilde{\varphi} \circ X: U \to \mathbb{R}^{2m}$ is continuous. Similarly X is smooth if and only if $\tilde{\varphi} \circ X$ is smooth. Now

$$\tilde{\varphi}(X_p) = (x^1(p), x^2(p), \ldots, x^n(p), v^1(p), v^2(p), \ldots, v^n(p))$$

for all $p \in U$. The result follows. ∎

Let M be a smooth manifold, let $X: U \to TM$ be a vector field defined over an open set U in M, and let $f: V \to \mathbb{R}$ be a continuously differentiable real-valued function defined over an open set V in M. We denote by $X[f]$ the real-valued function on $U \cap V$ defined such that $X[f](p) = X_p[f]$ for all $p \in U \cap V$. If $x^1, x^2, \ldots, x^n$ is a smooth coordinate system defined over $U \cap V$, and if

$$X = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j},$$

throughout $U \cap V$, where the components $v^1, v^2, \ldots, v^n$ of X on are real-valued functions on $U \cap V$, then

$$X[f] = \sum_{j=1}^{n} v^j \frac{\partial f}{\partial x^j}.$$

Lemma 7.2 *Let M be a smooth manifold, and let X be a vector field defined over an open subset U of M. Then the vector field X is smooth if and only if $X[f]$ is a smooth function on $U \cap V$ for any smooth real-valued function $f: V \to \mathbb{R}$ whose domain is an open set V in M.*

Proof It follows directly from Proposition 7.1 that if the vector field X is smooth, then so is $X[f]$ for all smooth real-valued functions f defined over open sets in M.

Conversely suppose that X is a vector field on U with the property that $X[f]$ is smooth for all smooth real-valued functions f defined over open sets in M. Let $(x^1, x^2, \ldots, x^n)$ be a smooth local coordinate system for M defined over an open subset V of M. Then $X[x^j]$ is a smooth function on $U \cap V$ for $j = 1, 2, \ldots, n$. Now there are real-valued functions $v^1, v^2, \ldots, v^n$ defined over $U \cap V$ such that

$$X = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}$$

on $U \cap V$. Then $X[x^j] = v^j | U \cap V$ for $j = 1, 2, \ldots, n$. But $X[x^j]$ is smooth for $j = 1, 2, \ldots, n$. Therefore the components $v^1, v^2, dots, v^n$ of X are smooth functions on $U \cap V$. It then follows from Proposition 7.1 that $X : U \to TM$ is a smooth map, and thus X is a smooth vector field on U, as required. ∎

Lemma 7.3 *Let M be a smooth manifold, let U and V be open sets in M, and let $X : U \to TM$ be a vector field over U. Then*

$$X[f \cdot g] = X[f] \cdot g + f \cdot X[g]$$

on $U \cap V$ for all smooth real-valued functions f and g defined over V, where $(f \cdot g)(v) = f(v)g(v)$ for all $v \in V$.

Proof This property of vector fields follows directly from the corresponding property characterizing the action of tangent vectors on smooth functions. ∎

7.2 Lie Brackets of Vector Fields

Proposition 7.4 *Let M be a smooth manifold and let X and Y be smooth vector fields on M. Then there is a well-defined smooth vector field $[X, Y]$ on M characterized by the property that*

$$[X, Y][f] = X[Y[f]] - Y[X[f]]$$

for all smooth real-valued functions f defined over open sets in M.

Proof Let L denote the linear differential operator on M that sends any smooth real-valued function f defined over an open subset U of M to the function $L(f)$ on U, where

$$L(f) = X[Y[f]] - Y[X[f]].$$

Let U be an open set in M. Then

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$$

for all smooth real-valued functions f and g on U and for all real numbers α and β. Also

$$\begin{aligned} X[Y[f \cdot g]] &= X[(Y[f] \cdot g + f \cdot Y[g])] \\ &= X[Y[f]] \cdot g + Y[f] \cdot X[g] + X[f] \cdot Y[g] + f \cdot X[Y[g]], \end{aligned}$$

and therefore

$$\begin{aligned} L(f \cdot g) &= X[Y[f \cdot g]] - Y[X[f \cdot g]] \\ &= X[Y[f]] \cdot g - Y[X[f] \cdot g + f \cdot X[Y[g]] - f \cdot Y[X[g]] \\ &= L(f) \cdot g + f \cdot L(g). \end{aligned}$$

for all smooth real-valued functions f and g on U, where $f \cdot g$ denotes the product of the functions f and g. Moreover if f and g are smooth real-valued functions on U that satisfy $f(w) = g(w)$ for all points w of some open subset W of U, then $L(f)(p) = L(g)(p)$ for all $p \in W$. It follows from the definition of tangent vectors that there is a well-defined tangent vector $[X, Y]_p$ at each point p of M which is characterized by the property that

$$[X, Y]_p[f] = X_p[Y[f]] - Y_p[X[f]]$$

for all smooth real-valued functions defined around the point p. The function sending each point p of M to the tangent vector $[X, Y]_p$ is a vector field on M. Moreover this vector field $[X, Y]$ is a smooth vector field since $[X, Y][f]$ is a smooth function for all smooth real-valued functions f defined over open subsets of M (see Lemma 7.2). The result follows. ∎

Let X and Y be smooth vector fields defined over some open set U in a smooth manifold M. Now U is itself a smooth manifold. It therefore follows from Proposition 7.4 that there is a well-defined vector field $[X, Y]$ on U which is characterized by the property that $[X, Y][f] = X[Y[f]] - Y[X[f]]$ for all smooth real-valued functions f whose domain is an open subset of U.

Definition Let U be an open set in a smooth manifold M, and let X and Y be smooth vector fields on U. The *Lie Bracket* $[X, Y]$ of the vector fields X and Y is the smooth vector field on U characterized by the property that

$$[X, Y][f] = X[Y[f]] - Y[X[f]]$$

for all smooth real-valued functions f whose domain is an open subset of U.

Lemma 7.5 (Jacobi Identity) *Let X, Y and Z be smooth vector fields on a smooth manifold M. Then*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

Proof Let f be a smooth real-valued function defined over some open subset of M. Then

$$\begin{aligned}
&[[X,Y],Z][f] + [[Y,Z],X][f] + [[Z,X],Y][f] \\
&\quad = [X,Y][Z[f]] - Z[[X,Y][f]] + [Y,Z][X[f]] - X[[Y,Z][f]] \\
&\qquad + [Z,X][Y[f]] - Y[[Z,X][f]] \\
&\quad = X[Y[Z[f]]] - Y[X[Z[f]]] - Z[X[Y[f]] + Z[Y[X[f]] \\
&\qquad + Y[Z[X[f]]] - Z[Y[X[f]]] - X[Y[Z[f]] + X[Z[Y[f]] \\
&\qquad + Z[X[Y[f]]] - X[Z[Y[f]]] - Y[Z[X[f]] + Y[X[Z[f]] \\
&\quad = 0.
\end{aligned}$$

The result follows. ∎

Lemma 7.6 *Let M be a smooth manifold. Let X and Y be smooth vector fields on M and let f and g be smooth real-valued functions on M. Then*

$$[fX, gY] = (f \cdot g)[X,Y] + (f \cdot X[g])Y - (g \cdot Y[f])X.$$

Proof Let h be a smooth real-valued function whose domain is some open subset of M. Then

$$\begin{aligned}
[fX, gY][h] &= f \cdot X[g \cdot Y[h]] - g \cdot Y[f \cdot X[h]] \\
&= (f \cdot g) \cdot X[Y[h]] + f \cdot X[g] \cdot Y[h] \\
&\quad - (f \cdot g) \cdot Y[X[h]] - g \cdot Y[f] \cdot X[h] \\
&= ((f \cdot g)[X,Y] + (f \cdot X[g])Y - (g \cdot Y[f])X)\,[h].
\end{aligned}$$

The result follows. ∎

Lemma 7.7 *Let M be a smooth manifold of dimension n, let $x^1, x^2, \ldots, x^n$ be a smooth coordinate system defined over some open subset U of M, and let X and Y be smooth vector fields on U. Suppose that*

$$X = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{i=1}^{n} w^i \frac{\partial}{\partial x^i}.$$

where $v^1, v^2, \ldots, v^n$ and $w^1, w^2, \ldots, w^n$ are smooth real-valued functions on U. Then

$$[X,Y] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

Proof Let f be a smooth real-valued function on U. Then

$$\begin{aligned}
[X,Y][f] &= X\left[\sum_{i=1}^{n} w^i \frac{\partial f}{\partial x^i}\right] - Y\left[\sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(v^j \frac{\partial}{\partial x^j}\left(w^i \frac{\partial f}{\partial x^i}\right) - w^j \frac{\partial}{\partial x^j}\left(v^i \frac{\partial f}{\partial x^i}\right)\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}\right)\frac{\partial f}{\partial x^i},
\end{aligned}$$

since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(v^j w^i - w^j v^i)\frac{\partial^2 f}{\partial x^j \partial x^i} = \sum_{i=1}^{n}\sum_{j=1}^{n} v^j w^i \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j}\right) = 0.$$

The result follows. ∎

Corollary 7.8 *Let M be a smooth manifold, and let $x^1, x^2, \ldots, x^n$ be a smooth coordinate system defined over some open subset U of M. Then*

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

on U for $i, j = 1, 2, \ldots, n$.

Example Let X and Y be the smooth vector fields on $\mathbb{R}^2$ defined by the equations

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

and let f be a smooth real-valued function defined over an open set of $\mathbb{R}^2$. We can calculate $[X,Y][f]$ directly using the definition of the Lie bracket of X and Y:

$$\begin{aligned}
[X,Y][f] &= X[Y[f]] - Y[X[f]] \\
&= X\left[-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}\right] - Y\left[x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right] \\
&= x\frac{\partial}{\partial x}\left(-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}\right) + y\frac{\partial}{\partial y}\left(-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}\right) \\
&\quad + y\frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) - x\frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -xy\frac{\partial^2 f}{\partial x^2} + x^2\frac{\partial^2 f}{\partial x\,\partial y} + x\frac{\partial f}{\partial y} - y^2\frac{\partial^2 f}{\partial y\,\partial x} - y\frac{\partial f}{\partial x} + yx\frac{\partial^2 f}{\partial y^2} \\
&\qquad + yx\frac{\partial^2 f}{\partial x^2} + y\frac{\partial f}{\partial x} + y^2\frac{\partial^2 f}{\partial x\,\partial y} - x^2\frac{\partial^2 f}{\partial y\,\partial x} - xy\frac{\partial^2 f}{\partial y^2} - x\frac{\partial f}{\partial y} \\
&= 0
\end{aligned}
$$

It follows that $[X, Y] = 0$. We can also calculate $[X, Y]$ using the equation established in Lemma 7.7:

$$
\begin{aligned}
[X, Y] &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)[-y]\frac{\partial}{\partial x} + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)[x]\frac{\partial}{\partial y} \\
&\qquad - \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)[x]\frac{\partial}{\partial x} - \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)[y]\frac{\partial}{\partial y} \\
&= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \\
&= 0.
\end{aligned}
$$

We now calculate $\left[\frac{\partial}{\partial x}, Y\right]$. Using Lemma 7.6 and Corollary 7.8, we find that

$$
\begin{aligned}
\left[\frac{\partial}{\partial x}, Y\right] &= \left[\frac{\partial}{\partial x}, -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right] \\
&= -y\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right] + \frac{\partial}{\partial x}[-y]\,\frac{\partial}{\partial x} + x\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] + \frac{\partial}{\partial x}[x]\,\frac{\partial}{\partial y} \\
&= \frac{\partial}{\partial y}.
\end{aligned}
$$

Similarly $\left[\frac{\partial}{\partial y}, Y\right] = -\frac{\partial}{\partial x}$.

Now let (r, θ) be polar coordinates on $\mathbb{R}^2$, so that

$$
x = r\cos\theta, \quad y = r\sin\theta.
$$

Then

$$
\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y} = \frac{x}{r}\,\frac{\partial}{\partial x} + \frac{y}{r}\,\frac{\partial}{\partial y} = \frac{1}{r}X.
$$

Thus $X = r\frac{\partial}{\partial r}$. Also

$$
\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} = -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y} = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} = Y.
$$

It follows that

$$[X, Y] = \left[r\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right] = 0.$$

Also

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{1}{r} \sin\theta \frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{1}{r} \cos\theta \frac{\partial}{\partial \theta}, \end{aligned}$$

Therefore

$$\begin{aligned} \left[\frac{\partial}{\partial x}, Y \right] &= \left[\cos\theta \frac{\partial}{\partial r} - \frac{1}{r} \sin\theta \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right] \\ &= -\frac{\partial}{\partial \theta}(\cos\theta) \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} \left(\frac{1}{r} \sin\theta \right) \frac{\partial}{\partial \theta} \\ &= \sin\theta \frac{\partial}{\partial r} + \frac{1}{r} \cos\theta \frac{\partial}{\partial \theta} \\ &= \frac{\partial}{\partial y} \end{aligned}$$

Definition Let M and N be smooth manifolds and let $\varphi\colon M \to N$ be a smooth map from M to N. Let X and $\tilde{X}$ be smooth vector fields on M and N respectively. We say that X and $\tilde{X}$ are *φ-related* if $\tilde{X}[g] \circ \varphi = X[g \circ \varphi]$ for all smooth real-valued functions g on N.

Proposition 7.9 *Let M and N be smooth manifolds and let $\varphi\colon M \to N$ be a smooth map from M to N. Let X and Y be smooth vector fields on M, and let $\tilde{X}$ and $\tilde{Y}$ be smooth vector fields on N. Suppose that the vector fields X and $\tilde{X}$ are φ-related and that the vector fields Y and $\tilde{Y}$ are φ-related. Then the vector fields $[X, Y]$ and $[\tilde{X}, \tilde{Y}]$ are also φ-related.*

Proof Let g be a smooth real-valued function defined over some open subset V of N. Then the composition function $g \circ \varphi$ is a smooth real-valued function defined over an open set U in M, where $U = \varphi^{-1}(V)$, and

$$(\varphi_*[X, Y])_{\varphi(p)}[g] = [X, Y]_p[g \circ \varphi]$$

for all $p \in U$. But

$$\begin{aligned} [\tilde{X}, \tilde{Y}][g] \circ \varphi &= \tilde{X}[\tilde{Y}[g]] \circ \varphi - \tilde{Y}[\tilde{X}[g]] \circ \varphi = X[\tilde{Y}[g] \circ \varphi] - Y[\tilde{X}[g] \circ \varphi] \\ &= X[Y[g \circ \varphi]] - Y[X[g \circ \varphi]] = [X, Y][g \circ \varphi]. \end{aligned}$$

The result follows. ∎

7.3 Integral Curves for Vector Fields

Definition Let M be a smooth manifold, let X be a smooth vector field defined over an open subset U of M, and let $\gamma: I \to M$ be a smooth curve defined over some open interval I, and mapping that interval into the open set U. The smooth curve γ is said to be an *integral curve* for the vector field X if

$$\gamma'(t) = \frac{d}{dt}\gamma(t) = X_{\gamma(t)}$$

for all $t \in I$.

When represented in smooth local coordinates on any smooth manifold, the components of any integral curve of a smooth vector field are solutions to a system of ordinary differential equations determined by that vector field. Indeed let $x^1, x^2, x^3, \ldots, x^n$ be smooth local coordinates defined over some open set U in a smooth manifold M of dimension n, and let X be a smooth vector field on U. Then there exist smooth functions $u^1, u^2, \ldots, u^n$, defined over the open set V in $\mathbb{R}^n$, where

$$V = \{x^1(p), x^2(p), \ldots, x^n(p) : p \in U\}$$

such that

$$X_p = \sum_{j=1}^{n} u^i(x^1(p), x^2(p), \ldots, x^n(p)) \left. \frac{\partial}{\partial x^i} \right|_p$$

for all $p \in U$. Let $\gamma: I \to M$ be a smooth curve in U, and let $g^j(t) = x^j(\gamma(t))$ for $j = 1, 2, \ldots, n$. Then

$$\gamma'(t) = \frac{d\gamma(t)}{dt} = \sum_{j=1}^{n} \frac{dg^j(t)}{dt} \left. \frac{\partial}{\partial x^j} \right|_{\gamma(t)}$$

for all $t \in I$. It follows the $\gamma: I \to M$ is an integral curve for the vector field X if and only if

$$\frac{dg^j(t)}{dt} = u^j(g^1(t), g^2(t), \ldots, g^n(t))$$

for $j = 1, 2, \ldots, n$. It follows from this that standard existence and uniqueness theorems for solutions of systems of ordinary differential equations give rise to existence and uniqueness theorems for integral curves of smooth vector fields on any smooth manifold.

The following result is an immediate consequence of a standard existence theorem for solutions of systems of ordinary differential equations.

Theorem 7.10 *Let M be a smooth manifold, let X be a smooth vector field defined over an open subset U of M, and let p be a point of U. Then there exists a smooth curve $\gamma: I \to M$, defined over some open interval I, which passes through the point p and is an integral curve for the vector field X.*

The following result is an immediate consequence of a standard uniqueness theorem for solutions of systems of ordinary differential equations.

Theorem 7.11 *Let M be a smooth manifold, let X be a smooth vector field defined over an open subset U of M, and let $\gamma_1: I_1 \to M$ and $\gamma_2: I_2 \to M$ be integral curves for X, defined over open intervals I_1 and I_2, where $I_1 \cap I_2 \neq \emptyset$. Suppose that $\gamma_1(t_0) = \gamma_2(t_0)$ for some $t_0 \in I_1 \cap I_2$. Then $\gamma_1 | I_1 \cap I_2 = \gamma_2 | I_1 \cap I_2$.*

7.4 Local Flows

Let M be a smooth manifold, and let $\varphi: W \times I \to M$ be a continuous map, where W is an open set in M and I be an open interval in $\mathbb{R}$. Suppose that the function from I to M that sends $t \in I$ to $\varphi(w, t)$ is differentiable. Given $w \in W$ and $t \in I$, we define $\dfrac{\partial \varphi(w,t)}{\partial t}$ to be the velocity vector to the curve $s \mapsto \varphi(w, s)$ at $s = t$. This velocity vector is an element of the tangent space $T_{\varphi(w,t)}$ to the smooth manifold M at the point $\varphi(w, t)$.

Definition Let X be a vector field defined over an open subset U of a smooth manifold M and let $\varphi: W \times I \to M$ be a continuous map into M defined on the product manifold $W \times I$, where W is an open subset of U, I is an open interval in $\mathbb{R}$, and $0 \in I$. The function φ is said to be a *local flow* for the vector field X if the following conditions are satisfied:

(i) $\varphi(W \times I) \subset U$;

(ii) $\varphi(w, 0) = w$ for all $w \in W$;

(iii) for each $w \in W$, the map $t \mapsto \varphi(w, t)$ is differentiable on I and satisfies

$$\frac{\partial \varphi(w,t)}{\partial t} = X_{\varphi(w,t)}$$

for all $(w, t) \in W \times I$.

Let X be a vector field defined over an open subset U of a smooth manifold M and let $\varphi: W \times I \to M$ be a continuous map into M defined on the product manifold $W \times I$, where W is an open subset of U, I is an open interval in $\mathbb{R}$, and $0 \in I$. It follows from the definition of local flows that this map φ is a local flow for the vector field X if and only if, for every $w \in W$, the map $t \mapsto \varphi(w, t)$ is an integral curve for the vector field X, defined for $t \in I$, which passes through the point w at time $t = 0$.

Example Let k be a real number. The function on $\mathbb{R} \times \mathbb{R}$ that sends (x, t) to $x + kt$ for all $x, t \in \mathbb{R}$ is a (local) flow for the vector field $k\dfrac{\partial}{\partial x}$ on the real line $\mathbb{R}$. This follows from the fact that

$$\frac{\partial}{\partial t} f(x + kt) = kf'(x + kt)$$

for all smooth real-valued functions f defined over open subsets of $\mathbb{R}$.

Example Let k be a real number. The function on $\mathbb{R} \times \mathbb{R}$ that sends (x, t) to xe^{kt} for all $x, t \in \mathbb{R}$ is a (local) flow for the vector field $kx\dfrac{\partial}{\partial x}$ on the real line $\mathbb{R}$. This follows from the fact that

$$\frac{\partial}{\partial t} f(xe^{kt}) = kxe^{kt} f'(xe^{kt})$$

for all smooth real-valued functions f defined over open subsets of $\mathbb{R}$.

Example Let Q be the vector field on $\mathbb{R}^3$ defined by the equation

$$Q = -y\,\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

where x, y, z are the standard Cartesian coordinates on $\mathbb{R}^3$. Let

$$\begin{aligned} \varphi((x, y, z), t) &= (\hat{x}(t), \hat{y}(t), \hat{z}(t)) \\ &= (x\,\cos t - y \sin t, x \sin t + y \cos t, z) \end{aligned}$$

for all $x, y, z, t \in \mathbb{R}$. If f is a smooth real-valued function defined over some open set in $\mathbb{R}^3$ then

$$\begin{aligned} \frac{df(\hat{x}(t), \hat{y}(t), \hat{z}(t))}{dt} &= \frac{\partial f(\varphi((x_0, y_0, z_0), t))}{\partial t} \\ &= \frac{\partial}{\partial t} f(x_0\,\cos t - y_0 \sin t, x_0 \sin t + y_0 \cos t, z_0) \\ &= -(x_0 \sin t + y_0 \cos t)\frac{\partial f}{\partial x}(\varphi((x_0, y_0, z_0), t)) \\ &\qquad + (x_0 \cos t - y_0 \sin t)\frac{\partial f}{\partial y}(\varphi((x_0, y_0, z_0), t)) \\ &= -\hat{y}(t)\,\left.\frac{\partial f}{\partial x}\right|_{(\hat{x}(t), \hat{y}(t), \hat{z}(t))} + \hat{x}(t)\,\left.\frac{\partial f}{\partial y}\right|_{(\hat{x}(t), \hat{y}(t), \hat{z}(t))} \\ &= Q_{(\hat{x}(t), \hat{y}(t), \hat{z}(t))} f. \end{aligned}$$

It follows that, for each point (x_0, y_0, z_0) of $\mathbb{R}^3$, the smooth curve in $\mathbb{R}^3$ that sends $t \in \mathbb{R}$ to $(\hat{x}(t), \hat{y}(t), \hat{z}(t))$ is an integral curve for the vector field X, and thus the map $\varphi: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a (local) flow for the vector field Q on $\mathbb{R}^3$.

The vector field Q is tangential to the unit sphere S^2 at each point of S^2, where

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

since

$$Q[x^2+y^2+z^2] = -y\frac{\partial}{\partial x}(x^2+y^2+z^2) + x\frac{\partial}{\partial y}(x^2+y^2+z^2) = -2yx + 2xy = 0.$$

Moreover $\varphi(\mathbf{p}, t) \in S^2$ for all $\mathbf{p} \in S^2$ and $t \in \mathbb{R}$. It follows that the restriction of the smooth vector field Q to the unit sphere S^2 is a vector field on S^2, and moreover this vector field generates a smooth flow on S^2 which is obtained by restricting the domain of the smooth map φ to $S^2 \times \mathbb{R}$.

Example Let X be the vector field on the real line $\mathbb{R}$ given by

$$X = x^2 \frac{\partial}{\partial x}$$

for all $x \in \mathbb{R}$. Let $u_c: I_c \to \mathbb{R}$ be an integral curve for this vector field, defined on some open interval I_c in $\mathbb{R}$, where $0 \in I_c$ and $u_c(0) = c$. Then u_c satisfies the differential equation

$$\frac{du_c(t)}{dt} = u_c(t)^2 \text{ for all } t \in I_c.$$

It follows on solving this differential equation that $u_c(t) = \dfrac{c}{1 - ct}$ for all $t \in I_c$. It follows that $I_c \subset (-\infty, c^{-1})$ when $c > 0$, and $I_c \subset (c^{-1}, +\infty)$ when $c < 0$.

Let W be a bounded open set in the real line $\mathbb{R}$. Choose $\varepsilon > 0$ small enough to ensure that $W \subset (-\varepsilon^{-1}, \varepsilon^{-1})$. Then there is a local flow $\varphi: W \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ for the vector field X defined over $W \times (-\varepsilon, \varepsilon)$, where

$$\varphi(c, t) = \frac{c}{1 - ct}$$

for all $c \in W$ and $t \in (-\varepsilon, \varepsilon)$. However it is not possible to define a flow for this vector field whose domain is $W \times \mathbb{R}$.

The theory of systems of ordinary differential equations guarantees the existence of smooth local flows for smooth vector fields on smooth manifolds. This result is significantly harder to prove than the existence and uniqueness theorems for integral curves of smooth vector fields. We state the result below.

Theorem 7.12 *Let X be a smooth vector field defined over an open subset U of a smooth manifold M and let p be a point of U. Then there exists an open set W, where $p \in W$ and $W \subset U$, and a smooth local flow $\varphi: W \times (-\varepsilon, \varepsilon) \to M$ for the smooth vector field X defined over $W \times (-\varepsilon, \varepsilon)$ for some positive real number ε.*

Let X be a smooth vector field defined over an open subset U of a smooth manifold M, and let $\varphi: W \times (-\varepsilon, \varepsilon) \to M$ be a smooth local flow for X defined over the product manifold $W \times (-\varepsilon, \varepsilon)$, where W is an open set in M and ε is a positive real number. Then, for each point w of W, the curve $t \mapsto \varphi(w, t)$ is an integral curve for the vector field X. Moreover each $t \in (-\varepsilon, \varepsilon)$ determines a smooth map $\varphi_t: W \to M$. We shall show that, for an appropriate choice of W and ε, the open set W is mapped by φ_t diffeomorphically onto an open set in M.

Let M be a smooth manifold, let X be a smooth vector field defined over some open subset U of M, let p be a point of U, and let $\varphi: U_1 \times (-\varepsilon_1, \varepsilon_1) \to M$ be a smooth local flow for the vector field X, where $p \in U_1$, $U_1 \subset U$, $\varepsilon_1 > 0$ and $\varphi(U_1 \times (-\varepsilon_1, \varepsilon_1)) \subset U$. The continuity of this local flow then ensures that there exist an open set U_2 in M and a positive real number ε_2 such that $p \in U_2$, $U_2 \subset U_1$, $0 < \varepsilon_2 < \varepsilon_1$ and $\varphi(U_2 \times (-\varepsilon_2, \varepsilon_2)) \subset U_1$.

Proposition 7.13 *Let M be a smooth manifold, let X be a smooth vector field defined over some open subset U of M, let p be a point of U, and let $\varphi: U_1 \times (-\varepsilon_1, \varepsilon_1) \to M$ be a smooth local flow for the vector field X, where $p \in U_1$, $U_1 \subset U$, $\varepsilon_1 > 0$ and $\varphi(U_1 \times (-\varepsilon_1, \varepsilon_1)) \subset U$. Also let U_2 be an open set in M, and let ε_2 be a positive real number such that $p \in U_2$, $U_2 \subset U_1$, $0 < \varepsilon_2 < \varepsilon_1$ and $\varphi(U_2 \times (-\varepsilon_2, \varepsilon_2)) \subset U_1$. Then $\varphi_t(\varphi_s(u)) = \varphi_{s+t}(u)$ for all $u \in U_2$ and $s, t \in (-\varepsilon_2, \varepsilon_2)$.*

Proof Let $v = \varphi_s(u) = \varphi(u, s)$ for some $u \in U_2$ and $s \in (-\varepsilon_2, \varepsilon_2)$. Then

$$\frac{\partial}{\partial t}\varphi(u, s+t) = X_{\varphi(u,s+t)}$$

for all $t \in (-\varepsilon_2, \varepsilon_2)$. But

$$\frac{\partial}{\partial t}\varphi(v, t) = X_{\varphi(v,t)}$$

for all $t \in (-\varepsilon_2, \varepsilon_2)$, and $\varphi(v, 0) = v = \varphi(u, s)$. It follows from the standard uniqueness theorem for solutions of differential equations determined by smooth vector fields that

$$\varphi_{s+t}(u) = \varphi(u, s+t) = \varphi(v, t) = \varphi(\varphi(u, s), t) = \varphi_t(\varphi(u, s)) = \varphi_t(\varphi_s(u))$$

for all $t \in (-\varepsilon_2, \varepsilon_2)$ (see Theorem 7.11). The result follows. ∎

Corollary 7.14 *Let M be a smooth manifold, let X be a smooth vector field defined over some open subset U of M, and let p be a point of U. Then there exists an open set W in M and a positive real number ε, where $p \in W$ and $W \subset U$ and a smooth local flow $\varphi: W \times (-\varepsilon, \varepsilon) \to M$ such that the map φ_t maps W diffeomorphically onto an open subset of U for all $t \in (-\varepsilon, \varepsilon)$, where $\varphi_t(w) = \varphi(w, t)$ for all $w \in W$ and $t \in (-\varepsilon, \varepsilon)$.*

Proof It follows from Proposition 7.13 that there exists an open set W containing the point p and a positive real number ε such that $\varphi(w, s+t) \in U$ and $\varphi_{s+t}(w) = \varphi_s(\varphi_t(w))$ for all $w \in W$ and $s, t \in (-\varepsilon, \varepsilon)$. Let $s = -t$. Then

$$\varphi_{-t}(\varphi_t(w)) = \varphi_{-t+t}(w) = \varphi_0(w) = w$$

for all $w \in W$ and $t \in (-\varepsilon, \varepsilon)$. It follows that, for all $t \in (-\varepsilon, \varepsilon)$, the smooth map φ_t maps W diffeormorphically onto an open set $\varphi_t(W)$ in M, and the inverse of this diffeomorphism is given by the restriction to $\varphi_t(W)$ of the map φ_{-t}. ∎

7.5 Global Flows

Definition Let X be a smooth vector field on a smooth manifold M. A smooth function $\varphi: M \times \mathbb{R} \to M$ is said to be a *(global) flow* for the vector field X if $\varphi(p, 0) = p$ for all $p \in M$ and

$$\frac{\partial}{\partial t}\varphi(p, t) = X_{\varphi(p,t)}.$$

Let M be a smooth manifold. If M is noncompact then smooth vector fields on M do not necessarily have global flows.

Corollary 7.15 *Let X be a smooth vector field on a smooth manifold M. Suppose there exists a global flow $\varphi: M \to \mathbb{R} \to M$ for the vector field X. Then $\varphi(\varphi(p, s), t) = \varphi(p, s + t)$ for all $p \in M$ and $s, t \in \mathbb{R}$. Thus $\varphi_t \circ \varphi_s = \varphi_{s+t}$ for all $s, t \in \mathbb{R}$, where $\varphi_t: M \to M$ is the smooth map that satisfies $\varphi_t(p) = \varphi(p, t)$ for all $p \in M$ and $t \in \mathbb{R}$.*

Proof This result follows directly from Proposition 7.13 since the conditions in the statement of that proposition are satisfied on taking $U = U_1 = U_2 = M$ (where U, U_1 and U_2 are the open sets in M referred to in the statement of that proposition) and on taking arbitrary large values of ε_1 and ε_2 (where $0 < \varepsilon_2 < \varepsilon_1$). ∎

Remark If a smooth manifold is noncompact then a smooth vector field on the manifold is not guaranteed to have a global flow. Indeed there is is no global flow for the vector field $x^2 \dfrac{\partial}{\partial x}$ on the real line $\mathbb{R}$, since an integral curves for this equation are of the form $t \mapsto \dfrac{c}{1 - ct}$, where c is some real constant, and an integral curve of this form is not defined over the whole real line unless $c = 0$.

Let X be a smooth vector field on a smooth manifold M. Suppose that there exists a global flow $\varphi: M \times \mathbb{R} \to M$ for X. Let $\varphi_t: M \to M$ be defined for all $t \in \mathbb{R}$ such that $\varphi_t(p) = \varphi(p, t)$ for all $p \in M$ and $t\mathbb{R}$. Then each smooth map $\varphi_t: M \to M$ is a diffeomorphism from M to itself with inverse φ_{-t}. The collection $(\varphi_t : t \in \mathbb{R})$ is referred to as the *one-parameter group of diffeomorphisms* of M *generated* by the vector field X.

A subset K of a topological space is said to be *compact* if every open cover of K has a finite subcover.

Theorem 7.16 *Let X be a smooth vector field on a smooth manifold M. Suppose that there exists a compact subset K of M such that $X_p = 0$ whenever $p \notin K$. Then there exists a smooth global flow $\varphi: M \times \mathbb{R} \to M$ for the vector field X. The smooth vector field X therefore generates a one-parameter group $(\varphi_t : t \in \mathbb{R})$ of diffeomorphisms of the smooth manifold M.*

Proof The existence theorem for smooth local flows (Theorem 7.12) and the compactness of the set K together ensure that there exists a finite collection

$$\varphi_i: W_i \times (-\varepsilon_i, \varepsilon_i) \to M \quad (i = 1, 2, \ldots, r)$$

of local flows for the vector field X, where W_i is an open set in M and ε_i is a positive real number for $i = 1, 2, \ldots, r$, and where

$$K \subset W_1 \cup W_2 \cup \cdots \cup W_r.$$

Let ε be the minimum of $\varepsilon_1, \varepsilon_2, \cdots \varepsilon_r$. Now $\varphi_i(p, 0) = p$ and

$$\frac{\partial \varphi_i(p, t)}{\partial t} = X_{\varphi_i(p,t)}$$

for all $p \in W_i$ and $t \in (-\varepsilon_i, \varepsilon_i)$. The uniqueness theorem for integral curves of smooth vector fields (Theorem 7.11) then ensures the existence of a smooth map

$$\psi: M \times (-\varepsilon, \varepsilon) \to M$$

such that

$$\psi(p,t) = \varphi_i(p,t) \text{ whenever } p \in W_i \text{ and } t \in (-\varepsilon, \varepsilon)$$

and

$$\psi(p,t) = p \text{ whenever } p \notin K \text{ and } t \in (-\varepsilon, \varepsilon).$$

Moreover

$$\frac{\partial \psi(p,t)}{\partial t} = X_{\psi(p,t)}$$

for all $p \in M$ and $t \in (-\varepsilon, \varepsilon)$. Choose some positive real number e satisfying $0 < e < \frac{1}{2}\varepsilon$. Proposition 7.13 then ensures that $\psi(\psi(p,s),t) = \psi(p,s+t)$ for all $p \in M$ and $s,t \in [-e,e]$. This is sufficient to ensure the existence of a well-defined smooth function $\varphi\colon M \times \mathbb{R} \to M$, where

$$\begin{aligned} \varphi(p,t) &= \psi(p,t) \text{ whenever } -e \leq t \leq e; \\ \varphi(p,t) &= \psi(\varphi(p,t-e),e) \text{ whenever } t > e; \\ \varphi(p,t) &= \psi(\varphi(p,t+e),-e) \text{ whenever } t < -e. \end{aligned}$$

This smooth function φ is then a global flow for the smooth vector field X. Thus if $\varphi_t(p) = \varphi(p,t)$ for all $p \in M$ and $t \in \mathbb{R}$ then $(\varphi_t : t \in \mathbb{R})$ is a one-parameter group of diffeomorphisms of the smooth manifold M generated by the vector field X. ∎

Corollary 7.17 *Let X be a smooth vector field on a compact smooth manifold M. Then there exists a smooth global flow $\varphi\colon M \times \mathbb{R} \to M$ for the vector field X. The smooth vector field X therefore generates a one-parameter group $(\varphi_t : t \in \mathbb{R})$ of diffeomorphisms of the smooth manifold M.*

Proof This result is a special case of Theorem 7.16. ∎

7.6 Lie Brackets and Commutativity of Flows

Proposition 7.18 *Let X and Y be smooth vector fields on a smooth manifold M, and let $\varphi\colon W_0 \times I \to M$ be a smooth local flow for the vector field X, where W_0 is an open set in M, I is an open interval in the real line, and $0 \in I$. Let W be an open subset of W_0 and let ε be a positive real number, where W and ε are chosen such that $(-\varepsilon, \varepsilon) \subset I$ and*

$$\varphi(W \times (-\varepsilon, \varepsilon)) \subset W_0.$$

Then

$$\frac{\partial}{\partial t}\left(\varphi_{-t*} Y_{\varphi(p,t)}\right) = \varphi_{-t*}[X,Y]_{\varphi(p,t)}$$

for all $p \in W$ *and* $t \in (-\varepsilon, \varepsilon)$*, where* $\varphi_t(p) = \varphi(p,t)$ *for all* $p \in W$*, and where*

$$\frac{\partial}{\partial t}\left(\varphi_{-t*} Y_{\varphi(p,t)}\right) = \lim_{h\to 0} \frac{1}{h}\left(\varphi_{-(t+h)*} Y_{\varphi(p,t+h)} - \varphi_{-t*} Y_{\varphi(p,t)}\right).$$

Proof Let $\tilde{W} = W \times (-\varepsilon, \varepsilon)$, and let $\pi: \tilde{W} \to W$ denote the projection map defined such that $\pi(p,t) = p$ for all $p \in W$ and $t \in (-\varepsilon, \varepsilon)$. Also for each $t \in (-\varepsilon, \varepsilon)$, let $\iota_t: W \to \tilde{W}$ be the smooth map defined such that $\iota_t(p) = (p,t)$ for all $p \in W$. Then $\pi(\iota_t(p)) = p$ and $\iota_t(\pi(p,t)) = (p,t)$ for all $(p,t) \in \tilde{W}$.

Let $Z_{(p,t)}$ be a tangent vector to $\tilde{W}$ at (p,t). Then there exists a real number c and a tangent vector $\hat{Z}_{(p,t)} \in T_{(p,t)}\tilde{W}$ such that $\hat{Z}_{(p,t)}$ is tangential to the submanifold $\iota_t(W)$ of $\tilde{W}$ and

$$Z_{(p,t)} = \hat{Z}_{(p,t)} + c \left.\frac{\partial}{\partial t}\right|_{(p,t)}.$$

Now $\iota_t \circ \pi$ is the identity map on $\iota_t(W)$. It follows that

$$\hat{Z}_{(p,t)} = \iota_{t*}\pi_* \hat{Z}_{(p,t)} = \iota_{t*}\pi_* Z_{(p,t)}.$$

Also $\hat{Z}_{(p,t)}[t] = 0$ and therefore $Z_{(p,t)}[t] = c$. Therefore

$$Z_{(p,t)} = \iota_{t*}\pi_* Z_{(p,t)} + Z_{(p,t)}[t] \left.\frac{\partial}{\partial t}\right|_{(p,t)}$$

for all $Z_{(p,t)} \in T_{(p,t)}\tilde{W}$.

Now

$$\varphi_*\left(\left.\frac{\partial}{\partial t}\right|_{(p,t)}\right) = \frac{\partial \varphi(p,t)}{\partial t} = X_{\varphi(p,t)}$$

for all $(p,t) \in \tilde{W}$. It follows that the vector field $\dfrac{\partial}{\partial t}$ on $\tilde{W}$ is φ-related to the vector field X on M.

There is a smooth vector field $\tilde{Y}$ on $\tilde{W}$ characterized by the property that

$$\tilde{Y}_{(p,t)} = \iota_{t*}\varphi_{-t*} Y_{\varphi(p,t)}.$$

for all $(p,t) \in \tilde{W}$. Then

$$\pi_* \tilde{Y}_{(p,t)} = \varphi_{-t*} Y_{\varphi(p,t)}.$$

for all $(p,t) \in \tilde{W}$. Also $\varphi_t(p) = \varphi(\iota_t(p))$ for all $p \in W$, and therefore

$$\varphi_* \tilde{Y}_{(p,t)} = \varphi_* \iota_{t*}\varphi_{-t*} Y_{\varphi(p,t)} = (\varphi \circ \iota_t)_* \varphi_{-t*} Y_{\varphi(p,t)} = \varphi_{t*}\varphi_{-t*} Y_{\varphi(p,t)} = Y_{\varphi(p,t)}$$

for all $(p,t) \in \tilde{W}$. The vector field $\tilde{Y}$ on $\tilde{W}$ is therefore φ-related to the vector field Y on M.

Now the vector fields $\dfrac{\partial}{\partial t}$ and $\tilde{Y}$ on $\tilde{W}$ are φ-related to the vector fields X and Y respectively on M. It follows from Proposition 7.9 that the vector field $\left[\dfrac{\partial}{\partial t}, \tilde{Y}\right]$ on $\tilde{W}$ is φ-related to the vector field $[X,Y]$ on M, and thus

$$\varphi_* \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)} = [X,Y]_{\varphi(p,t)}$$

for all $(p,t) \in \tilde{W}$. But

$$\left[\frac{\partial}{\partial t}, \tilde{Y}\right][t] = \frac{\partial}{\partial t}\left(\tilde{Y}[t]\right) - Y\left[\frac{\partial t}{\partial t}\right] = 0,$$

since $\tilde{Y}[t] = 0$ throughout $\tilde{W}$. It follows that

$$\left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)} = \iota_{t*}\pi_* \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)}.$$

Therefore

$$[X,Y]_{\varphi(p,t)} = \varphi_* \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)} = \varphi_*\iota_{t*}\pi_* \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)} = \varphi_{t*}\pi_* \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)},$$

and thus

$$\varphi_{-t*}[X,Y]_{\varphi(p,t)} = \pi_* \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)}.$$

But if f is any smooth function defined over some open neighbourhood of p in M then

$$\begin{aligned}
\left\langle df_p, \pi_* \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)} \right\rangle &= \pi_* \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)} [f] = \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)} [f \circ \pi] \\
&= \frac{\partial}{\partial t}\left(\tilde{Y}_{(p,t)}[f\circ\pi]\right) - \tilde{Y}_{(p,t)}\left[\frac{\partial (f\circ\pi)}{\partial t}\right] \\
&= \frac{\partial}{\partial t}\left(\pi_*\tilde{Y}_{(p,t)})[f]\right) = \frac{\partial}{\partial t}\left(\langle df_p, \pi_*\tilde{Y}_{(p,t)}\rangle\right) \\
&= \left\langle df_p, \frac{\partial}{\partial t}(\pi_*\tilde{Y}_{(p,t)})\right\rangle,
\end{aligned}$$

because

$$\frac{\partial (f\circ\pi)}{\partial t} = 0.$$

It follows that

$$\varphi_{-t*}[X,Y]_{\varphi(p,t)} = \pi_* \left[\frac{\partial}{\partial t}, \tilde{Y}\right]_{(p,t)} = \frac{\partial}{\partial t}\left(\pi_* \tilde{Y}_{(p,t)}\right) = \frac{\partial}{\partial t}\left(\varphi_{-t*} Y_{\varphi(p,t)}\right),$$

as required. ∎

Remark Let the smooth manifold M, the vector fields X and Y, the smooth local flow $\varphi: W_0 \times I \to M$ for X, the open set W, the positive real number ε and the maps $\varphi_t: W \to M$ satisfy the conditions set out in the statement of Proposition 7.18. Suppose also that the open set W is contained in the domain of a smooth local coordinate system $(x^1, x^2, \ldots, x^n)$ for M. Then there are smooth real-valued functions $v^1, v^2, \ldots, v^n$ on $W \times (-\varepsilon \times \varepsilon)$ such that

$$\varphi_{-t*} Y_{\varphi(p,t)} = \sum_{j=1}^{n} v^j(p,t) \left.\frac{\partial}{\partial x^j}\right|_p.$$

$\tilde{W} = W \times (-\varepsilon \times \varepsilon)$, let $\pi: \tilde{W} \to W$ be the projection function that satisfies $\pi(p,t) = p$ for all $(p,t) \in \tilde{W}$, and let $\tilde{x}^j = x^j \circ \pi$ for $j = 1, 2, \ldots, n$. Then $(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n, t)$ is a smooth coordinate system defined throughout $\tilde{W}$. The vector field $\tilde{Y}$ on $\tilde{W}$ employed in the proof of Proposition 7.18 is defined so that

$$\tilde{Y}_{(p,t)} = \sum_{j=1}^{n} v^j(p,t) \left.\frac{\partial}{\partial \hat{x}^j}\right|_{(p,t)}.$$

Now

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial \hat{x}^j}\right] = 0$$

for $j = 1, 2, \ldots, n$ (see Corollary 7.8). It therefore follows from Lemma 7.6 (or from Lemma 7.7) that

$$\left[\frac{\partial}{\partial t}, \tilde{Y}\right] = \sum_{j=1}^{n} \frac{\partial v^j}{\partial t} \frac{\partial}{\partial \hat{x}^j}.$$

The proof of Proposition 7.18 also exploits the fact that the vector fields $\dfrac{\partial}{\partial t}$ and $\tilde{Y}$ on W are φ-related to the vector fields X and Y on M, and therefore (as shown in Proposition 7.9) $\left[\dfrac{\partial}{\partial t}, \tilde{Y}\right]$ is φ-related to the Lie bracket $[X, Y]$. It follows that

$$\varphi_{-t*}[X,Y]_{\varphi(p,t)} = \sum_{j=1}^{n} \frac{\partial(v^j(p,t))}{\partial t} \left.\frac{\partial}{\partial x^j}\right|_p$$

$$= \frac{\partial}{\partial t}\left(\sum_{j=1}^{n} v^j(p,t) \left.\frac{\partial}{\partial x^j}\right|_p\right)$$
$$= \frac{\partial}{\partial t}\left(\varphi_{-t*}Y_{\varphi(p,t)}\right),$$

thus establishing the identity in the statement of Proposition 7.9.

Corollary 7.19 *Let X and Y be smooth vector fields on a smooth manifold M, and let $\varphi: W \times I \to M$ be a smooth local flow for the vector field X, where W is an open set in M, I is an open interval in the real line, and $0 \in I$. Then*

$$\left.\frac{\partial}{\partial t}\left(\varphi_{-t*}Y_{\varphi(p,t)}\right)\right|_{t=0} = [X, Y]_p$$

for all $p \in W$, where $\varphi_t(p) = \varphi(p,t)$ for all $p \in W$, and where

$$\left.\frac{\partial}{\partial t}\left(\varphi_{-t*}Y_{\varphi(p,t)}\right)\right|_{t=0} = \lim_{h\to 0}\frac{1}{h}\left(\varphi_{-h*}Y_{\varphi(p,h)} - Y_p\right).$$

Proof Given any point p of W, we can find some open neighbourhood W_0 of p and some positive real number ε such that $(-\varepsilon, \varepsilon) \subset I$ and

$$\varphi(W_0 \times (-\varepsilon, \varepsilon)) \subset W.$$

The result therefore follows directly from Proposition 7.18. ∎

Remark Let the smooth manifold M, the vector fields X and Y, the smooth local flow $\varphi: W_0 \times I \to M$ for X, the open set W, the positive real number ε and the maps $\varphi_t: W \to M$ satisfy the conditions set out in the statement of Proposition 7.18. Then

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\varphi_{-t*}Y_{\varphi(p,t)}\right) &= \lim_{h\to 0}\frac{1}{h}\left(\varphi_{-(t+h)*}Y_{\varphi(p,t+h)} - \varphi_{-t*}Y_{\varphi(p,t)}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\varphi_{-t*}\varphi_{-h*}Y_{\varphi(p,t+h)} - \varphi_{-t*}Y_{\varphi(p,t)}\right) \\
&= \varphi_{-t*}\left(\lim_{h\to 0}\frac{1}{h}\left(\varphi_{-h*}Y_{\varphi(\varphi(p,t),h)} - Y_{\varphi(p,t)}\right)\right).
\end{aligned}$$

It follows that if the identity in the statement of Corollary 7.19 has been established (e.g., by some alternative proof to that given above), then the result of Proposition 7.18 may be deduced from it.

Corollary 7.20 *Let X and Y be smooth vector fields on a smooth manifold M, and let $\varphi: W_0 \times I \to M$ be a smooth local flow for the vector field X, where W_0 is an open set in M, I is an open interval in the real line, and $0 \in I$. Let W be an open subset of W_0 and let ε be a positive real number, where W and ε are chosen such that $(-\varepsilon, \varepsilon) \subset I$ and*

$$\varphi(W \times (-\varepsilon, \varepsilon)) \subset W_0.$$

Suppose that $[X, Y] = 0$ on M. Then

$$Y_{\varphi(p,t)} = \varphi_{t*} Y_p$$

for all $p \in W$ and $t \in (-\varepsilon, \varepsilon)$, where $\varphi_t(p) = \varphi(p, t)$ for all $p \in W$.

Proof This result follows immediately from Proposition 7.18. ∎

Corollary 7.21 *Let X and Y be smooth vector fields on a smooth manifold M, let p_0 be a point of M, and let $\varphi_X: W \times I \to M$ and $\varphi_Y: \hat{W} \times \hat{I} \to M$ be smooth local flow for the vector fields X and Y respectively, where W and $\hat{W}$ are open sets in M, I and $\hat{I}$ are open intervals in the real line, $p_0 \in W \cap \hat{W}$, $0 \in I$ and $0 \in \hat{I}$, and let $\varphi_{X,s}(p) = \varphi_X(p, s)$ for all $(p, s) \in W \times I$ and $\varphi_{Y,t}(p) = \varphi_Y(p, t)$ for all $(p, t) \in \hat{W} \times \hat{I}$. Suppose that $[X, Y] = 0$. Then there exists some open set W_0 and some positive real number ε such that $p \in W_0$, $W_0 \subset W \cap \hat{W}$, $\varphi_{Y,t} \circ \varphi_{X,s}$ and $\varphi_{X,s} \circ \varphi_{Y,t}$ are defined throughout W_0 for all $s, t \in (-\varepsilon, \varepsilon)$, and*

$$\varphi_{X,s}(\varphi_{Y,t}(p)) = \varphi_{Y,t}(\varphi_{X,s}(p))$$

for all $p \in W_0$ and $s, t \in (-\varepsilon, \varepsilon)$.

Proof It follows from Corollary 7.20 that there exists an open set W_1 and a positive real number ε_1 such that $Y_{\varphi_X(p,s)} = \varphi_{X,s*} Y_p$ for all $p \in W_1$ and $s \in (-\varepsilon_1, \varepsilon_1)$. Let $\gamma: I_\gamma \to M$ be an integral curve for the vector field Y, where I_γ is an open interval in $\mathbb{R}$ and $\gamma(I_\gamma) \subset W_1$. Then

$$\frac{\partial}{\partial t}(\varphi_{X,s}(\gamma(t))) = \varphi_{X,s*} \frac{d\gamma(t)}{dt} = \varphi_{X,s*} Y_{\gamma(t)} = Y_{\varphi_{X,s}(\gamma(t))}.$$

It follows that $\hat{\gamma}: I_\gamma \to M$ is also an integral curve for the vector field Y, where $\hat{\gamma}(t) = \varphi_{X,s}(\gamma(t))$ for all $t \in I_\gamma$. $0 \in I_\gamma$ and $\gamma(0) = p$ then

$$\begin{aligned} \varphi_{X,s}(\varphi_{Y,t}(p)) &= \varphi_{X,s}(\varphi_{Y,t}(\gamma(0))) = \varphi_{X,s}(\gamma(t)) = \hat{\gamma}(t) \\ &= \varphi_{Y,t}(\hat{\gamma}(0)) = \varphi_{Y,t}(\varphi_{X,s}(\gamma(0))) \\ &= \varphi_{Y,t}(\varphi_{X,s}(p)), \end{aligned}$$

as required. ∎

Theorem 7.22 *Let $X_1, X_2, \ldots, X_r$ be smooth vector fields on a smooth manifold M. Suppose that*

$$[X_i, X_j] = 0 \quad \text{for } i, j = 1, 2, \ldots, r.$$

Then, given any point p_0 of M, there exists a smooth map

$$F: W \times (-\varepsilon, \varepsilon)^r \to M,$$

defined over $W \times (-\varepsilon, \varepsilon)^r$, where W is an open neighbourhood of p_0 in M and $\varepsilon > 0$, such that

$$\frac{\partial}{\partial t^i} F(p, t^1, t^2, \ldots, t^r) = (X_i)_{F(p,t^1,t^2,\ldots,t^r)}$$

for all $p \in W$, $t^1, t^2, \ldots, t^r \in (-\varepsilon, \varepsilon)$ and $i \in \{1, 2, \ldots, r\}$.

Proof There exists an open set W_1, where $p_0 \in W_1$, a positive real number ε_1, and smooth maps $\varphi_{X_i}: W_1 \times (-\varepsilon_1, \varepsilon_1) \to M$ such that $\varphi_{X,i}$ is a smooth local flow for the vector field X_i for $i = 1, 2, \ldots, r$. Then

$$\frac{\partial}{\partial t} \varphi_{X_i}(p, t) = (X_i)_{\varphi_{X_i}(p,t)}$$

for all $p \in W_1$ and $t \in (-\varepsilon_1, \varepsilon_1)$. Let $\varphi_{X_i,t}(p) = \varphi_{X_i}(p, t)$ for all $p \in W_1$ and $t \in (-\varepsilon_1, \varepsilon_1)$.

Let us define

$$F_1(p, t^1) = \varphi_{X_1}(p, t^1)$$

for all $p \in W_1$ and $t^1 \in (-\varepsilon_1, \varepsilon_1)$. Then

$$\frac{\partial}{\partial t^1} F_1(p, t^1) = (X_1)_{F_1(p,t^1)}$$

for all $w \in W_1$ and $t^1 \in (-\varepsilon_1, \varepsilon_1)$.

Suppose that, for some integer k satisfying $1 < k \leq r$ there exists an open set W_{k-1}, where $p_0 \in W_{k-1}$ and $W_{k-1} \subset W_1$, a positive real number ε_{k-1}, where $0 < \varepsilon_{k-1} \leq \varepsilon_1$ and a smooth map

$$F_{k-1}: W_{k-1} \times (-\varepsilon_{k-1}, \varepsilon_{k-1})^{k-1} \to M$$

with the property that

$$\frac{\partial}{\partial t^i} F_{k-1}(p, t^1, t^2, \ldots, t^{k-1}) = (X_i)_{F_{k-1}(p,t^1,t^2,\ldots,t^{k-1})}$$

for all $w \in W_{k-1}$, $t^1, t^2, \ldots, t^{k-1} \in (-\varepsilon_{k-1}, \varepsilon_{k-1})$ and $i \in \{1, 2, \ldots, k-1\}$. Choose an open set W_k and a positive number ε_k such that $p_0 \in W_k$, $W_k \subset W_{k-1}$ $0 < \varepsilon_k < \varepsilon_{k-1} \leq \varepsilon_1$ and

$$F_{k-1}(W_k \times (-\varepsilon_{k-1}, \varepsilon_k)^{k-1}) \subset W_1.$$

Define

$$F_k(p, t^1, t^2, \ldots, t^k) = \varphi_{X_k, t^k}(F_{k-1}(p, t^1, t^2, \ldots, t^{k-1}))$$

for all $p \in W_k$ and $t^1, t^2, \ldots, t^k \in (-\varepsilon_k, \varepsilon_k)$. Then $t \mapsto F_k(p, t^1, t^2, \ldots, t^{k-1}, t)$ is an integral curve for the vector field X_k, and therefore

$$\frac{\partial}{\partial t^k} F_k(p, t^1, t^2, \ldots, t^k) = (X_k)_{F_k(p, t^1, t^2, \ldots, t^k)}.$$

If $i < k$ then it follows from Corollary 7.20 that

$$(X_i)_{\varphi_{X_k}(q, t^k)} = \varphi_{X_k, t^k *}(X_i)_q$$

for all $q \in W_1$ and $t \in (-\varepsilon_k, \varepsilon_k)$. Therefore

$$\begin{aligned}
\frac{\partial}{\partial t^i} F_k(p, t^1, t^2, \ldots, t^k) &= \frac{\partial}{\partial t^i}\left(\varphi_{X_k, t^k}(F_{k-1}(p, t^1, t^2, \ldots, t^{k-1}))\right) \\
&= \varphi_{X_k, t^k *}\left(\frac{\partial}{\partial t^i}\left(F_{k-1}(p, t^1, t^2, \ldots, t^{k-1})\right)\right) \\
&= \varphi_{X_k, t^k *}\left((X_i)_{F_{k-1}(p, t^1, t^2, \ldots, t^{k-1})}\right) \\
&= (X_i)_{\varphi_{X_k, t^k}(F_{k-1}(p, t^1, t^2, \ldots, t^{k-1}))} \\
&= (X_i)_{F_k(p, t^1, t^2, \ldots, t^k)}.
\end{aligned}$$

The result therefore follows by induction on k. ∎

Remark The function F constructed in the proof of Theorem 7.22 can be represented in the form

$$F(p, t^1, t^2, \ldots, t^r) = (\varphi_{X_r, t^r} \circ \varphi_{X_{r-1}, t^{r-1}} \circ \cdots \circ \varphi_{X_2, t^2} \circ \varphi_{X_1, t^1})(p)$$

(employing the notation in the statement and proof of that theorem), where the point p lies within some sufficiently small open neighbourhood W of the point p_0, and where $t^1, t^2, \ldots, t^r$ have absolute values small enough to ensure that the map represented by the above formula is well-defined. The result of Corollary 7.21 may be used in place of Corollary 7.20 to verify that this function F satisfies the required differential equations. Indeed Corollary 7.21 ensures that, if the absolute values of $t^1, t^2, \ldots, t^r$ are sufficiently small, then the order in which the maps φ_{X_i,t^i} are composed is immaterial, and therefore, given any value of i between 1 and r, we can write

$$F(p, t^1, t^2, \ldots, t^r) = \varphi_{X_i,t^i}(G(p, t^1, \ldots, t^{i-1}, t^{i+1}, \ldots, t^r))$$

for some smooth function G that does not involve the real variable t^i. It follows directly from this that

$$\frac{\partial}{\partial t^i} F(p, t^1, t^2, \ldots, t^r) = (X_i)_{F(p,t^1,t^2,\ldots,t^r)},$$

as required.

Theorem 7.23 *Let $X_1, X_2, \ldots, X_n$ be smooth vector fields on a smooth manifold M of dimension n. Suppose that*

$$[X_i, X_j] = 0 \quad \textit{for } i, j = 1, 2, \ldots, n.$$

Suppose also that the values of these vector fields at some point p_0 of M constitute a basis of the tangent space $T_{p_0}M$ to M at p_0. Then there exists a smooth coordinate system $(x^1, x^2, \ldots, x^n)$, defined throughout some open neighbourhood W of the point p_0, such that

$$X_i = \frac{\partial}{\partial x^i}$$

on W for $i = 1, 2, \ldots, n$.

Proof It follows from Theorem 7.22 that there exists a smooth map

$$F \colon W_0 \times (-\varepsilon, \varepsilon)^n \to M,$$

defined over $W_0 \times (-\varepsilon, \varepsilon)^n$, where W_0 is an open neighbourhood of p_0 in M and $\varepsilon > 0$, such that

$$\frac{\partial}{\partial t^i} F(p, t^1, t^2, \ldots, t^n) = (X_i)_{F(p,t^1,t^2,\ldots,t^n)}$$

for all $p \in W_0$, $t^1, t^2, \ldots, t^n \in (-\varepsilon, \varepsilon)$ and $i \in \{1, 2, \ldots, n\}$. Define

$$\psi \colon (-\varepsilon, \varepsilon)^n \to M$$

such that

$$\psi(t^1, t^2, \ldots, t^n) = F(p_0, t^1, t^2, \ldots, t^n)$$

for all $t^1, t^2, \ldots, t^n \in (-\varepsilon, \varepsilon)$. Then

$$\psi_* \left(\left. \frac{\partial}{\partial t^i} \right|_{(t^1,t^2,\ldots,t^n)} \right) = \frac{\partial}{\partial t^i} \left(\psi(t^1, t^2, \ldots, t^n) \right) = (X_i)_{\psi(t^1,t^2,\ldots,t^n)}$$

for $i = 1, 2, \ldots, n$. Now the values $(X_1)_{p_0}, (X_2)_{p_0}, \ldots, (X_0)_{p_0}$ of the vector fields $X_1, X_2, \ldots, X_n$ at the point p_0 constitute a basis of the tangent space $T_{p_0}M$. Therefore the derivative $\psi_*: T_{(0,0,\ldots,0)}\mathbb{R}^n \to T_{p_0}M$ of the map ψ at $(0, 0, \ldots, 0)$ is an isomorphism. It follows from the Inverse Function Theorem of multivariable real analysis that there exists an open neighbourhood of the origin in $(-\varepsilon, \varepsilon)^n$ that is mapped diffeomorphically onto an open set W in M. The inverse $\varphi: W \to \mathbb{R}^n$ of this diffeomorphism is then a smooth chart on M, and

$$\varphi_*(X_i) = \frac{\partial}{\partial t^i}$$

throughout W for $i = 1, 2, \ldots, n$. Let $\varphi(p) = (x^1(p), x^2(p), \ldots, x^n(p))$ for all $p \in W$. Then $(x^1, x^2, \ldots, x^n)$ is a smooth coordinate system defined over W, and it follows from the definition of $\dfrac{\partial}{\partial x^i}$ that

$$\varphi_* \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial t^i}$$

for $i = 1, 2, \ldots, n$. Therefore $X_i = \dfrac{\partial}{\partial x^i}$ throughout W for $i = 1, 2, \ldots, n$, as required. ∎

Theorem 7.24 *Let $X_1, X_2, \ldots, X_r$ be smooth vector fields on a smooth manifold M of dimension n. Suppose that*

$$[X_i, X_j] = 0 \quad \text{for } i, j = 1, 2, \ldots, r.$$

Suppose also that the values of these vector fields at some point p_0 of M are linearly independent elements of the tangent space $T_{p_0}M$. Then there exists a smooth coordinate system $(x^1, x^2, \ldots, x^n)$, defined throughout some open neighbourhood W of the point p_0, such that

$$X_i = \frac{\partial}{\partial x^i}$$

on W for $i = 1, 2, \ldots, r$.

Proof It follows from Theorem 7.22 that there exists a smooth map

$$F: W_0 \times (-\varepsilon, \varepsilon)^r \to M,$$

defined over $W_0 \times (-\varepsilon, \varepsilon)^r$, where W_0 is an open neighbourhood of p_0 in M and $\varepsilon > 0$, such that

$$\frac{\partial}{\partial t^i} F(p, t^1, t^2, \ldots, t^r) = (X_i)_{F(p,t^1,t^2,\ldots,t^n)}$$

for all $p \in W_0$, $t^1, t^2, \ldots, t^r \in (-\varepsilon, \varepsilon)$ and $i \in \{1, 2, \ldots, r\}$.

Now there exists a diffeomorphism $G: U \to M$, where U is an open neighbourhood of the origin $\mathbf{0}$ in $\mathbb{R}^n$, such that $G(\mathbf{0}) = p_0$ and

$$G_* \left(\left. \frac{\partial}{\partial t^i} \right|_{\mathbf{0}} \right) = (X_i)_{p_0} \text{ for } i = 1, 2, \ldots, r.$$

(This diffeomorphism may be constructed as the inverse of a smooth chart around p_0, composed with an appropriate non-singular linear transformation of $\mathbb{R}^\ltimes$.) Then the vectors

$$G_* \left(\left. \frac{\partial}{\partial t^i} \right|_{\mathbf{0}} \right) \qquad (i = 1, 2, \ldots, n)$$

constitute a basis of the tangent space $T_{p_0}M$ to M at p_0. Define

$$H(t^1, t^2, \ldots, t^n) = F(G(0, 0, \ldots, 0, t^{r+1}, \cdots, t^n), t^1, t^2, \ldots, t^r)$$

for all points $(t^1, t^2, \ldots, t^n)$ that lie within a sufficiently small neighbourhood U_0 of the origin in $\mathbb{R}^n$.

Given $\mathbf{u} \in U_0$, where $\mathbf{u} = (u_1, u^2, \ldots, u^n)$ and given an integer i satisfying $1 \leq i \leq n$, let $\lambda_{\mathbf{u},i}: \mathbb{R} \to \mathbb{R}^n$ be defined such that the ith component of $\lambda_{\mathbf{u},i}(t)$ is $u^i + t$ and the jth component is u^j for $j \neq i$. Thus $t \mapsto \lambda_{\mathbf{u},i}(t)$ is a path which follows a straight line parallel to the ith coordinate axis, and passes through the point $\mathbf{u}$ at time $t = 0$.

Now if $1 \leq i \leq r$ and if $\mathbf{u} \in U_0$ then

$$H(\lambda_{\mathbf{u},i}(t)) = F(G(0, 0, u^{r+1}, \ldots, u^n), u^1, \ldots, u^i + t, \ldots, u^r)$$

and therefore the curve $t \mapsto H(\lambda_{\mathbf{u},i}(t))$ is an integral curve for the vector field X_i. It follows that

$$H_* \left(\left. \frac{\partial}{\partial t^i} \right|_{\mathbf{u}} \right) = \left. \frac{\partial H}{\partial t^i} \right|_{\mathbf{u}} = \left. \frac{dH(\lambda_{\mathbf{u},t})}{dt} \right|_{t=0} = (X_i)_{H(\mathbf{u})}$$

for $i = 1, 2, \ldots, r$. In particular

$$H_*\left(\left.\frac{\partial}{\partial t^i}\right|_{\mathbf{0}}\right) = (X_i)_{p_0} = G_*\left(\left.\frac{\partial}{\partial t^i}\right|_{\mathbf{0}}\right)$$

for $i = 1, 2, \ldots, r$. Moreover if $i > r$ then $H(\lambda_{\mathbf{0},i}(t)) = G(\lambda_{\mathbf{0},i}(t))$.

$$H_*\left(\left.\frac{\partial}{\partial t^i}\right|_{\mathbf{0}}\right) = G_*\left(\left.\frac{\partial}{\partial t^i}\right|_{\mathbf{0}}\right)$$

and thus the derivatives H_* and G_* of the maps H and G coincide at $\mathbf{0}$, and thus the derivative $H_*: T_{\mathbf{0}}\mathbb{R}^n \to T_{p_0}M$ of the map H at $\mathbf{0}$ is an isomorphism of vector spaces. It follows from the Inverse Function Theorem that H maps some open neighbourhood U of $\mathbf{0}$ in $\mathbb{R}^n$ diffeomorphically onto some open neighbourhood W of p_0 in M. Let $x^1, x^2, \ldots, x^n$ be the Cartesian components of the inverse of the diffeomorphism from U to V determined by H, so that

$$\mathbf{u} = (x^1(H(\mathbf{u})), x^1(H(\mathbf{u})), \ldots, x^n(H(\mathbf{u})))$$

for all $\mathbf{u} \in U$. Then

$$H_*\left(\left.\frac{\partial}{\partial t^i}\right|_{\mathbf{u}}\right) = \left.\frac{\partial}{\partial x^i}\right|_{H(\mathbf{u}}$$

for $i = 1, 2, \ldots, n$. But

$$H_*\left(\left.\frac{\partial}{\partial t^i}\right|_{\mathbf{u}}\right) = \left.\frac{\partial H}{\partial t^i}\right|_{\mathbf{u}} = \left.\frac{dH(\lambda_{\mathbf{u},t})}{dt}\right|_{t=0} = (X_i)_{H(\mathbf{u})}$$

for $i = 1, 2, \ldots, r$. It follows that $X_i = \dfrac{\partial}{\partial x^i}$ throughout the open set W for $i = 1, 2, \ldots, r$. ∎

Example Let X and Y be the smooth vector fields on $\mathbb{R}^3$ defined by the equations

$$\begin{aligned} X &= xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - (x^2 + y^2)\frac{\partial}{\partial z} \\ Y &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \end{aligned}$$

A short computation shows that $[X, Y] = 0$. Moreover these vector fields are linearly independent throughout the complement of the z-axis. It transpires that $X = \dfrac{\partial}{\partial u}$ and $Y = \dfrac{\partial}{\partial \varphi}$, where (u, φ, r) is the smooth coordinate system on the complement of $\{(x, y, z) : y = 0 \text{ and } x \leq 0\}$ defined such that

$$x = \frac{2re^{ru}\cos\varphi}{1 + e^{2ru}}, \quad y = \frac{2re^{ru}\sin\varphi}{1 + e^{2ru}}, \quad z = \frac{r(1 - e^{2ru})}{1 + e^{2ru}}.$$

Further calculation shows that, in the (u, φ, r) coordinate system,

$$
\begin{aligned}
\frac{\partial}{\partial r} &= \left(\frac{x}{r} + \frac{xz}{2r^2}\log\frac{r-z}{r+z}\right)\frac{\partial}{\partial x} + \left(\frac{y}{r} + \frac{yz}{2r^2}\log\frac{r-z}{r+z}\right)\frac{\partial}{\partial y} \\
&= + \left(\frac{z}{r} - \frac{x^2+y^2}{2r^2}\log\frac{r-z}{r+z}\right)\frac{\partial}{\partial y} \\
&= \frac{1}{r}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) + \left(\frac{1}{2r^2}\log\frac{r-z}{r+z}\right) X
\end{aligned}
$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Note that this vector is not directed radially outwards away from the origin. This is a consequence of the fact that curves along which the functions φ and u are constant do not lie wholly within straight lines passing through the origin. Indeed the cosine of the angle between the z-axis and the line joining the origin to the point with coordinates r, φ, u in this curvilinear coordinate coordinate system is $\dfrac{1 - e^{2ru}}{1 + e^{2ru}}$, and this angle clearly varies along any curve along which the functions u and φ are both constant.

The special case of Theorem 7.24 involving just one smooth vector field on the smooth manifold is an important result in its own right, which we now state.

Corollary 7.25 *Let X be a smooth vector field on a smooth manifold M of dimension r. Suppose that that the vector field X is non-zero at some point p_0 of M. Then there exists a smooth coordinate system $(x^1, x^2, \ldots, x^n)$, defined throughout some open neighbourhood W of the point p_0, such that $X = \dfrac{\partial}{\partial x^1}$ on W.*

Remark Note that, in the special case addressed in Corollary 7.25, where there is only one vector field involved, the function F employed in the proof of Theorem 7.22 is a smooth local flow for this vector field X. Thus the proof of Theorem 7.22 in this special case requires only the existence theorem for smooth local flows (Theorem 7.12) and the Inverse Function Theorem. Therefore the proof of the existence of the required smooth coordinate system in the situation described in the statement of Corollary 7.25 does not require the use of the results stated in Proposition 7.18 and its corollaries.

Example Let B be the vector field on $\{(x, y, z) \in \mathbb{R}^3 : x \neq 0\}$ defined by the equation

$$B = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \frac{-xz - x^2 y + c^2 y}{x^2}\frac{\partial}{\partial z},$$

where c is a real constant. Let $\gamma: I \to \mathbb{R}^3$ be an smooth integral curve for the vector field B, where I is some open interval in $\mathbb{R}$, and let $\gamma(t) = (u(t), v(t), w(t))$ for all $t \in I$, where u, v and w are smooth real-valued functions on I. Then $\gamma'(t) = B_{\gamma(t)}$ for all $t \in I$, where

$$\gamma'(t) = \frac{d\gamma(t)}{dt} = u'(t)\frac{\partial}{\partial x} + v'(t)\frac{\partial}{\partial y} + w'(t)\frac{\partial}{\partial z},$$

and therefore

$$u'(t) = 1, \quad v'(t) = w(t), \quad w'(t) = \frac{-u(t)w(t) - u(t)^2 v(t) + c^2 v(t)}{u(t)^2}.$$

Then $u(t) = t + t_0$ for some constant t_0. We may reparameterize the integral curve I so that $t_0 = 0$, and $u(t) = t$. Then

$$v''(t) = w'(t) = -\frac{1}{t}w(t) - \frac{t^2 - c^2}{t^2}\, v(t),$$

and thus

$$t^2\frac{d^2v(t)}{dt^2} + t\frac{dv(t)}{dt} + (t^2 - c^2)v(t) = 0.$$

The function $v(t)$ thus satisfies *Bessel's differential equation.* The solutions of this equation are Bessel functions.

Examples such as we have seen should indicate that, whilst it may be a fairly trivial exercise to compute Lie brackets of smooth vector fields and draw conclusions concerning the existence and behaviour of smooth coordinate systems and flows, it may not be as easy to compute the flows and find explicit formulae defining these coordinate systems.