# Module MA3429: Differential Geometry Michaelmas Term 2010 Part I: Sections 1 to 4

David R. Wilkins



## Contents

# 1 Topological Spaces and Smooth Manifolds

## 1.1 Euclidean Spaces

We denote by $\mathbb{R}^n$ the set consisting of all $n$-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents $n$-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let $\lambda$ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\
\lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Now

$$\left( \sum_{j=1}^{n} x_j y_j \right)^2 \leq \left( \sum_{j=1}^{n} x_j^2 \right) \left( \sum_{j=1}^{n} x_j^2 \right)$$

for all real numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. It follows that $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. This basic inequality is known as Schwarz's Inequality. It follows easily from Schwarz' Inequality that $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Indeed

$$\begin{aligned}
|\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}).(\mathbf{x} + \mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x} \cdot \mathbf{y} \\
&\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2.
\end{aligned}$$

It follows that

$$|\mathbf{z} - \mathbf{x}| \leq |\mathbf{z} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of $\mathbb{R}^n$. This important inequality is known as the *Triangle Inequality.* It expresses the geometric fact that the length of any side of a triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

## 1.2 Continuity

**Definition** Let $X$ and $Y$ be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from $X$ to $Y$ is said to be *continuous* at a point $\mathbf{p}$ of $X$ if and only if the following criterion is satisfied:—

> given any real number $\varepsilon$ satisfying $\varepsilon > 0$ there exists some real number $\delta$ satisfying $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of $X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f: X \to Y$ is said to be continuous on $X$ if and only if it is continuous at every point $\mathbf{p}$ of $X$.

It is a straightforward exercise to verify from this formal definition of continuity that any composition of continuous functions is continuous (in the particular case under consideration here where the domains and codomains of the functions in question are subsets of Euclidean spaces). Moreover sums, differences, products and quotients of continuous real-valued function defined over subsets of of Euclidean spaces are themselves continuous.

Let $X$ and $Y$ be a subset of Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a function from $X$ to $Y$. Then the function $f$ is determined by its components $f_1, f_2, \ldots, f_n$, where each component $f_j: X \to \mathbb{R}$ is a real-valued function on $X$, and where

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$. It is a straightforward exercise to verify from the definition of continuity that the function $f: X \to Y$ is continuous if and only if all its components $f_1, f_2, \ldots, f_n$ are continuous.

## 1.3 Limits of Functions

The concept of the limit of a function is closely related to continuity. Let $f: X \to \mathbb{R}^n$ be a function defined on some subset $X$ of $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of $\mathbb{R}^m$. We seek to define the concept of the *limit* of $f(\mathbf{x})$ as the point $\mathbf{x}$ tends to $\mathbf{p}$ within the set $X$. Now, in order to get a sensible and useful definition of the limit, we must impose a restriction on the location of the point $\mathbf{p}$ in relation to the domain $X$ of the function. Specifically, we require the point $\mathbf{p}$ to be a *limit point* of $X$: a point $\mathbf{p}$ of $\mathbb{R}^m$ is a *limit point* of $X$ if and only if, given any positive real number $\delta$, there exists at least one point of the set $X$ which is not equal to $\mathbf{p}$ but which lies within a distance $\delta$ of the point $\mathbf{p}$. It is not difficult to prove that a point $\mathbf{p}$ of $\mathbb{R}^m$ is a limit point of a subset $X$ of $\mathbb{R}^m$ if and only if there exists an infinite sequence of distinct points of $X$ which converges to the point $\mathbf{p}$.

**Example** Let $B$ be the open unit ball in $\mathbb{R}^3$ consisting of all points that lie within the sphere of unit radius centred on the origin, so that

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}.$$

Then all points of the ball $B$ are limit points of $B$. Also all points on the unit sphere

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

are limit points of $B$, though these points do not belong to $B$ itself.

**Definition** Let $X$ be a subset of a Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ mapping $X$ into a Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of $X$, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. We say that $\mathbf{q}$ is the *limit* of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in $X$ if, given any real number $\varepsilon$ satisfying $\varepsilon > 0$, there exists some real number $\delta$ satisfying $\delta > 0$ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ for all points $\mathbf{x}$ of $X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. If the point $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in $X$, then we denote this fact by writing: $\mathbf{q} = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$.

Let $X$ and $Y$ be subsets of Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ of dimensions $m$ and $n$ respectively, let $f\colon X \to Y$ be a function from $X$ to $Y$, let $\mathbf{p}$ be a limit point of the domain $X$ of this function, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. Then the function $f\colon X \to Y$ determines a function $\tilde{f}\colon X \cup \{\mathbf{p}\} \to Y \cup \{\mathbf{q}\}$, where

$$\tilde{f}(\mathbf{x}) = \begin{cases} \mathbf{q} & \text{if } \mathbf{x} = \mathbf{p}; \\ f(\mathbf{x}) & \text{if } \mathbf{x} \in X \setminus \{\mathbf{p}\}. \end{cases}$$

On comparing the definition of limits of functions with the definition of continuity, one can verify that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$ if and only if the associated function $\tilde{f}$ is continuous at the point $\mathbf{p}$. In consequence of this, many standard results concerning limits of functions can be deduced as consequences of corresponding results that concern continuity.

## 1.4 Open Sets in Euclidean Spaces

Given a point $\mathbf{p}$ of $\mathbb{R}^n$ and a non-negative real number $r$, the *open ball* $B(\mathbf{p}, r)$ of *radius* $r$ about $\mathbf{p}$ is defined to be the subset of $\mathbb{R}^n$ given by

$$B(\mathbf{p}, r) \equiv \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B(\mathbf{p}, r)$ is the set consisting of all points of $\mathbb{R}^n$ that lie within a sphere of radius $r$ centred on the point $\mathbf{p}$.)

**Definition** A subset $V$ of $\mathbb{R}^n$ is said to be *open* in $\mathbb{R}^n$ if and only if, given any point $\mathbf{p}$ of $V$, there exists some $\delta > 0$ such that $B(\mathbf{p}, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of $\mathbb{R}^n$. (The criterion given above is satisfied vacuously in the case when $V$ is the empty set.)

**Example** Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where $c$ is some real number. Then $H$ is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of $H$. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point $(x, y, z)$ to the point $(u, v, w)$ is less than $\delta$ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore $H$ is an open set.

**Example** Let $\mathbf{p}$ be a point of $n$-dimensional Euclidean space $\mathbb{R}^n$. Then, for any positive real number $r$, the open ball $B(\mathbf{p}, r)$ of radius $r$ about $\mathbf{p}$ is an open set in $\mathbb{R}^n$. Indeed let $\mathbf{x}$ be an element of $B(\mathbf{p}, r)$. If we set $\delta = r - |\mathbf{x} - \mathbf{p}|$ then $\delta > 0$, and $B(\mathbf{x}, \delta) \subset B(\mathbf{p}, r)$. Indeed if $\mathbf{y} \in B(\mathbf{x}, \delta)$ then the Triangle Inequality ensures that

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

and therefore $\mathbf{y} \in B(\mathbf{p}, r)$. This shows that the open ball $B(\mathbf{p}, r)$ is indeed an open set.

**Example** Let $\mathbf{p}$ be a point of $n$-dimensional Euclidean space $\mathbb{R}^n$. Then, for any non-negative real number $r$, the set $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in $\mathbb{R}^n$. Indeed let $\mathbf{x}$ be a point of $\mathbb{R}^n$ satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then $\delta > 0$ and the Triangle Inequality can be used to show that $B(\mathbf{x}, \delta) \subset B(\mathbf{p}, r)$.

**Proposition 1.1** *The collection of open sets in $n$-dimensional Euclidean space $\mathbb{R}^n$ has the following properties:—*

(i) *the empty set $\emptyset$ and the whole space $\mathbb{R}^n$ are both open sets;*

(ii) *the union of any collection of open sets is itself an open set;*

(iii) *the intersection of any* finite *collection of open sets is itself an open set.*

**Proof** The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole space $\mathbb{R}^n$. Thus (i) is satisfied.

Let $\mathcal{A}$ be any collection of open sets in $\mathbb{R}^n$, and let $U$ denote the union of all the open sets belonging to $\mathcal{A}$. We must show that $U$ is itself an open set. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some open set $V$ belonging to the collection $\mathcal{A}$.

Therefore there exists some $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B(\mathbf{x}, \delta) \subset U$. This shows that $U$ is open. Thus (ii) is satisfied.

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of open sets in $\mathbb{R}^n$, and let $V = V_1 \cap V_2 \cap \cdots \cap V_k$. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for all $j$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let $\delta$ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of open sets.) Moreover $B(\mathbf{x}, \delta) \subset B(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B(\mathbf{x}, \delta) \subset V$. This shows that the intersection $V$ of the open sets $V_1, V_2, \ldots, V_k$ is itself open. Thus (iii) is satisfied. ∎

**Example** The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

**Example** The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

**Example** The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n, 0, 0)$ for all integers $n$.

**Example** For each natural number $k$, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set $V_k$ is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets $V_k$ for all natural numbers $k$ is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets $V_k$ for all natural numbers $k$ is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

## 1.5 Continuous Functions and Open Sets

Let $f\colon X \to Y$ be a function from a set $X$ to a set $Y$. Given any subset $V$ of $Y$, we denote by $f^{-1}(V)$ the *preimage* of $V$ under the map $f$. This preimage is defined such that

$$f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}.$$

**Proposition 1.2** *Let $X$ be an open subset of $\mathbb{R}^m$, and let $f: X \to \mathbb{R}^n$ be a function from $X$ to $\mathbb{R}^n$. The function $f$ is continuous if and only if $f^{-1}(V)$ is an open subset of $\mathbb{R}^m$ for every open subset $V$ of $\mathbb{R}^n$.*

**Proof** Suppose that $f: X \to \mathbb{R}^n$ is continuous. Let $V$ be an open set in $\mathbb{R}^n$. We must show that the subset $f^{-1}(V)$ of $X$ is an open set. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But $V$ is open, hence there exists some $\varepsilon > 0$ with the property that the open ball $B(f(\mathbf{p}), \varepsilon)$ of radius $\varepsilon$ centred on $f(\mathbf{p})$ is contained in $V$. But $X$ is an open set in $X$, and the function $f: X \to \mathbb{R}^n$ is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that the open ball $B(\mathbf{p}, \delta)$ of radius $\delta$ is contained in $X$ and $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of $X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B(\mathbf{p}, \delta)$ and therefore $B(\mathbf{p}, \delta) \subset f^{-1}(V)$. This proves that $f^{-1}(V)$ is an open subset of $\mathbb{R}^m$ for every open subset $V$ of $\mathbb{R}^n$.

Conversely suppose that $f: X \to \mathbb{R}^n$ is a function with the property that $f^{-1}(V)$ is an open set for every open subset $V$ of $\mathbb{R}^n$. Let $\mathbf{p} \in X$. We must show that $f$ is continuous at $\mathbf{p}$. Let some positive real number $\varepsilon$ be given. Then the open $B(f(\mathbf{p}), \varepsilon)$ of radius $\varepsilon$ about $\mathbf{p}$ is an open set in $\mathbb{R}^n$ and therefore the subset $f^{-1}(B(f(\mathbf{p}), \varepsilon))$ of $X$ is an open set in $\mathbb{R}^n$ which contains the point $\mathbf{p}$. It then follows that there exists some positive real number $\delta$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}(B(f(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $f$ maps $B_X(\mathbf{p}, \delta)$ into $B(f(\mathbf{p}), \varepsilon)$. But then $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of $X$ that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. We conclude that the function $f: X \to \mathbb{R}^n$ is continuous at $\mathbf{p}$, as required.

Proposition 1.2 is applicable to functions whose domain is an open set in a Euclidean space $\mathbb{R}^m$. The result can be generalized so as to apply to functions whose domain is a subset of $\mathbb{R}^m$, irrespective of whether or not that subset is open in $\mathbb{R}^m$. It can be shown that a function $f: X \to \mathbb{R}^n$, defined over some subset $X$ of $\mathbb{R}^m$, and mapping that subset into $\mathbb{R}^n$, is continuous if and only if, given any open set $V$ in $\mathbb{R}^n$, there exists some open set $W$ of $\mathbb{R}^m$ such that $f^{-1}(V) = X \cap W$.

The relationship between open sets and continuous functions described in Proposition 1.2 motivates the introduction of the concept of a *topological space*. A topological space is a set which is provided with a special collection of subsets. This collection of subsets is required to satisfy certain conditions that satisfied by the collection of open sets in a Euclidean space. A function $f: X \to Y$ between topological spaces $X$ and $Y$ is then said to be *continuous* if and only if the preimage of every open set in $Y$ is an open set in $X$.

## 1.6 Topological Spaces

**Definition** A *topological space* $X$ consists of a set $X$ together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

(i) the empty set $\emptyset$ and the whole set $X$ are open sets,

(ii) the union of any collection of open sets is itself an open set,

(iii) the intersection of any *finite* collection of open sets is itself an open set.

The collection consisting of all the open sets in a topological space $X$ is referred to as a *topology* on the set $X$.

**Definition** A function $f: X \to Y$ from a topological space $X$ to a topological space $Y$ is said to be *continuous* if $f^{-1}(V)$ is an open set in $X$ for every open set $V$ in $Y$, where

$$f^{-1}(V) \equiv \{x \in X : f(x) \in V\}.$$

A continuous function from $X$ to $Y$ is often referred to as a *map* from $X$ to $Y$.

Let $X$, $Y$ and $Z$ be topological spaces, and let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. It follows directly from the definition of continuity given above that the composition $g \circ f: X \to Z$ of the functions $f$ and $g$ is continuous.

Let $X$ be a topological space, and let $A$ be a subset of $X$. We say that a subset $U$ of $A$ is *open in* $A$, if and only if there exists some open set $W$ in $X$ such that $U = A \cap W$. The collection consisting of all subsets of $A$ that are open in $A$ satisfies all the requirements stated in the definition of a topological space, and thus constitutes a topology on $A$. This topology is referred to as the *subspace topology* on $A$. In this fashion, every subset of a topological space may be regarded as a topological space in its own right.

Let $A$ be a subset of a topological space $X$. A subset of $A$ that is open in $A$ need not be open in the larger topological space $X$. However, in the special case where the subset $A$ of $X$ is itself open in $X$, a subset of $A$ is open in $A$ if and only if it is open in $X$. This is an immediate consequence of the fact that the intersection of any two open sets in a topological space is guaranteed to be itself an open set.

In particular, let $X$ be a subset of a Euclidean space $\mathbb{R}^m$ of dimension $m$. Then $X$ carries a natural topology, which is the subspace topology defined in the manner described above: a subset $U$ of $X$ is *open in* $X$ if and only if there exists some open set $W$ in $\mathbb{R}^m$ such that $U = X \cap W$. This subspace topology is referred to as the *usual topology* on $X$. In the special case where $X$ is an open set in the ambient Euclidean space $\mathbb{R}^m$ the subsets of $X$ that are open in $X$ are those that are open in $\mathbb{R}^m$.

## 1.7 Closed Sets

**Definition** Let $X$ be a topological space. A subset $F$ of $X$ is said to be a *closed set* if and only if its complement $X \setminus F$ is an open set.

Now the complement of the union of some collection of subsets of some set $X$ is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of $X$ is the union of the complements of those sets. The following result therefore follows directly from the definition of a topological space.

**Proposition 1.3** *Let $X$ be a topological space. Then the collection of closed sets of $X$ has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set $X$ are closed sets,*

(ii) *the intersection of any collection of closed sets is itself a closed set,*

(iii) *the union of any finite collection of closed sets is itself a closed set.*

A function $f: X \to Y$ between topological spaces $X$ to $Y$ is continuous if and only if the preimage of every open set in $Y$ is an open set in $X$. Now $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ for all subsets $B$ of $Y$ (i.e., the preimage of the complement of a subset of $Y$ is the complement of the preimage of that subset). Also subsets of a topological spaces are open if and only if their complements are closed. On combining these observations, it follows directly that a function $f: X \to Y$ between topological spaces $X$ and $Y$ is continuous if and only if the preimage of every closed set in $Y$ is a closed set in $X$.

## 1.8 Hausdorff Spaces

**Definition** A topological space $X$ is said to be a *Hausdorff space* if and only if it satisfies the following *Hausdorff Axiom*:

- if $x$ and $y$ are distinct points of $X$ then there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

A subset of a Euclidean space (with the usual topology) is guaranteed to be a Hausdorff space. Indeed there is a more general class of topological spaces known as *metric spaces* in which the collection of open sets is determined by a distance function satisfying appropriate axioms. The definition of an open set in a metric space generalizes that of an open set in a Euclidean space. All metric spaces are Hausdorff spaces.

Many basic properties of metric spaces are shared by Hausdorff spaces, but may not hold in more general topological spaces. One such property is the uniqueness property of limits of convergent infinite sequences. An infinite sequence $x_1, x_2, x_3, \ldots$ of points of a topological space $X$ is said to *converge* to some point $p$ of that space if and only if, given any open set $V$ containing the point $p$, there exists some natural number $N$ such that $x_j \in V$ whenever $j \geq N$. The limit of a convergent sequence in a Hausdorff topological space is guaranteed to be unique. Indeed let $p$ and $q$ be distinct points of a Hausdorff topological space $X$. Then there exist open sets $U$ and $V$ such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$. A member $x_j$ of an infinite sequence $x_1, x_2, x_3, \ldots$ cannot belong to both $U$ and $V$, since these open sets are disjoint. Therefore the infinite sequence $x_1, x_2, x_3, \ldots$ cannot simultaneously converge to both $p$ and $q$. Thus the uniqueness property of limits of convergent sequences is guaranteed to hold in any Hausdorff space. But this property does not hold in all topological spaces.

## 1.9 Homeomorphisms

Let $f: X \to Y$ be a function from a set $X$ to a set $Y$. A function $f^{-1}: Y \to X$ from $Y$ to $X$ is said to be the *inverse* of the function $f$ if and only if the composition function $f^{-1} \circ f$ is the identity function of the set $X$ and the composition function $f \circ f^{-1}$ is the identity function of the set $Y$.

A function $f: X \to Y$ has a well-defined inverse if and only if it is both *injective* and *surjective*. A function $f: X \to Y$ is said to be *injective* if it maps distinct elements of the set $X$ to distinct elements of $Y$. Thus, for the function $f: X \to Y$ to be injective, we require that $f(u) \neq f(v)$ for all elements $u$ and $v$ of $X$ with $u \neq v$. A function $f: X \to Y$ is said to be *surjective* if $f(X) = Y$. Thus $f: X \to Y$ is surjective if and only if, given any element $y$ of $Y$, there exists some element $x$ of $X$ such that $f(x) = y$.

**Definition** Let $X$ and $Y$ be topological spaces. A function $h: X \to Y$ is said to be a *homeomorphism* if and only if the following conditions are satisfied:

- the function $h: X \to Y$ is invertible;
- the function $h: X \to Y$ is continuous;

- the inverse function $h^{-1}: Y \to X$ is also continuous.

Two topological spaces $X$ and $Y$ are said to be *homeomorphic* if there exists a homeomorphism $h: X \to Y$ from $X$ to $Y$.

If $h: X \to Y$ is a homeomorphism between topological spaces $X$ and $Y$ then $h$ induces a one-to-one correspondence between the open sets of $X$ and the open sets of $Y$. Thus the topological spaces $X$ and $Y$ can be regarded as being essentially identical as topological spaces.

When two topological spaces are homeomorphic, topological properties of one space are replicated in the other. Let $X$ and $Y$ be topological spaces that are homeomorphic, let $h: X \to Y$ be a homeomorphism between $X$ and $Y$, and let $Z$ be some topological space. Then, each continuous map $f: X \to Z$ from $X$ to $Z$ determines a corresponding continuous map $g: Y \to Z$ from $Y$ to $Z$, where $f = g \circ h$ and $g = f \circ h^{-1}$. A similar correspondence exists between continuous maps into the topological spaces $X$ and $Y$. And any convergent sequence in the topological space $X$ corresponds under the homeomorphism $h: X \to Y$ to a convergent sequence in the topological space $Y$, and vice versa.

## 1.10 Countability

Sets may be categorized as *finite sets* and *infinite sets*. A *finite set* is only that has only finitely many elements.

The mathematician Dedekind (1831–1916) came up with a characterization of finiteness expressed in terms of functions from a set to itself.

We recall that a function $f: X \to Y$ is said to be *injective* if it maps distinct elements of $X$ to distinct elements of $Y$ (so that if $u, v \in X$ satisfy $u \neq v$ then $f(u) \neq f(v)$). Now an injective function $f: X \to Y$ determines a one-to-one correspondence between the elements of the domain $X$ of the function and the elements of the range $f(X)$ of the function. It follows that if the domain $X$ of the injective function $f: X \to Y$ is a finite set with $m$ elements, then the range $f(X)$ of the function is also a finite set with $m$ elements. Thus if $Y$ is a finite set with the same number of elements as $X$, and if $f: X \to Y$ is an injective function, then this function is surjective, since the range $f(X)$ of the function is a subset of the codomain $Y$ which has exactly the same number of elements as the codomain, and is therefore the whole of the codomain. In particular, any injective function from a finite set to itself is guaranteed also to be surjective. It is also the case that any surjective function from a finite set to itself is guaranteed also be injective.

The properties of functions from a finite set to itself stated above do not hold for functions from an infinite set to itself. Let $X$ be an infinite set.

Then there exists an infinite sequence $x_1, x_2, x_3, \ldots$ of distinct elements of $X$. We can define a function $f: X \to X$, such that $f(x_j) = x_{j+1}$ for all natural numbers $j$, and $f(z) = z$ for all elements $z$ of $X$ that are not included in the infinite sequence $x_1, x_2, x_3, \ldots$. The function $f: X \to X$ defined in this fashion is an injective function. But it is not surjective since the element $x_1$ of $X$ does not belong to the range of the function.

From these observations it follows that a set is infinite if and only if there exists an injective function from the set to itself which is not surjective. This is Dedekind's characterization of infinite sets, expressed in contemporary mathematical terminology.

Now, in addition to the dichotomy between finite sets and infinite sets, there is an important dichotomy between *countable sets* and *uncountable sets*.

A set $X$ is said to be *countable* if there exists some infinite sequence $x_1, x_2, x_3, \ldots$ that includes every element of the set. In other words, a set is countable if there exists a surjection $f: \mathbb{N} \to X$ from the set $\mathbb{N}$ of natural numbers to the set $X$. All finite sets are countable. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ of natural numbers, integers and rational numbers are also countable. Sets that are not countable are said to be *uncountable*. Important examples of uncountable sets are the sets $\mathbb{R}$ and $\mathbb{C}$ of real numbers and complex numbers respectively. This distinction between countable and uncountable sets was first explored by the mathematician Georg Cantor (1845–1918) who developed an extensive theory concerned with infinite sets and transfinite cardinal and ordinal numbers. The proof of the uncountability of the set $\mathbb{R}$ of real numbers is an important result that is due to Cantor.

Any superset of an uncountable set is uncountable. It follows that all Euclidean spaces $\mathbb{R}^n$ with positive dimension $n$ are uncountable sets. Also any non-empty open set in a Euclidean space of positive dimension contains an non-empty open ball, and such an open ball is homeomorphic to the entire Euclidean space. It follows that any non-empty open set in a Euclidean space of positive dimension is an uncountable set.

A collection may be considered to be a set whose elements are the members of the collection. Thus collections can be countable or uncountable. It can be shown that the union of any countable collection of sets is itself a countable set. Also the Cartesian product of any finite number of countable sets is a countable set. Subsets of countable sets are countable sets. And if $f: X \to Y$ is a surjective function whose domain $X$ is a countable set, then the range $Y$ of the function is also a countable set. With these results, it becomes a trivial exercise to verify that the set $\mathbb{Z}$ of integers and the set $\mathbb{Q}$ of rational numbers are countable sets.

## 1.11 Topological Manifolds

**Definition** A *topological manifold* of dimension $n$ is a Hausdorff topological space $M$ which is the union of a countable collection of open sets, where each of the open sets in the collection is homeomorphic to an open set in $n$-dimensional Euclidean space $\mathbb{R}^n$.

**Example** The unit sphere $S^n$ of dimension $n$ is representable as the unit sphere in Euclidean space $\mathbb{R}^{n+1}$ of dimension $n+1$, and is defined as follows:

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

This space $S^n$ is a topological manifold of dimension $n$. Indeed let $\mathbf{n}$ and $\mathbf{s}$ be the points of $S^n$ defined such that

$$\mathbf{n} = (0, 0, \ldots, 0, 1), \quad \mathbf{s} = (0, 0, \ldots, 0, -1),$$

and let $P$ be the hyperplane in $\mathbb{R}^{n+1}$, defined so that

$$P = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}.$$

Now $S^n$ is the union of the complements $S^n \setminus \{\mathbf{n}\}$ and $S^n \setminus \{\mathbf{s}\}$ of the points $\mathbf{n}$ and $\mathbf{s}$. And these two subsets of $S^n$ are both homeomorphic to $\mathbb{R}^n$. Indeed there is a well-defined homeomorphism $\varphi\colon S^n \setminus \{\mathbf{n}\} \to P$ defined so that, for each $\mathbf{x} \in S^n \setminus \{\mathbf{n}\}$, the point $\varphi(\mathbf{x})$ is the point where the line passing through $\mathbf{n}$ and $\mathbf{x}$ intersects the hyperplane $P$. Similarly there is a well-defined homeomorphism $\psi\colon S^n \setminus \{\mathbf{s}\} \to P$ defined so that, for each $\mathbf{x} \in S^n \setminus \{\mathbf{s}\}$, the point $\varphi(\mathbf{x})$ is the point where the line passing through $\mathbf{s}$ and $\mathbf{x}$ intersects the hyperplane $P$. The geometric construction whereby the open subsets $S^n \setminus \{\mathbf{n}\}$ and $S^n \setminus \{\mathbf{s}\}$ are mapped onto the hyperplane $P$ by means of the homeomorphisms $\varphi$ and $\psi$ is known as *stereographic projection.* Straightforward calculations show that

$$\varphi(x_1, x_2, \ldots, x_n, x_{n+1}) = \left( \frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}}, 0 \right)$$

and

$$\varphi^{-1}(y_1, y_2, \ldots, y_n, 0) = \left( \frac{2y_1}{|\mathbf{y}|^2 + 1}, \frac{2y_2}{|\mathbf{y}|^2 + 1}, \ldots, \frac{2y_n}{|\mathbf{y}|^2 + 1}, \frac{|\mathbf{y}|^2 - 1}{|\mathbf{y}|^2 + 1} \right)$$

(where $|\mathbf{y}|^2 = y_1^2 + y_2^2 + \cdots + y_n^2$).

Let $M$ be a topological manifold of dimension $n$. Then, given any point $m$ of $M$, there exists an open set $U$ containing the point $m$ which is homeomorphic to an open set in $\mathbb{R}^n$. Thus every point of a topological manifold has open neighbourhoods that have the same topological properties as open sets in $\mathbb{R}^n$.

A topological manifold is also required to be a Hausdorff space. Were this not the case, it would be possible to construct examples of topological manifolds in which an infinite sequence of points converges simultaneously to two distinct limits. In classical mechanics and general relativity, one determines the trajectory of moving particles by solving systems of ordinary differential equations on a smooth manifold. (A smooth manifold is a topological manifold with extra structure that allows one to differentiate functions and solve differential equations.) Provided that appropriate technical conditions are met, one expects the solution of such a system of ordinary differential equations to be completely determined by appropriate initial conditions. For example, in classical mechanics and in general relativity, the trajectory of a particle is determined by its initial position, its initial velocity, and by the forces which act on it as it moves. There are basic theorems that ensure that, under appropriate technical conditions, solutions to systems of ordinary differential equations are uniquely determined by initial conditions. But if the manifold upon which the motion takes place were not required to be a Hausdorff space, then such theorems would fail to hold. And, as a result, many standard results and techniques used in the study of differential geometry and its applications to mechanics would no longer be valid. Thus the requirement that topological manifolds be Hausdorff spaces is necessary to ensure the validity of many results that play an important role in the study of both geometrical problems and applications to mechanics.

A further topological requirement is that it should be possible to cover a topological manifold with a countable collection of open sets, where each of these open sets is homeomorphic to an open set in a Euclidean space. Given the other topological conditions which a topological manifold must satisfy, this countability condition is equivalent to the requirement that there must exist a countable basis for the topology of the manifold. A collection of open sets in a topological space is a *basis* for the topology if every other open set can be expressed as a union of open sets belonging to the basis. For example the collection of all open balls in $n$-dimensional Euclidean space $\mathbb{R}^n$ is a basis for the (usual) topology on $\mathbb{R}^n$. However this basis is not a *countable* basis for the topology on $\mathbb{R}^n$. But if we consider the collection of all such open balls where the centres of the balls have Cartesian coordinates that are rational numbers, and where the radii of the balls are positive rational numbers, then the collection of all such open balls is a countable collection, and is a countable basis for the topology on $\mathbb{R}^n$. Moreover the subcollection consisting of those open balls that are contained in some open subset $V$ of $\mathbb{R}^n$ is a countable basis for the topology on $V$. It is a straightforward exercise to verify from this that if a topological space is a union of a countable collection of open sets, where each of those sets is homeomorphic to an open set in $\mathbb{R}^n$, then it has a countable basis. Conversely, if a Hausdorff space has a countable basis, and if every point is contained in some open set that is homeomorphic to an open subset of $\mathbb{R}^n$, then it is possible to cover the space by countably many such open sets, and therefore the Hausdorff space is a topological manifold of dimension $n$. Topological manifolds are required to satisfy this countability requirement in order to ensure the validity of certain technical results that are necessary in order to develop the theory of integration over manifolds, and to ensure certain other useful results.

## 1.12 Differentiability

The definition of a topological manifold gives meaning to the notion of a continuous function that is defined on a manifold, or that maps into the manifold. It also gives meaning to the notion of convergence of sequences of points in a topological manifold. It also places restrictions on the topological structure of the manifold, requiring in particular that sufficiently small portions of the manifold have the topological properties of open subsets of Euclidean spaces. However the definition of a topological manifold as a topological space with specified properties does not in itself provide the concept of a topological manifold with sufficient structure to apply notions of *differentiability* or *smoothness* to functions defined on a topological manifold, or mapping into a topological manifold. The manifold needs to be endowed with additional structure that enables the application of concepts of differentiability and smoothness to functions between manifolds.

We first review the definition of differentiability for functions of several real variables.

**Definition** Let $U$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$, let $\varphi\colon U \to \mathbb{R}^n$ be a function mapping $U$ into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of $U$. The function $\varphi$ is said to be *differentiable* at $\mathbf{p}$ if there exists some linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ with the property that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|}\left(\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}\right) = \mathbf{0}.$$

If a function $\varphi\colon U \to \mathbb{R}^n$ is differentiable at some point $\mathbf{p}$ of its domain $U$, where $U$ is an open set in $\mathbb{R}^m$, then there is a unique linear transformation

$T: \mathbb{R}^m \to \mathbb{R}^n$ for which

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}) = \mathbf{0}.$$

Indeed suppose that $T: \mathbb{R}^m \to \mathbb{R}^n$ and $S: \mathbb{R}^m \to \mathbb{R}^n$ are linear transformations, and that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}) = \lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - S\mathbf{h}) = \mathbf{0}.$$

Then

$$\begin{aligned} \lim_{\mathbf{h}\to\mathbf{0}} & \frac{1}{|\mathbf{h}|} (S\mathbf{h} - T\mathbf{h}) \\ &= \lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}) \\ &\qquad - \lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - S\mathbf{h}) \\ &= \mathbf{0}, \end{aligned}$$

and therefore

$$\lim_{t\to 0^+} \frac{1}{t|\mathbf{h}|} (S(t\mathbf{h}) - T(t\mathbf{h})) = 0.$$

But $S$ and $T$ are linear transformations, and therefore

$$\frac{1}{t|\mathbf{h}|} (S(t\mathbf{h}) - T(t\mathbf{h})) = \frac{1}{t|\mathbf{h}|} (tS(\mathbf{h}) - tT(\mathbf{h})) = \frac{1}{|\mathbf{h}|} (S(\mathbf{h}) - T(\mathbf{h})).$$

It follows that $S(\mathbf{h}) - T(\mathbf{h}) = \mathbf{0}$ for all $\mathbf{h} \in \mathbb{R}^m$, and therefore $S = T$.

**Definition** Let $U$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$, let $\varphi: U \to \mathbb{R}^n$ be a function mapping $U$ into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of $U$. Suppose that the function $\varphi$ is differentiable at $\mathbf{p}$. The *derivative* (or *total derivative*) of the function $\varphi$ at $\mathbf{p}$ is defined to be the unique linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ with the property that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}) = \mathbf{0}.$$

Let $\varphi: U \to \mathbb{R}^n$ a function, defined over an open subset $U$ of $\mathbb{R}^m$, which is differentiable, with derivative $T: \mathbb{R}^m: \mathbb{R}^n$, at some point $\mathbf{p}$ of $U$, and let $\mathbf{v} \in \mathbb{R}^m$. Then

$$\begin{aligned} \lim_{t\to 0} \frac{1}{t}\Big(\varphi(\mathbf{p}+t\mathbf{v}) - \varphi(\mathbf{p})\Big) - T\mathbf{v} &= \lim_{t\to 0} \frac{1}{t}\Big(\varphi(\mathbf{p}+t\mathbf{v}) - \varphi(\mathbf{p}) - T(t\mathbf{v})\Big) \\ &= |\mathbf{v}| \lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}) \\ &= \mathbf{0}, \end{aligned}$$

and therefore

$$T\mathbf{v} = \lim_{t\to 0} \frac{1}{t}\Bigl(\varphi(\mathbf{p}+t\mathbf{v}) - \varphi(\mathbf{p})\Bigr).$$

Linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$ can be represented by $n \times m$ matrices in the usual fashion. The entries of the matrix representing the derivative of a differentiable function $\varphi: U \to \mathbb{R}^n$ at a point $\mathbf{p}$ of $U$ are the partial derivatives of the components of the function $\varphi$. Indeed suppose that

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})),$$

for all $\mathbf{x} \in U$, where $f_1, f_2, \ldots, f_n$ are real-valued functions on $U$. Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be the derivative of $\varphi$ at $\mathbf{p}$, and let $T_{ij}$ denote the entry in the $i$th row and $j$th column of the matrix representing the linear transformation $T$. Then $T_{ij}$ is the $i$th component of the vector $T\mathbf{e_j}$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \ldots \mathbf{e}_m$ is the standard basis of the vector space $\mathbb{R}^m$, defined such that

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \quad \ldots \quad \mathbf{e}_m = (0, 0, \ldots, 0, 1).$$

But

$$T\mathbf{e}_j = \lim_{t\to 0} \frac{1}{t}\Bigl(\varphi(\mathbf{p}+t\mathbf{e_j}) - \varphi(\mathbf{p})\Bigr).$$

It follows that

$$T_{ij} = \lim_{t\to 0} \frac{f_i(\mathbf{p}+t\mathbf{e_j}) - f_i(\mathbf{p})}{t} = \left.\frac{\partial f_i(x_1, x_2, \ldots, x_m)}{\partial x_j}\right|_{\mathbf{x}=\mathbf{p}}.$$

Thus the derivative $T: \mathbb{R}^m \to \mathbb{R}^n$ of the function $\varphi: U \to \mathbb{R}^n$ at the point $\mathbf{p}$ is the linear transformation represented by the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_2} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_m} \\ \dfrac{\partial f_2}{\partial x_2} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_2} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_m} \end{pmatrix}$$

evaluated at the point $\mathbf{p}$.

If the function $\varphi: U \to \mathbb{R}^n$ is differentiable at some point $\mathbf{p}$ of the open set $U$ then all partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ are well-defined at the point $\mathbf{p}$. The converse result is not generally true: there are examples of functions where all these partial derivatives are well-defined at some point $\mathbf{p}$, though the function itself is not differentiable at that point. Nevertheless it can be shown that if the partial derivatives $\frac{\partial f_i}{\partial x_j}$ of the components $f_1, f_2, \ldots, f_n$ of the function $\varphi: U \to \mathbb{R}^n$ are continuous functions defined throughout the open set $U$, then the function $\varphi$ is differentiable at each point of $\mathbf{p}$.

Let $f: U \to \mathbb{R}$ be a real-valued function defined over some open set $U$ in $\mathbb{R}^m$. Suppose that the partial derivatives $\frac{\partial f}{\partial x_j}$ are defined throughout $U$ and themselves have partial derivatives $\frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right)$ that are continuous functions defined throughout $U$. Then

$$\frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right)$$

for all values of $j$ and $k$ between 1 and $m$, i.e.,

$$\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k}.$$

## 1.13 Smoothness

**Definition** A real-valued function $f: U \to \mathbb{R}$ defined on an open subset $U$ of a Euclidean space $\mathbb{R}^m$ is said to be *smooth* if the partial derivatives of $f$ of all orders are defined throughout $U$.

A function $\varphi: U \to \mathbb{R}^n$ mapping an open subset $U$ of $\mathbb{R}^m$ into $\mathbb{R}^n$ is said to be smooth if its components are smooth functions.

A smooth function is differentiable. Indeed the fact that the second order partial derivatives of a smooth function $f: U \to \mathbb{R}$ are defined throughout its domain $U$ ensures that the first order partial derivatives of the function are continuous, and this in turn is enough to ensure that the function is differentiable.

The space $L(\mathbb{R}^m, \mathbb{R}^n)$ of linear transformations from a Euclidean space $\mathbb{R}^m$ to a Euclidean space $\mathbb{R}^n$ may itself be regarded as a Euclidean space of dimension $mn$. Thus if $\varphi: U \to \mathbb{R}^n$ is a function defined over an open subset $U$ of $\mathbb{R}^m$, and if $\varphi$ is differentiable at each point of $\mathbf{p}$, then the function which maps points $\mathbf{p}$ to $U$ to the derivative of $\varphi$ at those points is itself a function mapping $U$ into a Euclidean space $L(\mathbb{R}^m, \mathbb{R}^n)$. If this derivative function is itself differentiable, then the original function $\varphi$ is twice differentiable. This function $\varphi: U \to \mathbb{R}$ is smooth if and only if it is $k$-times differentiable for all positive integers $k$.

## 1.14 Diffeomorphisms

**Definition** Let $U$ and $V$ be open sets in $n$-dimensional Euclidean space $\mathbb{R}^n$. A function $\varphi: U \to V$ is said to be a *diffeomorphism* if it satisfies the following conditions:

- the function $\varphi: U \to V$ is invertible;
- the function $\varphi: U \to V$ is smooth;
- the inverse function $\varphi^{-1}: V \to U$ is also smooth.

Note that any diffeomorphism is a homeomorphism. Moreover a homeomorphism is a diffeomorphism if and only if both it and its inverse are smooth.

## 1.15 Continuous Charts and Transition Functions

**Definition** Let $M$ be a topological manifold of dimension $n$. A *continuous chart* $(V, \varphi)$ for $M$ consists of an open subset $V$ of $M$ together with a continuous map $\varphi: V \to \mathbb{R}^n$, where the range $\varphi(V)$ of the function is an open set in $\mathbb{R}^n$, and where the function $\varphi$ establishes a homeomorphism between $V$ and $\varphi(V)$.

Let $M$ be a topological manifold of dimension $n$, and let $(V, \varphi)$ be a continuous chart for $M$. Then $V$ is an open set in $M$, and the continuous map $\varphi: V \to \mathbb{R}^n$ gives rise to an $n$-tuple of continuous real-valued functions $y^1, y^2, \ldots, y^n$ defined over the open set $V$, where

$$\varphi(m) = (y^1(m), y^2(m), \ldots, y^n(m)).$$

for all $m \in V$. Note that these functions are indexed by superscripts: this is a standard notational convention in differential geometry; the reasons for employing it should become apparent when we discuss tensor fields on smooth manifolds. These functions $y^1, y^2, \ldots, y^n$ constitute a continuous coordinate system defined over $V$. If $u$ and $v$ are distinct points of $V$ then at least one of the functions $y^1, y^2, \ldots, y^n$ takes on distinct values at $u$ and $v$. (This ensures that $\varphi: V \to \mathbb{R}^n$ is injective.) Moreover, given any point $\mathbf{p}$ of $\varphi(V)$, there is a unique point $m$ of $V$ for which

$$(y^1(m), y^2(m), \ldots, y^n(m)) = \mathbf{p},$$

and moreover the point $m$ of $V$ determined by $\mathbf{p}$ in this fashion depends continuously on $\mathbf{p}$ as $\mathbf{p}$ ranges over the open set $\varphi(V)$.

Given any continuous chart $(V, \varphi)$ on a topological manifold $M$ of dimension $n$, there is a one-to-one correspondence between open subsets of the domain $V$ of the chart and open subsets of the range $\varphi(V)$ of the chart: a subset $W$ of $V$ is an open set in $M$ if and only if $\varphi(W)$ is an open set in $\mathbb{R}^n$. This is an immediate consequence of the requirement that the function mapping $m \in V$ to $\varphi(m) \in \varphi(V)$ be a homeomorphism between $V$ and $\varphi(V)$.

Let $y^1, y^2, \ldots, y^n$ be continuous real-valued functions defined over an open set $V$ which determine a continuous chart $(V, \varphi)$, where

$$\varphi(m) = (y^1(m), y^2(m), \ldots, y^n(m))$$

for all $u \in V$, and let $g\colon V \to \mathbb{R}$ be a real-valued function defined over the domain $V$ of this continuous chart. Then the function $g$ determines a real-valued function $G\colon \varphi(V) \to \mathbb{R}$ defined over $\varphi(V)$, where

$$g(m) = G(y^1(m), y^2(m), \ldots, y^n(m))$$

for all $m \in V$. This function $G\colon \varphi(V) \to \mathbb{R}$ is continuous if and only if the function $g\colon V \to \mathbf{R}$ is continuous.

Let $(V, \varphi)$ and $(W, \psi)$ be continuous charts on a topological manifold $M$ of dimension $n$. Then then $V \cap W$ is an open subset of both $V$ and $W$, and therefore $\varphi(V \cap W)$ and $\psi(V \cap W)$ are both open subsets of $\mathbb{R}^n$, and the function $\varphi$ and $\psi$ determine homeomorphisms from $V \cap W$ to $\varphi(V \cap W)$ and $\psi(V \cap W)$. It follows that there is a homeomorphism $\theta\colon \varphi(V \cap W) \to \psi(V \cap W)$ between $\varphi(V \cap W)$ and $\psi(V \cap W)$ characterized by the requirement that $\theta(\varphi(m)) = \psi(m)$ for all $m \in V \cap W$. This homeomorphism $\theta$ is the *transition function* determined by the charts $\varphi$ and $\psi$. Now these charts can be represented in terms of continuous real-valued functions $y^1, y^2, \ldots, y^n$ and $z^1, z^2, \ldots, z^n$, where

$$\varphi(m) = (y^1(m), y^2(m), \ldots, y^n(m))$$

for all $m \in V$, and

$$\psi(m) = (z^1(m), z^2(m), \ldots, z^n(m))$$

for all $m \in W$. Then the transition function $\theta\colon \varphi(V \cap W) \to \psi(V \cap W)$ is characterized by the requirement that

$$\theta(y^1(m), y^2(m), \ldots, y^n(m)) = (z^1(m), z^2(m), \ldots, z^n(m))$$

for all $m \in V \cap W$.

## 1.16 Smooth Atlases

In order to employ manifolds in the study of both differential geometry and physics, where calculus is an essential tool, it is neccessary to formulate concepts of *differentiability* and *smoothness* that can be applied to functions defined over manifolds, and to functions mapping into manifolds. This would in particular allow one to study trajectories on manifolds that are determined by systems of ordinary differential equations, and to study functions over manifolds that are solutions of partial differential equations. The necessary structure is obtained by associating with the manifold an *atlas* of *smoothly compatible charts*.

**Definition** Let $(V, \varphi)$ and $(W, \psi)$ be continuous charts on a topological manifold $M$ of dimension $n$, and let $\theta\colon \varphi(V \cap W) \to \psi(V \cap W)$ be the homeomorphism from $\varphi(V \cap W)$ to $\psi(V \cap W)$ characterized by the requirement that $\theta(\varphi(m)) = \psi(m)$ for $m \in V \cap W$. The continuous charts $(V, \varphi)$ and $(W, \psi)$ are said to be *smoothly compatible* if and only if this transition function $\theta\colon \varphi(V \cap W) \to \psi(V \cap W)$ is a diffeomorphism.

If the domains $V$ and $W$ of the continuous charts $(V, \varphi)$ and $(W, \psi)$ are disjoint (i.e., if $V \cap W = \emptyset$) then the charts are considered to be smoothly compatible.

Let $M$ be a topological manifold of dimension $n$, and let $(V, \varphi)$ and $(W, \psi)$ be continuous charts on $M$ which are smoothly compatible. Then there is a diffeomorphism $\theta\colon \varphi(V \cap W) \to \psi(V \cap W)$ between $\varphi(V \cap W)$ and $\psi(V \cap W)$ characterized by the requirement that $\theta(\varphi(m)) = \psi(m)$ for all $m \in V \cap W$. Now the charts $\varphi$ and $\psi$ can be represented in terms of continuous real-valued functions $y^1, y^2, \ldots, y^n$ and $z^1, z^2, \ldots, z^n$, where

$$\varphi(m) = (y^1(m), y^2(m), \ldots, y^n(m))$$

for all $m \in V$, and

$$\psi(m) = (z^1(m), z^2(m), \ldots, z^n(m))$$

for all $m \in W$. Also there are smooth real-valued functions $H^1, H^2, \ldots, H^n$ defined on $\varphi(V \cap W)$ such that

$$\theta(\mathbf{x}) = (H^1(\mathbf{x}), H^2(\mathbf{x}), \ldots, H^n(\mathbf{x}))$$

for all $x \in \varphi(V \cap W)$. Then

$$(z^1(m), z^2(m), \ldots, z^n(m)) = \theta(y^1(m), y^2(m), \ldots, y^n(m))$$

for all $m \in V \cap W$, and therefore

$$z^j(m) = H^j(y^1(m), y^2(m), \ldots, y^n(m))$$

for $j = 1, 2, \ldots, n$ and for all $m \in V \cap W$. Thus if two continuous coordinate charts are smoothly compatible then the values of coordinate functions determining one chart can be expressed as smooth functions of the values of the coordinate function determining the other chart. Indeed two continuous coordinate charts are smoothly compatible if and only if the values of the coordinate functions of either one of the charts can be expressed as smooth functions of the values of coordinate functions of the other over the intersection of the domains of the charts.

**Definition** Let $M$ be a topological manifold of dimension $n$. A *smooth atlas* on $M$ is a collection of continuous charts on $M$ where the domains of the charts cover $M$, and where any two charts belonging to the atlas are smoothly compatible.

Let $M$ be a smooth manifold of dimension $n$. The requirement that the domains of the charts belonging a smooth atlas on $M$ cover the manifold $M$ ensures that each point of $M$ belongs to the domain of at least one chart belonging to the atlas.

Let $\mathcal{A}$ be a smooth atlas on a smooth manifold $M$ of dimension $n$, let $f\colon U \to \mathbb{R}$ be a real-valued function defined over some open subset $U$ of $M$, and let $p \in U$. Then the point $p$ belongs to the domain $V$ of some chart $(V, \varphi)$ belonging to the smooth atlas. There is then a real-valued function $F\colon \varphi(U \cap V) \to \mathbb{R}$, defined over the open subset $\varphi(U \cap V)$ of $\mathbb{R}^n$, such that $f(m) = F(\varphi(m))$ for all $m \in U \cap V$.

**Definition** Let $M$ be a manifold of dimension $n$ that is provided with some smooth atlas $\mathcal{A}$, let $f\colon U \to \mathbb{R}$ be a real-valued function defined over some open set $U$ in $M$, and let $p \in U$. The function $f$ is said to be *smooth* around $p$ with respect to the smooth atlas $\mathbb{A}$ if and only if there is a chart $(V, \varphi)$ belonging to this smooth atlas and a smooth function $F\colon \varphi(U \cap V) \to \mathbb{R}$ such that $p \in V$ and $f(m) = F(\varphi(m))$ for all $m \in U \cap V$.

**Lemma 1.4** *Let $M$ be a manifold of dimension $n$ that is provided with some smooth atlas $\mathcal{A}$, and let $f\colon U \to \mathbb{R}$ be a real-valued function defined over some open set $U$ in $M$ that is smooth around some point $p$ of $U$. Let $(V, \varphi)$ be a chart belonging to the smooth atlas $\mathcal{A}$, and let $F\colon \varphi(U \cap V) \to \mathbb{R}$ be the real valued function on $\psi(U \cap V)$ characterized by the requirement that $F(\varphi(m)) = f(m)$ for all $m \in U \cap V$. Then the function $F$ is smooth around $\varphi(p)$.*

**Proof** The definition of smoothness ensures the existence of a chart $(W, \psi)$ belonging to the smooth atlas $\mathcal{A}$ and a smooth function $G: \psi(U \cap W) \to \mathbb{R}$ such that $f(m) = G(\psi(m))$ for all $m \in U \cap W$. Moreover the charts $(V, \varphi)$ and $(W, \psi)$ are smoothly compatible, since both charts belong to the smooth atlast $\mathcal{A}$. It follows that the transition function $\theta: \varphi(V \cap W) \to \psi(V \cap W)$ determined by this pair of charts is a diffeomorphism, where $\theta(\varphi(m)) = \psi(m)$ for all $m \in V \cap W$. Then

$$F(\varphi(m)) = f(m) = G(\psi(m)) = G(\theta(\varphi(m)))$$

for all $m \in U \cap V \cap W$, and thus $F(\mathbf{x}) = G(\theta(\mathbf{x}))$ for all $\mathbf{x} \in \varphi(U \cap V \cap W)$. Thus the restriction of the function $F$ to $\varphi(U \cap V \cap W)$ can be expressed as the composition of smooth functions, and therefore this function is smooth around $\varphi(p)$, as required. ∎

**Lemma 1.5** *Let $M$ be a manifold of dimension $n$ that is provided with some smooth atlas $\mathcal{A}$, and let $(V, \varphi)$ and $(W, \psi)$ be continuous charts on $M$. Suppose that these charts are smoothly compatible with all charts belonging to the atlas $\mathcal{A}$. Then they are smoothly compatible with each other.*

**Proof** If $V \cap W = \emptyset$ then there is nothing to prove. Suppose that $V \cap W \neq \emptyset$. Let $p \in V \cap W$. Then there exists a chart $(U, \chi)$ belonging to the atlas $\mathcal{A}$ such that $p \in U$. Let $\theta: \varphi(V \cap W) \to \psi(V \cap W)$ be the transition function relating the charts $\varphi$ and $\psi$, let $\xi: \varphi(U \cap V) \to \chi(U \cap V)$ be the transition function relating the charts $\varphi$ and $\chi$, and let $\eta: \chi(U \cap W) \to \psi(U \cap W)$ be the transition function between the clarts $\chi$ and $\varphi$, so that

$$\begin{aligned}
\psi(m) &= \theta(\varphi(m)) \text{ for all } m \in V \cap W, \\
\chi(m) &= \xi(\varphi(m)) \text{ for all } m \in U \cap V, \\
\psi(m) &= \eta(\chi(m)) \text{ for all } m \in U \cap W,
\end{aligned}$$

Then

$$\theta(\varphi(m)) = \eta(\chi(m)) = \eta(\xi(\varphi(m))$$

for all $m \in U \cap V \cap W$, and therefore

$$\theta|_{\varphi(U \cap V \cap W)} = \eta|_{\chi(U \cap V \cap W)} \circ \xi|_{\varphi(U \cap V \cap W)}.$$

But the functions $\xi$ and $\eta$ are smooth around $\varphi(p)$ and $\chi(p)$ respectively, because the charts $(V, \varphi)$ and $(W, \psi)$ are both compatible with $(U, \chi)$. Also any composition of smooth functions between open sets in Euclidean space is itself smooth. It follows that the transition function $\theta$ is smooth around $\varphi(p)$. Similarly the inverse $\theta^{-1}$ of this transition function is smooth around $\psi(p)$. Therefore the charts $(V, \varphi)$ and $(W, \psi)$ are smoothly compatible, as required. ∎

## 1.17 Smooth Manifolds

Let $M$ be a topological manifold, let $\mathcal{A}$ be a smooth atlas on $M$, and let $\mathcal{A}_{\max}$ be the collection of all continuous charts on $M$ that are smoothly compatible with all charts that belong to the smooth atlas $\mathcal{A}$. It follows immediately from Lemma 1.5 that any two charts that belong to this collection $\mathcal{A}_{\max}$ are smoothly compatible with each other. Moreover $\mathcal{A} \subset \mathcal{A}_{\max}$, the collection $\mathcal{A}_{\max}$ of continuous charts on $M$ is a smooth atlas, and any continuous chart on $M$ that is smoothly compatible with all charts belonging to the atlas $\mathcal{A}_{\max}$ must itself belong to this atlas.

**Definition** Let $M$ be a smooth manifold. A smooth atlas on $M$ is said to be *maximal* if every chart that is smoothly compatible with all charts in the atlas itself belongs to the atlas.

It follows easily from the results and remarks described above that any smooth atlas $\mathcal{A}$ on a topological manifold $M$ is contained in a unique maximal smooth atlas $\mathcal{A}_{\max}$. This maximal smooth atlas is the collection of all continuous charts on $M$ that are smoothly compatible with all charts belonging to the smooth atlas $\mathcal{A}$. The uniqueness of this maximal smooth atlas is an immediate consequence of the following result.

**Lemma 1.6** *Let $M$ be a topological manifold, and let $\mathcal{A}$ and $\mathcal{B}$ be smooth atlases on $M$. Suppose that $\mathcal{A} \subset \mathcal{B}$. Then $\mathcal{B} \subset \mathcal{A}_{\max}$, where $\mathcal{A}_{\max}$ denotes the maximal smooth atlas on $M$ consisting of all continuous charts on $M$ that are smoothly compatible with all the charts that belong to the smooth atlas $\mathcal{A}$.*

**Proof** Any continuous chart on $M$ that belongs to the smooth atlas $\mathcal{B}$ is smoothly compatible with all continuous charts belonging to $\mathcal{B}$. In particular it is smoothly compatible with all continuous charts belonging to $\mathcal{A}$, and therefore belongs to $\mathcal{A}_{\max}$. ∎

**Definition** A *smooth manifold* $(M, \tau, \mathcal{A})$ of dimension $n$ consists of a topological manifold $M$ of dimension $n$, with topology $\tau$, which is provided with a maximal smooth atlas $\mathcal{A}$. This maximal smooth atlas represents the *differentiable structure* on the smooth manifold $M$.

It is customary to refer to a smooth manifold $(M, \tau, \mathcal{A})$ as 'the smooth manifold $M$', unless it is necessary to make explicit reference to the topology or maximal smooth atlas on $M$.

**Definition** Let $M$ be a smooth manifold. A continuous chart $(V, \varphi)$ on $M$ is said to be *smooth* if and only if it belongs to the maximal smooth atlas that represents the differentiable structure on $M$.

Note that any two smooth charts on a smooth manifold $M$ are guaranteed to be smoothly compatible, since they both belong to the maximal smooth atlas that represents the differentiable structure of $M$.

## 1.18 Smooth Maps between Smooth Manifolds

Let $f\colon M \to N$ be a function between smooth manifolds $M$ and $N$, let $(V, \varphi)$ be a smooth chart on $M$, and let $(W, \psi)$ be a smooth chart on $M$. Then the function $f$ and the charts $\varphi$ and $\psi$ determine a function

$$F\colon \varphi(V \cap f^{-1}(W)) \to \psi(W)$$

characterized by the property that $F(\varphi(v)) = \psi(v)$ for all $v \in V \cap f^{-1}(W)$. We shall refer to this function $F$ as the function which *represents* $f\colon M \to N$ with respect to the smooth charts $(V, \varphi)$ and $(W, \psi)$. Note that the domain $\varphi(V \cap f^{-1}(W))$ and the codomain $\psi(W)$ of this function $F$ are open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, where $m = \dim M$ and $n = \dim N$.

**Definition** Let $M$ and $N$ be smooth manifolds. A function $f\colon M \to N$ is said to be *smooth* around a point $p$ of $M$ if and only if there exists a smooth chart $(V, \varphi)$ on $M$ and a smooth chart $(W, \psi)$ on $N$, where $p \in V$ and $f(p) \in W$, such that the function $F\colon \varphi(V \cap f^{-1}(W)) \to \psi(W)$ which represents the function $f$ with respect to the smooth charts $(V, \varphi)$ and $(W, \psi)$ is smooth around the point $\varphi(p)$.

**Lemma 1.7** *Let $f\colon M \to N$ be a function between smooth manifolds $M$ and $N$, and let $p$ be a point of $M$, let $(V, \varphi)$ be a smooth chart on $M$, where $p \in V$, and let $(W, \psi)$ be a smooth chart on $N$, where $f(p) \in W$. Then the function $f$ is smooth around $p$ if and only if the function $F\colon \varphi(V \cap f^{-1}(W)) \to \psi(W)$ that represents $f$ with respect to the smooth charts $(V, \varphi)$ and $(W, \psi)$ is smooth around $\varphi(p)$.*

**Proof** The definition of smoothness ensures that if the representing function $F$ is smooth, then the function $f\colon M \to N$ is smooth, where $F(\varphi(v)) = \psi(v)$ for all $v \in V \cap f^{-1}(W)$

Conversely suppose that the function $f\colon M \to N$ is smooth around $p$. Then there exist smooth charts $(V_1, \varphi_1)$ and $(W_1, \psi_1)$, where $p \in V_1$ and $f(p) \in W_1$, such that the function $F_1\colon \varphi_1(V_1 \cap f^{-1}(W_1)) \to \psi_1(W_1)$ representing $f\colon M \to N$ with respect to the smooth charts $(V_1, \varphi_1)$ and $(W_1, \psi_1)$ is smooth. Now the function $F$ may be represented around $\varphi(p)$ as the composition of the transition function relating the charts $(V, \varphi)$ and $(V_1, \varphi_1)$, the smooth function $F_1$ that represents $f$ with respect to the charts $(V_1, \varphi_1)$ and

$(W_1, \psi_1)$, and the transition function relating the charts $(W_1, \psi_1)$ and $(W, \psi)$. The transition functions involved are smooth around the relevant points, and have smooth inverses. Also compositions of smooth functions are smooth. It follows that the function $F$ is also smooth, as required. ∎

Note that a function $f: M \to N$ between smooth manifolds $M$ and $N$ is smooth throughout $M$ if and only if, given any smooth chart $(V, \varphi)$ on $M$, and given any smooth chart $(W, \psi)$ on $N$, the function that represents $f$ with respect to these charts is a smooth function.

Let $f: M \to N$ be a function between smooth manifolds $M$ and $N$, let $p$ be a point of $M$, let $(V, \varphi)$ be a smooth chart around $p$, and let $(W, \psi)$ be a smooth chart around $f(p)$. The smooth chart $(V, \varphi)$ is represented by coordinate functions $y^1, y^2, \ldots, y^m$ defined over $V$, where $m = \dim M$ and

$$\varphi(v) = (y^1(v), y^2(v), \ldots, y^m(v)) \quad \text{for all } v \in V.$$

Similarly the smooth chart $(W, \psi)$ is represented by by coordinate functions $z^1, z^2, \ldots, z^m$ defined over $W$, where $n = \dim N$ and

$$\psi(w) = (z^1(w), z^2(w), \ldots, z^n(w)) \quad \text{for all } w \in W.$$

There are then real-valued functions $F^k: \varphi(V \cap f^{-1}(W)) \to \mathbb{R}$ for $k = 1, 2, \ldots, n$ characterized by the property that

$$z^k(f(v)) = F^k(y^1(v), y^2(v), \ldots, y^m(v))$$

for all $v \in V$. The function $f$ is then represented with respect to the smooth charts $(V, \varphi)$ and $(W, \psi)$ by the vector-valued function $F: \varphi(V \cap f^{-1}(W)) \to \psi(W)$ where

$$F(\mathbf{x}) = \Big(F^1(\mathbf{x}), F^2(\mathbf{x}), \ldots, F^n(\mathbf{x})\Big)$$

for all $\mathbf{x} \in \varphi(V \cap f^{-1}(W))$. Then the function $f$ is smooth at $p$ if and only if the function $F$ is smooth at $\varphi(p)$, and this is the case if and only if the components $F^1, F^2, \ldots, F^n$ of $F$ are smooth functions at $\varphi(p)$.

# 2 Tangent Spaces and Derivatives

## 2.1 Partial Derivatives with respect to Local Coordinates

Let $\varphi: U \to \mathbb{R}^n$ be a smooth chart defined over an open set $U$ in $M$. Then there are smooth real-valued functions $x^1, x^2, \ldots, x^n$ on $U$ such that

$$\varphi(u) = (x^1(u), x^2(u), \ldots, x^n(u))$$

for all $u \in U$. The functions $x^1, x^2, \ldots, x^n$ determined by the chart $(U, \varphi)$ constitute smooth *local coordinate functions* defined over the domain $U$ of the chart. Now the function $\varphi$ maps the open set $U$ homeomorphically onto an open set $V$ in $\mathbb{R}^n$, where

$$V = \varphi(U) = \{(x^1(u), x^2(u), \ldots, x^n(u)) : u \in U\}.$$

Every real-valued function $f: U \to \mathbb{R}$ on $U$ determines a corresponding real-valued function $F: V \to \mathbb{R}$ on $\varphi(U)$, where

$$f(u) = F(\varphi(u)) = F(x^1(u), x^2(u), \ldots, x^n(u))$$

for all $u \in U$. Moreover the function $f$ is smooth on $U$ if and only if if the corresponding function $F$ is smooth on $V$. This set $V$ is an open set in the Euclidean space $\mathbb{R}^n$, and therefore the function $F$ has well-defined partial derivatives with respect to the standard Cartesian coordinates $s^1, s^2, \ldots, s^n$.

**Definition** Let $x^1, x^2, \ldots, x^n$ be smooth local coordinates defined over an open set $U$ in a smooth manifold $M$ of dimension $n$, and let $V$ be the corresponding open set in $\mathbb{R}^n$ defined such that

$$V = \{(x^1(u), x^2(u), \ldots, x^n(u)) : u \in U\}.$$

The *partial derivatives*

$$\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots \frac{\partial f}{\partial x^n}$$

of $f$ with respect to the coordinate functions $x^1, x^2, \ldots, x^n$ are defined to be the real-valued smooth functions on $U$ that satisfy

$$\left.\frac{\partial f}{\partial x^j}\right|_u = \left.\frac{\partial F(s^1, s^2, \ldots, s^n)}{\partial s^j}\right|_{(s^1,s^2,\ldots,s^n)=\mathbf{p}}$$

for all $u \in U$, where $F: V \to \mathbb{R}$ is the smooth real-valued function on the open set $V$ characterized by the property that

$$f(u) = F(x^1(u), x^2(u), \ldots, x^n(u))$$

for all $u \in U$, and where

$$\mathbf{p} = (x^1(u), x^2(u), \ldots, x^n(u)).$$

**Lemma 2.1** *Let $x^1, x^2, \ldots, x^n$ be a smooth local coordinate system defined over an open set $U$ in a smooth manifold $M$ of dimension $n$, let $\gamma\colon I_\gamma \to M$ be a smooth curve in $M$, defined over some open interval $I_\gamma$ in the real line, where $\gamma(I_\gamma) \subset U$, and let $f\colon U \to \mathbb{R}$ be a smooth function defined over $U$. Then*

$$\frac{d}{dt}\Bigl(f(\gamma(t))\Bigr) = \sum_{j=1}^{n} \left.\frac{\partial f}{\partial x^j}\right|_{\gamma(t)} \frac{dx^j(\gamma(t))}{dt}$$

**Proof** There exists a smooth function $F\colon V \to \mathbb{R}$, where

$$V = \{(x^1(u), x^2(u), \ldots, x^n(u)) : u \in U\},$$

such that $f(u) = F(x^1(u), x^2(u), \ldots, x^n(u))$ for all $u \in I$. It follows that

$$\begin{aligned}
\frac{d}{dt}\Bigl(f(\gamma(t))\Bigr) &= \frac{d}{dt}\Bigl(F(x^1(\gamma(t)), x^2(\gamma(t)), \ldots, x^n(\gamma(t)))\Bigr) \\
&= \sum_{j=1}^{n} \left.\frac{\partial F(s^1, s^2, \ldots, s^n)}{\partial s^j}\right|_{(x^1(\gamma(t)),\ldots,x^n(\gamma(t))} \frac{dx^j(\gamma(t))}{dt} \\
&= \sum_{j=1}^{n} \left.\frac{\partial f}{\partial x^j}\right|_{\gamma(t)} \frac{dx^j(\gamma(t))}{dt},
\end{aligned}$$

as required. ∎

We see therefore that the value of the partial derivative $\dfrac{\partial f}{\partial x^j}$ of a smooth function at some point of the domain $U$ of a smooth local coordinate system $x^1, x^2, \ldots, x^n$ is (as the notation would suggest) the rate of change of the value of the function $f$ along a curve passing through that point, where the coordinate functions $x^k$ are constant along the curve for $k \neq j$, and where the coordinate function $x^j$ increases at a uniform unit rate along the curve.

## 2.2 Smooth Submanifolds of Euclidean Spaces

Smooth functions $y^1, y^2, \ldots, y^k$ defined over an open set $U$ in $\mathbb{R}^k$ are said to constitute a *smooth curvilinear coordinate system* over $U$ if and only if these functions are the component functions of a smooth chart with domain $U$. It follows that the smooth functions $y^1, y^2, \ldots, y^k$ constitute a smooth curvilinear coordinate system over $U$ if and only if the function from $U$ to $\mathbb{R}^k$ that sends $\mathbf{u} \in U$ to $(y^1(\mathbf{u}), y^2(\mathbf{u}), \ldots, y^k(\mathbf{u}))$ defines a diffeomorphism mapping $U$ onto an open set in $\mathbb{R}^k$.

**Definition** Let $M$ be a subset of $k$-dimensional Euclidean space $\mathbb{R}^k$. Then $M$ is said to be a *smooth submanifold* of $\mathbb{R}^k$ of dimension $n$ (where $n \leq k$), if and only if, given any point $\mathbf{p}$ of $M$, there exists an open set $U$ in $\mathbf{R}^k$, and a smooth curvilinear coordinate system $y^1, y^2, \ldots, y^k$ defined over $U$ such that $\mathbf{p} \in U$ and

$$M \cap U = \{\mathbf{u} \in U : y^j(\mathbf{u}) = 0 \text{ for } j > n\}.$$

Let $M$ be a smooth submanifold of $\mathbb{R}^k$ of dimension $n$, let $\mathbf{p} \in M$, and let $y^1, y^2, \ldots, y^k$ be a smooth curvilinear coordinate system defined over some open set $U$ in $\mathbb{R}^k$ such that $\mathbf{p} \in U$ and

$$M \cap U = \{\mathbf{u} \in U : y^j(\mathbf{u}) = 0 \text{ for } j > n\}.$$

The restrictions of the functions $y^1, y^2, \ldots, y^n$ to $M \cap U$ determine a continuous chart with domain $M \cap U$. Moreover any two charts on $M$ constructed in this fashion are smoothly compatible, and thus the collection consisting of all such charts constitutes a smooth atlas on $M$. Moreover it is not difficult to show that $M$ can be covered by the domains of countably many such charts. It follows that any submanifold of $\mathbb{R}^k$ of dimension $n$ is a smooth manifold of dimension $n$.

Let $x^1, x^2, \ldots, x^k$ be the standard Cartesian coordinate system on $\mathbb{R}^k$, characterized by the property that

$$\mathbf{p} = (x^1(\mathbf{p}), x^2(\mathbf{p}), \ldots, x^k(\mathbf{p}))$$

for all $\mathbf{p} \in \mathbb{R}^k$. Now the function that sends each point $\mathbf{u}$ of $U$ to

$$(x^1(\mathbf{u}), x^2(\mathbf{u}), \ldots, x^k(\mathbf{u}))$$

maps $U$ diffeomorphically into an open set in $\mathbb{R}^k$. It follows that the Jacobian matrix of this diffeomorphism is a non-singular matrix at each point of $U$. This Jacobian matrix is the $k \times k$ matrix whose entry in the $i$th row and $j$th column is $\dfrac{\partial x^i}{\partial y^j}$. Its inverse is the matrix whose entry in the $j$th row and $i$th column is $\dfrac{\partial y^j}{\partial x^i}$.

Let $M$ be a smooth $n$-dimensional submanifold of $\mathbb{R}^k$. A *smooth curve* in $M$ is represented by a smooth function $\gamma\colon I_\gamma \to M$ defined over some open interval $I_\gamma$ in $\mathbb{R}$. This function parameterizes points along the curve by elements of the interval $I_\gamma$. It is convenient to conceptualize this smooth curve as being parameterized by *time*, so that, for each value $t$ of the time parameter in $I_\gamma$, the point $\gamma(t)$ represents the position at time $t$ of a particle traversing the curve, Now, because the manifold $M$ is embedded in $\mathbb{R}^k$, we can differentiate this function. The derivative $\dfrac{d\gamma(t)}{dt}$ then represents the *velocity vector* of the smooth curve at position $\gamma(t)$ and time $t$.

**Definition** Let $M$ be a smooth $n$-dimensional submanifold of $\mathbb{R}^k$, and let $\mathbf{p}$ be a point of $M$. The *tangent space* $T_{\mathbf{p}}M$ to $M$ at the point $\mathbf{p}$ is defined to be the subspace of $\mathbb{R}^k$ consisting of all vectors in $\mathbb{R}^k$ that can be represented at the velocity vectors

$$\left.\frac{d\gamma(t)}{dt}\right|_{t=0}$$

of smooth curves $\gamma\colon I_\gamma \to M$ in $M$ for which $0 \in I_\gamma$ and $\gamma(0) = \mathbf{p}$.

**Lemma 2.2** *Let $M$ be a smooth submanifold of $\mathbb{R}^k$ of dimension $n$, and let $y^1, y^2, \ldots, y^k$ be a smooth curvilinear coordinate system defined over some open set $U$ in $\mathbb{R}^k$ such that*

$$M \cap U = \{\mathbf{u} \in U : y^j(\mathbf{u}) = 0 \text{ for } j > n\}.$$

*Given any point $\mathbf{p}$ of $M \cap U$, the tangent space to $M$ at $\mathbf{p}$ is an $n$-dimensional subspace of $\mathbb{R}^k$, and is the vector subspace of $\mathbb{R}^k$ that is spanned by the vectors*

$$\left.\frac{\partial \mathcal{P}}{\partial y_1}\right|_{\mathbf{p}}, \quad \left.\frac{\partial \mathcal{P}}{\partial y_2}\right|_{\mathbf{p}}, \ldots, \quad \left.\frac{\partial \mathcal{P}}{\partial y_n}\right|_{\mathbf{p}},$$

*where $\mathcal{P}\colon M \to \mathbb{R}^k$ is the inclusion function of $M$ in $\mathbb{R}^k$. These vectors are linearly independent at each point of $U$, and*

$$\frac{d\gamma(t)}{dt} = \sum_{j=1}^{n} \left.\frac{\partial \mathcal{P}}{\partial y^j}\right|_{\gamma(t)} \frac{dy^j(\gamma(t))}{dt}$$

*for all smooth curves $\gamma\colon I_\gamma \to M \cap U$ in $M \cap U$.*

**Proof** The inclusion function $\mathcal{P}\colon M \to \mathbb{R}^k$ satisfies

$$\mathcal{P}(\mathbf{p}) = (x^1(\mathbf{p}), x^2(\mathbf{p}), \ldots, x^k(\mathbf{p}))$$

for all $\mathbf{p} \in M$, where $x^1, x^2, \ldots, x^k$ is the standard Cartesian coordinate system on $\mathbb{R}^k$. Therefore

$$\frac{\partial \mathcal{P}}{\partial y_j} = \left(\frac{\partial x^1}{\partial y^j}, \frac{\partial x^2}{\partial y^j}, \ldots, \frac{\partial x^k}{\partial y^j}\right)$$

for $j = 1, 2, \ldots, n$. It follows that, for each $\mathbf{p} \in M \cap U$, the vectors $\left.\dfrac{\partial \mathcal{P}}{\partial y_j}\right|_{\mathbf{p}}$ for $j = 1, 2, \ldots, n$ represent the first $n$ columns of the non-singular $k \times k$ matrix with entries $\dfrac{\partial x^i}{\partial y^j}$ that is the Jacobian matrix at $\mathbf{p}$ of the transition function between the smooth curvilinear oordinate systems $(u^j)$ on $U$ and the Cartesian coordinate system $(x^k)$. Therefore these vectors are linearly independent, and thus span an $n$-dimensional subspace of $\mathbb{R}^k$.

Let $\gamma: I_\gamma \to M$ be a smooth curve in $M$, where $0 \in I_\gamma$ and $\gamma(I_\gamma) \subset M \cap U$, and let $\gamma(0) = \mathbf{p}$. Then $\gamma(t) = \mathcal{P}(\gamma(t))$ for all $t \in I_\gamma$, and

$$\frac{d\gamma(t)}{dt} = \frac{d\mathcal{P}(\gamma(t))}{dt} = \left( \frac{dx^1(\gamma(t))}{dt}, \ \frac{dx^2(\gamma(t))}{dt}, \ldots, \ \frac{dx^k(\gamma(t))}{dt}, \right).$$

Now

$$\frac{dx^i(\gamma(t))}{dt} = \sum_{j=1}^n \left.\frac{\partial x^i}{\partial y^j}\right|_{\gamma(t)} \frac{dy^j(\gamma(t))}{dt}.$$

It follows that

$$\frac{d\gamma(t)}{dt} = \frac{d\mathcal{P}(\gamma(t))}{dt} = \sum_{j=1}^n \left.\frac{\partial \mathcal{P}}{\partial y^j}\right|_{\gamma(t)} \frac{dy^j(\gamma(t))}{dt}.$$

This establishes the formula for the velocity vector $\dfrac{d\gamma(t)}{dt}$ to the curve $\gamma$ at time $t$.

Now

$$\left.\frac{d\gamma(t)}{dt}\right|_{t=0} = \sum_{j=1}^n \left.\frac{\partial \mathcal{P}}{\partial y^j}\right|_{\mathbf{p}} \left.\frac{dy^j(\gamma(t))}{dt}\right|_{t=0}.$$

It follows that the velocity vector $\left.\dfrac{d\gamma(t)}{dt}\right|_{t=0}$ belongs to the $n$-dimensional vector subspace of $\mathbb{R}^k$ spanned by the vectors

$$\left.\frac{\partial \mathcal{P}}{\partial y_1}\right|_{\mathbf{p}}, \ \left.\frac{\partial \mathcal{P}}{\partial y_2}\right|_{\mathbf{p}}, \ldots, \ \left.\frac{\partial \mathcal{P}}{\partial y_n}\right|_{\mathbf{p}}.$$

Let $a^1, a^2, \ldots, a^n$ be real numbers. Then there exists a smooth curve $\gamma: I_\gamma \to M$ in $M$, defined over some sufficiently small open interval $I_\gamma$ containing zero, such that

$$y^j(\gamma(t)) = y^j(\mathbf{p}) + a^j t$$

for $t \in I_\gamma$ and $j = 1, 2, \ldots, n$. Then $\left.\dfrac{dy^j(\gamma(t))}{dt}\right|_{t=0} = a^j$ for $j = 1, 2, \ldots, n$, and therefore

$$\left.\frac{d\gamma(t)}{dt}\right|_{t=0} = \sum_{j=1}^{n} a^i \left.\frac{\partial \mathcal{P}}{\partial y^j}\right|_{\mathbf{p}}.$$

Thus every element of the vector subspace of $\mathbb{R}^k$ spanned by

$$\left.\frac{\partial \mathcal{P}}{\partial y_1}\right|_{\mathbf{p}}, \left.\frac{\partial \mathcal{P}}{\partial y_2}\right|_{\mathbf{p}}, \ldots, \left.\frac{\partial \mathcal{P}}{\partial y_n}\right|_{\mathbf{p}}.$$

is the velocity vector of some smooth curve in $M$ passing through the point $\mathbf{p}$. The result follows. ∎

**Lemma 2.3** *Let $M$ be a smooth submanifold of $\mathbb{R}^k$ of dimension $n$, let $f\colon V \to \mathbb{R}$ be a smooth real-valued function defined over some subset $V$ of $M$ that is open in $M$, and let $\mathbf{p} \in V$. Then the function $f$ determines a linear functional $df_{\mathbf{p}}\colon T_{\mathbf{p}}M \to \mathbb{R}$ on the tangent space $T_{\mathbf{p}}M$ to $M$ at the point $\mathbf{p}$ characterized by the property that*

$$df_{\mathbf{p}}\left(\left.\frac{d\gamma(t)}{dt}\right|_{t=0}\right) = \left.\frac{df(\gamma(t))}{dt}\right|_{t=0}$$

*for all smooth curves $\gamma\colon I_\lambda \to V$ in $V$ for which $0 \in I_\lambda$ and $\gamma(0) = \mathbf{p}$.*

**Proof** Let $y^1, y^2, \ldots, y^k$ be a smooth curvilinear coordinate system defined over some open set $U$ in $\mathbb{R}^k$, where $\mathbf{p} \in U$ and

$$M \cap U = \{\mathbf{u} \in U : y^j(\mathbf{u}) = 0 \text{ for } j > n\},$$

and let $\gamma\colon I_\gamma \to M$ be a smooth curve in $M$, where $0 \in I_\gamma$, $\gamma(0) = \mathbf{p}$ and $\gamma(I_\gamma) \subset M \cap U \cap V$. Then it follows from Lemma 2.1 and Lemma 2.2 that

$$\left.\frac{d}{dt}\Big(f(\gamma(t))\Big)\right|_{t=0} = \sum_{j=1}^{n} a^j \left.\frac{\partial f}{\partial y^j}\right|_{\mathbf{p}},$$

where

$$a^j = \left.\frac{dy^j(\gamma(t))}{dt}\right|_{t=0}$$

for $j = 1, 2, \ldots, n$. But

$$\left.\frac{d\gamma(t)}{dt}\right|_{t=0} = \sum_{j=1}^{n} a^j \left.\frac{\partial \mathcal{P}}{\partial y^j}\right|_{\mathbf{p}},$$

where $\mathcal{P}\colon M \to \mathbb{R}^k$ denotes the inclusion function of $M$ in $\mathbb{R}^k$. Then

$$\frac{d}{dt}\Big(f(\gamma(t))\Big)\bigg|_{t=0} = \sum_{j=1}^{n} a^j \left.\frac{\partial f}{\partial y^j}\right|_{\mathbf{p}}.$$

It follows that

$$\left.\frac{df(\gamma(t))}{dt}\right|_{t=0} = df_{\mathbf{p}}\left(\left.\frac{d\gamma(t)}{dt}\right|_{t=0}\right),$$

where $df_{\mathbf{p}}\colon T_{\mathbf{p}}M \to \mathbb{R}$ is the function from $T_{\mathbf{p}}M$ to $\mathbb{R}$ defined such that

$$df_{\mathbf{p}}\left(\sum_{j=1}^{n} a^j \left.\frac{\partial \mathcal{P}}{\partial y^j}\right|_{\mathbf{p}}\right) = \sum_{j=1}^{n} a^j \left.\frac{\partial f}{\partial y^j}\right|_{\mathbf{p}}$$

for all $(a^1, a^2, \ldots, a^n) \in \mathbb{R}^n$. Now the tangent space $T_{\mathbf{p}}M$ to $M$ at $p$ is an $n$-dimensional vector space with basis

$$\left.\frac{\partial \mathcal{P}}{\partial y_1}\right|_{\mathbf{p}}, \left.\frac{\partial \mathcal{P}}{\partial y_2}\right|_{\mathbf{p}}, \ldots, \left.\frac{\partial \mathcal{P}}{\partial y_n}\right|_{\mathbf{p}},$$

and therefore the function from $T_{\mathbf{p}}M$ to $\mathbb{R}^n$ that sends the tangent vector $\displaystyle\sum_{j=1}^{n} a^j \left.\frac{\partial \mathcal{P}}{\partial y^j}\right|_{\mathbf{p}}$ to $(a^1, a^2, \ldots, a^n)$ for all $a^1, a^2, \ldots, a^n \in \mathbb{R}$ is an isomorphism of real vector spaces. Also the function from $\mathbb{R}^n$ to $\mathbb{R}$ that sends the $n$-tuple $(a^1, a^2, \ldots, a^n)$ to $\displaystyle\sum_{j=1}^{n} a^j \left.\frac{\partial f}{\partial y^j}\right|_{\mathbf{p}}$ is a linear functional on $\mathbb{R}^n$. It follows that $df_{\mathbf{p}}\colon T_{\mathbf{p}}M \to \mathbb{R}$ is a linear functional on the tangent space $T_{\mathbf{p}}M$. This linear functional is independent of the choice of the curvilinear coordinate system $y^1, y^2, \ldots, y^k$ on the open set $U$, since it can be characterized (without reference to any such curvilinear coordinate system) as the linear functional on $T_{\mathbf{p}}M$ that maps the velocity vector $\left.\frac{d\gamma(t)}{dt}\right|_{t=0}$ to $\displaystyle\left.\frac{df(\gamma(t))}{dt}\right|_{t=0}$ for all smooth curves $\gamma\colon I_\gamma \to M$ in $M$ for which $0 \in I_\gamma$ and $\gamma(0) = \mathbf{p}$. ∎

The set of all linear functionals on a real vector space is itself a real vector space; it is the *dual space* of the vector space on which the linear functionals are defined. The dual of the tangent space $T_{\mathbf{p}}M$ to a smooth submanifold $M$ of a Euclidean space at a point $\mathbf{p}$ of $M$ is referred to as the *cotangent space* to $M$ at the point $\mathbf{p}$, and is denoted by $T^*_{\mathbf{p}}M$. The elements of the cotangent space $T^*_{\mathbf{p}}M$ therefore represent linear functionals on the tangent space $T_{\mathbf{p}}M$. It follows from Lemma 2.3 that any smooth function $f\colon V \to \mathbb{R}$ defined over an open set $V$ in $M$ determines at each point $\mathbf{p}$ of $M$ an element $df_{\mathbf{p}}$ of the cotangent space $T^*_{\mathbf{p}}M$; this element of $T^*_{\mathbf{p}}M$ is referred to as the *differential* of the function $f$ at the point $\mathbf{p}$.

Let $f$ and $g$ be smooth real-valued functions defined over an open set $V$ in the smooth submanifold $M$ of $\mathbb{R}^k$, and let $f \cdot g$ denote the product of the functions $f$ and $g$, where $(f \cdot g)(\mathbf{v}) = f(\mathbf{v})g(\mathbf{v})$ for all $\mathbf{v} \in V$. It follows directly from Lemma 2.3 and the Product Rule for differentiation that

$$d(f \cdot g)_{\mathbf{p}} = g(\mathbf{p})\, df_{\mathbf{p}} + f(\mathbf{p})\, dg_{\mathbf{p}}$$

for all $\mathbf{p} \in V$. Moreover if $f$ and $g$ are smooth real-valued functions on this open set $V$ which satisfy $f(\mathbf{v}) = g(\mathbf{v})$ at all points $\mathbf{v}$ of some open neighbourhood of a given point $\mathbf{p}$ of $V$ then $df_{\mathbf{p}} = dg_{\mathbf{p}}$.

Let $f\colon V \to \mathbb{R}$ be a smooth real-valued function defined over some open set $V$ in a smooth submanifold $M$ of some Euclidean space, and, given any point $\mathbf{p}$ of $V$, let $df_{\mathbf{p}} \in T^*_{\mathbf{p}}M$ be the differential of $f$ at the point $\mathbf{p}$. Then $df_{\mathbf{p}}$ is a linear functional that maps tangent vectors at the point $\mathbf{p}$ to real numbers. Given any vector $X_{\mathbf{p}}$ belonging to the tangent space $T_{\mathbf{p}}M$ to $M$ at the point $\mathbf{p}$ we define $X_{\mathbf{p}}[f]$ and $\langle df_{\mathbf{p}}, X_p\rangle$ such that

$$X_p[f] = \langle df_{\mathbf{p}}, X_p\rangle = df_{\mathbf{p}}(X_p)$$

for all tangent vectors $X_p \in T_{\mathbf{p}}M$ and for all smooth real-valued functions $f$ defined around the point $\mathbf{p}$ of the submanifold $M$. We refer to $X_p[f]$ as the *directional derivative* of the smooth function $f$ along the vector $X_p$.

One can readily verify that the directional derivatives of real-valued functions along a tangent vector $X_p$ at a point $\mathbf{p}$ of a smooth submanifold of some Euclidean space satisfy the following properties:

(i) $X_{\mathbf{p}}[\alpha f + \beta g] = \alpha X_{\mathbf{p}}[f] + \beta X_{\mathbf{p}}[g]$ for all real numbers $\alpha$ and $\beta$ and smooth functions $f$ and $g$ defined around the point $\mathbf{p}$,

(ii) $X_{\mathbf{p}}[f \cdot g] = X_{\mathbf{p}}[f]\, g(\mathbf{p}) + f(\mathbf{p})\, X_{\mathbf{p}}[g]$ for all smooth functions $f$ and $g$ defined around the point $\mathbf{p}$,

(iii) if $f$ and $g$ are smooth real-valued functions defined around $\mathbf{p}$, and if $f|V = g|V$ for some open set $V$ containing the point $\mathbf{p}$, then $X_{\mathbf{p}}[f] = X_{\mathbf{p}}[g]$.

## 2.3 Tangent Spaces to Smooth Manifolds

Let $M$ be a smooth $n$-dimensional manifold of some Euclidean space $\mathbb{R}^k$. Then each point $\mathbf{p}$ of $M$ determines an $n$-dimensional vector subspace $T_{\mathbf{p}}M$

of the ambient space $\mathbb{R}^k$ that represents the *tangent space* to the smooth submanifold $M$ at the point $\mathbf{p}$. The elements of this tangent space are *tangent vectors* to $M$ at the point $\mathbf{p}$, and may be represented as velocity vectors of smooth curves in $M$ passing through the point $\mathbf{p}$. Moreover, given any smooth real-valued function $f$ defined over an open subset of $M$ that contains the point $\mathbf{p}$, and given any tangent vector $X_{\mathbf{p}}$ at the point $\mathbf{p}$, we have defined a quantity $X_{\mathbf{p}}[f]$ that represents the *directional derivative* of the function $f$ along the tangent vector $X_p$.

The tangent space $T_{\mathbf{p}}M$ to a smooth submanifold $M$ of $\mathbb{R}^k$ at a point $\mathbf{p}$ of $M$ has thus been represented as a vector subspace of the ambient space $\mathbb{R}^k$. It is desirable however to extend the notions and basic properties of tangent spaces and tangent vectors to smooth manifolds that do not come embedded as submanifolds of Euclidean spaces. In particular one may wish to apply the concepts of differential geometry to the study of General Relativity and String Theory. But the four-dimensional curved space-time underlying the theory of General Relativity does not have a natural embedding in any flat finite-dimensional Euclidean space. And many examples of smooth manifolds occur in mathematics that have no natural representation as submanifolds of Euclidean spaces.

In order to effect this generalization, we represent tangent vectors at a point $p$ of a smooth $n$-dimensional manifold $M$ as operators (or functions) that map smooth functions defined around the point $p$ to real numbers. These tangent vectors satisfy certain basic properties that suffice to characterize linear first order differential operators at the point $p$. We shall then prove that these tangent vectors are the elements of an $n$-dimensional real vector space $T_pM$. This vector space is the *tangent space* to the smooth manifold $M$ at the point $p$.

We now set out this construction of tangent spaces in more detail.

**Definition** Let $M$ be a smooth manifold of dimension $n$, and let $p$ be a point of $M$. We define a *tangent vector* $X_p$ at the point $p$ to be a operator that associates a real number $X_p[f]$ to each smooth real-valued function $f$ defined throughout some open neigbourhood of $p$, where this operator satisfies the following conditions:—

(i) $X_p[\alpha f + \beta g] = \alpha X_p[f] + \beta X_p[g]$ for all real numbers $\alpha$ and $\beta$ and smooth functions $f$ and $g$ defined around the point $p$;

(ii) $X_p[f \cdot g] = X_p[f]\, g(p) + f(p)\, X_p[g]$ for all smooth functions $f$ and $g$ defined around the point $p$;

(iii) if $f$ and $g$ are smooth real-valued functions defined around $p$, and if $f|V = g|V$ for some open set $V$ that contains the point $p$, then $X_p[f] = X_p[g]$.

(Here $f \cdot g$ denotes the product of the functions $f$ and $g$, defined such that $(f \cdot g)(m) = f(m)g(m)$ for all $m \in M$, and $f|V$ and $g|V$ denote the restrictions of the functions $f$ and $g$ to the open set $V$.)

The quantity $X_p[f]$ is referred to as the *directional derivative* of the function $f$ along the vector $X_p$.

We say that a real-valued function $f$ is defined *around* a point $p$ of a smooth manifold $M$ if $f$ is defined throughout some open neighbourhood of $p$ in $M$. A tangent vector to $M$ at the point $p$ is thus an operator $X_p$ that sends each smooth real-valued function $f$ defined around the point $p$ to a real number $X_p[f]$, where this operator satisfies conditions (i), (ii) and (iii) above.

If $X_p$ and $Y_p$ are tangent vectors at the point $p$ then, for any real numbers $\lambda$ and $\mu$, $\lambda X_p + \mu Y_p$ is also a tangent vector at the point $p$, where $(\lambda X_p + \mu Y_p)[f] = \lambda X_p[f] + \mu Y_p[f]$ for all smooth real-valued functions $f$ defined around $p$. Moreover the operations of addition of tangent vectors and of multiplication of tangent vectors by real numbers satisfy all the axioms that must be satisfied by the algebraic operations on a real vector space. It follows that the collection of all tangent vectors at the point $p$ constitutes a real vector space $T_pM$, referred to as the *tangent space* to $M$ at the point $p$.

**Lemma 2.4** *Let $M$ be a smooth manifold, and let $X_p$ be a tangent vector at some point $p$ of $M$. let $c$ be a real-valued function on $M$ that is constant throughout $M$. Then $X_p[c] = 0$.*

**Proof** Let $c_1$ denote the constant function on $M$ with value 1. Then

$$X_p[c_1] = X_p[c_1 \cdot c_1] = 2c_1(p)X_p[c_1] = 2X_p[c_1],$$

and therefore $X_p[c_1] = 0$. Thus if $c = \lambda c_1$ for some $\lambda \in \mathbb{R}$ then $X_p[c] = \lambda X_p[c_1] = 0$. The result follows. ∎

Let $\varphi\colon U \to \mathbb{R}^n$ be a smooth chart defined over an open set $U$ in $M$. Then there are smooth real-valued functions $x^1, x^2, \ldots, x^n$ on $U$ such that

$$\varphi(u) = (x^1(u), x^2(u), \ldots, x^n(u))$$

for all $u \in U$.

Given any real numbers $a^1, a^2, \ldots, a^n$, the operator sending any smooth real-valued function $f$ defined around some point $p$ of $U$ to

$$a^1 \left.\frac{\partial f}{\partial x^1}\right|_p + a^2 \left.\frac{\partial f}{\partial x^2}\right|_p + \cdots + a^n \left.\frac{\partial f}{\partial x^n}\right|_p$$

satisfies conditions (i)–(iii) and therefore represents a tangent vector at $p$ which we denote by

$$a^1 \left.\frac{\partial}{\partial x^1}\right|_p + a^2 \left.\frac{\partial}{\partial x^2}\right|_p + \cdots + a^n \left.\frac{\partial}{\partial x^n}\right|_p .$$

Conversely, we shall show that any tangent vector at $p$ is of this form for suitable real numbers $a^1, \ldots, a^n$. The following lemma establishes the basic result needed to prove this fact.

**Lemma 2.5** *Let $M$ be a smooth manifold of dimension $n$ and let $p$ be a point of $M$. Let $f$ be a smooth function defined over some neighbourhood of the point $p$. Let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system defined around the point $p$. Then there exist smooth functions $g_1, g_2, \ldots, g_n$, defined over some suitable open set $U$ containing the point $p$, such that*

$$f(u) = f(p) + \sum_{i=1}^{n} \left(x^i(u) - x^i(p)\right) g_i(u)$$

*for all $u \in U$. Moreover*

$$g_i(p) = \left.\frac{\partial f}{\partial x^i}\right|_p \qquad \textit{for } i = 1, 2, \ldots, n.$$

**Proof** Without loss of generality, we may assume that $f$ is a real-valued function defined over some open ball $B$ about the origin in $\mathbb{R}^n$. We must show that there exist smooth real-valued functions $g_1, g_2, \ldots, g_n$ on $B$ such that

$$f(\mathbf{x}) = f(\mathbf{0}) + x^1 g_1(\mathbf{x}) + x^2 g_2(\mathbf{x}) + \cdots + x^n g_n(\mathbf{x})$$

for all $\mathbf{x} \in B$. Now

$$f(\mathbf{x}) - f(\mathbf{0}) = \int_0^1 \frac{d}{dt} \left(f(t\mathbf{x})\right) \, dt = \sum_{i=1}^{n} x^i \int_0^1 (\partial_i f)(t\mathbf{x}) \, dt,$$

where $\partial_i f = \dfrac{\partial f}{\partial x^i}$. Let

$$g_i(\mathbf{x}) = \int_0^1 (\partial_i f)(t\mathbf{x}) \, dt$$

for $i = 1, 2, \ldots, n$. Then $g_1, g_2, \ldots, g_n$ satisfy the required conditions. ∎

**Proposition 2.6** *Let $M$ be a smooth manifold of dimension $n$, and let $X_p$ be a tangent vector at some point $p$ of $M$. Let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system around the point $p$. Then*

$$X_p = a^1 \left.\frac{\partial}{\partial x^1}\right|_p + a^2 \left.\frac{\partial}{\partial x^2}\right|_p + \cdots + a^n \left.\frac{\partial}{\partial x^n}\right|_p .$$

*where $a^i = X_p[x^i]$. If $(y^1, y^2, \ldots, y^n)$ is another smooth coordinate system around $p$ then*

$$X_p = b^1 \left.\frac{\partial}{\partial y^1}\right|_p + b^2 \left.\frac{\partial}{\partial y^2}\right|_p + \cdots + b^n \left.\frac{\partial}{\partial y^n}\right|_p ,$$

*where*

$$b^j = \sum_{i=1}^n a^i \left.\frac{\partial y^j}{\partial x^i}\right|_p \qquad (j = 1, 2, \ldots, n).$$

**Proof** Let $f$ be a smooth real-valued function defined around $p$. It follows from Lemma 2.5 that there exist smooth functions $g_1, g_2, \ldots, g_n$ defined around $p$ such that

$$f(u) = f(p) + \sum_{i=1}^n \bigl(x^i(u) - x^i(p)\bigr)\, g_i(u)$$

for all points $u$ belonging to some sufficiently small open set containing $p$. Moreover

$$g_i(p) = \left.\frac{\partial f}{\partial x^i}\right|_p \qquad \text{for } i = 1, 2, \ldots, n.$$

Let $h^i(u) = x^i(u) - x^i(p)$. Now the operator $X_p$ annihilates constant functions, by Lemma 2.4. Therefore $X_p[h^i] = X_p[x^i] = a^i$ for all $i$, and hence

$$X_p[f] = \sum_{i=1}^n \bigl(X_p[h^i]\, g_i(p) + h^i(p)\, X_p[g^i]\bigr) = \sum_{i=1}^n a^i \left.\frac{\partial f}{\partial x^i}\right|_p .$$

If $(y^1, y^2, \ldots, y^n)$ is another smooth coordinate system around $p$ then

$$\frac{\partial f}{\partial x^i} = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i} \frac{\partial f}{\partial y^j},$$

by the Chain Rule for partial derivatives, and therefore

$$X_p = b^1 \left.\frac{\partial}{\partial y^1}\right|_p + b^2 \left.\frac{\partial}{\partial y^2}\right|_p + \cdots + b^n \left.\frac{\partial}{\partial y^n}\right|_p ,$$

where

$$b^j = \sum_{i=1}^{n} a^i \left.\frac{\partial y^j}{\partial x^i}\right|_p \qquad (j = 1, 2, \ldots, n),$$

as required. ∎

**Corollary 2.7** *Let $M$ be a smooth manifold of dimension $n$. Then the tangent space $T_pM$ to $M$ at any point $p$ of $M$ has dimension $n$. Moreover, given any smooth coordinate system $(x^1, x^2, \ldots, x^n)$ around $p$, the tangent vectors*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

*constitute a basis for the tangent space $T_pM$.*

**Proof** It follows immediately from Proposition 2.6 that these tangent vectors span the tangent space $T_pM$. It thus suffices to show that they are linearly independent. Suppose that

$$\sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial x^i}\right|_p = 0$$

for some real numbers $a^1, a^2, \ldots, a^n$. Then

$$0 = \left( \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial x^i}\right|_p \right) [x^j] = \sum_{i=1}^{n} a^i \left.\frac{\partial x^j}{\partial x^i}\right|_p = a^j$$

for $j = 1, 2, \ldots, n$. Thus the tangent vectors

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

are linearly independent, as required. ∎

**Definition** Let $\gamma: I \to M$ be a smooth curve in the smooth manifold $M$, where $I$ is some open interval in $\mathbb{R}$. Then Given $t \in I$, we define the *velocity vector* of the curve $\gamma$ at $\gamma(t)$ to be the tangent vector $\gamma'(t)$ at the point $\gamma(t)$, defined such that

$$\gamma'(t)[f] = \frac{df(\gamma(t))}{dt}$$

for all smooth real-valued functions $f$ whose domain is an open subset of $M$ containing the point $\gamma(t)$.

**Lemma 2.8** *Let $M$ be a smooth manifold, and let $p$ be a point of $M$. Then every tangent vector at $p$ is the velocity vector of some smooth curve passing through the point $p$.*

**Proof** Let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system around the point $p$ chosen such that $x^i(p) = 0$ for $i = 1, 2, \ldots, n$. Let $X_p$ be a tangent vector at the point $p$. It follows from Proposition 2.6 that

$$X_p = a^1 \left.\frac{\partial}{\partial x^1}\right|_p + a^2 \left.\frac{\partial}{\partial x^2}\right|_p + \cdots + a^n \left.\frac{\partial}{\partial x^n}\right|_p,$$

where $a^i = X_p[x^i]$ for $i = 1, 2, \ldots, n$. Let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be the smooth curve in $M$, defined on the open interval $(-\varepsilon, \varepsilon)$ determined by some suitably small positive number $\varepsilon$, which satisfies $x^i(\gamma(t)) = a^i t$ for $i = 1, 2, \ldots, n$ and for all $t \in (-\varepsilon, \varepsilon)$. It follows from the Chain Rule that

$$\gamma'(0)[f] = \left.\frac{df(\gamma(t))}{dt}\right|_{t=0} = \sum_{i=1}^n \left.\frac{\partial f}{\partial x^i}\right|_p \left.\frac{d(x^i(\gamma(t)))}{dt}\right|_{t=0} = \sum_{i=1}^n a^i \left.\frac{\partial f}{\partial x^i}\right|_p = X_p[f]$$

for all smooth real-valued functions $f$ defined around $p$. Thus $\gamma'(0) = X_p$. The result follows. ∎

## 2.4 Cotangent Spaces and Differentials

Let $M$ be a smooth manifold of dimension $n$, let $f\colon M \to \mathbb{R}$ be a smooth real-valued function on $M$, and let $p$ be a point of $M$. Then $f$ determines a linear functional $df_p\colon T_pM \to \mathbb{R}$ on the tangent space $T_pM$ to $M$ at the point $p$, where $df_p(X_p) = X_p[f]$ for all $X_p in T_p M$. Now the linear functionals on $T_pM$ constitute a real vector space $T_p^*M$ which is the dual space of $T_pM$. This vector space $T_p^*M$ is referred to as the *cotangent space* to $M$ at the point $p$, and its elements are often referred to as *covectors*. The linear functional $df_p$ on $T_pM$ is thus an element of the cotangent space $T_p^*M$ to $M$ at the point $p$.

We use the notation $\langle ., . \rangle$ to denote the natural *pairing* between the cotangent space $T_p^*M$ and the tangent space $T_pM$ to $M$ at the point $p$, which is defined such that

$$\langle \theta_p, X_p \rangle = \theta_p(X_p)$$

all $\theta_p \in T_p^*M$ and $X_p \in T_pM$.

**Definition** Let $M$ be a smooth manifold of dimension $n$, let $p$ be a point of $M$, and let $f$ be a smooth real-valued function defined over some open neighbourhood of the point $p$ in $M$. The *differential* $df_p$ of the function $f$ at the point $p$ is the element of the cotangent space $T_p^* M$ which is defined such that

$$\langle df_p, X_p \rangle = df_p(X_p) = X_p[f]$$

for all $X_p \in T_p M$.

**Lemma 2.9** *Let $M$ be a smooth manifold of dimension $n$, let $p$ be a point of $M$, let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system defined over an open neighbourhood of $p$ in $M$, and let $f$ be a smooth real-valued function that is also defined over an open neighbourhood of $p$ in $M$. Then*

$$df_p = \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x^i} \right|_p dx_p^i,$$

*where $df_p$ is the differential of the smooth function $f$ at $p$, and where $dx_p^i$ is the differential of the coordinate function $x^i$ at $p$.*

**Proof** Let $X_p \in T_p M$. Then there exist real numbers $a^1, a^2, \ldots, a^n$ such that

$$X_p = \sum_{j=1}^{n} a^j \left. \frac{\partial}{\partial x^j} \right|_p .$$

It follows from the definition of the differential $df_p$ of $f$ at $p$ that

$$\langle df_p, X_p \rangle = \sum_{j=1}^{n} a^j \left. \frac{\partial f}{\partial x^j} \right|_p .$$

On replacing the smooth function $f$ by the coordinate function $x^i$, we find that

$$\langle dx_p^i, X_p \rangle = \sum_{j=1}^{n} a^j \left. \frac{\partial x^i}{\partial x^j} \right|_p = a^i.$$

It follows that

$$\left\langle \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x^i} \right|_p dx_p^i, \;\; X_p \right\rangle = \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x^i} \right|_p \langle dx_p^i, X_p \rangle = \sum_{i=1}^{n} a^i \left. \frac{\partial f}{\partial x^i} \right|_p = \langle df_p, X_p \rangle.$$

We have thus shown that

$$\left\langle \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x^i} \right|_p dx_p^i, \;\; X_p \right\rangle = \langle df_p, X_p \rangle$$

for all $X_p \in T_pM$. It follows that

$$df_p = \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x^i} \right|_p dx_p^i,$$

as required. ∎

**Lemma 2.10** *Let $M$ be a smooth manifold of dimension $n$, let $p$ be a point of $M$, and let $(x^1, x^2, \ldots, x^n)$ be a smooth coordinate system defined over an open neighbourhood of $p$ in $M$. Then*

$$\left\langle \sum_{i=1}^{n} b_i \, dx_p^i, \ \sum_{j=1}^{n} a^j \left. \frac{\partial}{\partial x^j} \right|_p \right\rangle = \sum_{i=1}^{n} b_i a^i$$

*for all $(a^1, a^2, \ldots, a^n)$, $(b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$.*

**Proof**

$$\begin{aligned} \left\langle \sum_{i=1}^{n} b_i \, dx_p^i, \ \sum_{j=1}^{n} a^j \left. \frac{\partial}{\partial x^j} \right|_p \right\rangle &= \sum_{i=1}^{n} b_i \left\langle dx_p^i, \ \sum_{j=1}^{n} a^j \left. \frac{\partial}{\partial x^j} \right|_p \right\rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} b_i a^j \left. \frac{\partial x^i}{\partial x^j} \right|_p \\ &= \sum_{i=1}^{n} b_i a^i, \end{aligned}$$

as required. ∎

## 2.5 Derivatives of Smooth Maps

**Lemma 2.11** *Let $\varphi\colon M \to N$ be a smooth map between smooth manifolds $M$ and $N$, and let $X_p$ be a tangent vector at some point $p$ of $M$. Let*

$$(\varphi_* X_p)[g] = X_p[g \circ \varphi]$$

*for all smooth real-valued functions $g$ that are defined throughout some open neighbourhood of $\varphi(p)$ in $N$. Then the operator $\varphi_* X_p$ is a tangent vector at $\varphi(p)$.*

**Proof** Let $g$ and $h$ be smooth real-valued functions defined around $\varphi(p)$, and let $\alpha$ and $\beta$ be real numbers. Then

$$\begin{aligned}(\varphi_* X_p)[\alpha g + \beta h] &= X_p[\alpha(g \circ \varphi) + \beta(h \circ \varphi)] = \alpha X_p[g \circ \varphi] + \beta X_p[h \circ \varphi] \\ &= \alpha(\varphi_* X_p)[g] + \beta(\varphi_* X_p)[h], \\ (\varphi_* X_p)[g \cdot h] &= X_p[g \circ \varphi] h(\varphi(p)) + g(\varphi(p)) X_p[h \circ \varphi] \\ &= \varphi_* X_p[g] h(\varphi(p)) + g(\varphi(p))(\varphi_* X_p)[h]\end{aligned}$$

Moreover if the functions $g$ and $h$ agree on some open set in $N$ containing $\varphi(p)$ then the functions $g \circ \varphi$ and $h \circ \varphi$ agree on some open set containing $p$ (since $\varphi\colon M \to N$ is continuous), and therefore $(\varphi_* X_p)[g] = (\varphi_* X_p)[h]$. Thus the operator $\varphi_* X_p$ is a tangent vector at $\varphi(p)$. ∎

**Definition** Let $\varphi\colon M \to N$ be a smooth map between smooth manifolds $M$ and $N$, and let $p$ be a point of $M$. The *derivative* of the smooth map $\varphi$ at the point $p$ is defined to be the linear transformation $\varphi_*\colon T_p M \to T_{\varphi(p)} N$ characterized by the property that

$$(\varphi_* X_p)[g] = X_p[g \circ \varphi]$$

for all smooth real-valued functions $g$ that are defined throughout some open neighbourhood of $\varphi(p)$ in $N$.

**Lemma 2.12** *Let $\varphi\colon M \to N$ be a smooth map between smooth manifolds $M$ and $N$, let $p$ be a point of $M$, let $(x^1, \ldots, x^n)$ be a smooth coordinate system defined throughout some open neighbourhood $U$ of the point $p$ in $M$, let $(y^1, \ldots, y^k)$ be a smooth coordinate system defined throughout some open neighbourhood $V$ of the point $\varphi(p)$ in $N$, and let $F^1, F^2, \ldots, F^k$ be the smooth functions of $n$ real variables, defined throughout some open neighbourhood of the point $(x^1(p), x^2(p), \ldots, x^n(p))$ in $\mathbb{R}^n$, that represent the smooth map $\varphi$ around $p$ with respect to the coordinate systems on $M$ and $N$, so that*

$$y^j(\varphi(u)) = F^j(x^1(u), x^2(u), \ldots, x^n(u))$$

*for $j = 1, 2, \ldots, k$ and for all $u \in U \cap \varphi^{-1}(V)$. Then*

$$\varphi_* \left( \sum_{i=1}^n a^i \left. \frac{\partial}{\partial x^i} \right|_p \right) = \sum_{j=1}^k b^j \left. \frac{\partial}{\partial y^j} \right|_{\varphi(p)},$$

*where*

$$b^j = \sum_{i=1}^n a^i \left. \frac{\partial F^j}{\partial x^i} \right|_p.$$

**Proof** Let $g$ be a smooth real-valued function defined throughout some open neighbourhood of $\varphi(p)$ in $N$. Then

$$\begin{aligned}
\varphi_*\left(\sum_{i=1}^n a^i \left.\frac{\partial}{\partial x^i}\right|_p\right)[g] &= \left(\sum_{i=1}^n a^i \left.\frac{\partial}{\partial x^i}\right|_p\right)[g\circ\varphi] \\
&= \sum_{i=1}^n a^i \left.\frac{\partial (g\circ\varphi)}{\partial x^i}\right|_p \\
&= \sum_{i=1}^n \sum_{j=1}^k a^i \left.\frac{\partial g}{\partial y^j}\right|_{\varphi(p)} \left.\frac{\partial (y^j\circ\varphi)}{\partial x^i}\right|_p \\
&= \sum_{i=1}^n \sum_{j=1}^k a^i \left.\frac{\partial g}{\partial y^j}\right|_{\varphi(p)} \left.\frac{\partial F^j}{\partial x^i}\right|_p \\
&= \sum_{j=1}^k b^j \left.\frac{\partial g}{\partial y^j}\right|_{\varphi(p)},
\end{aligned}$$

where

$$b^j = \sum_{i=1}^n a^i \left.\frac{\partial F^j}{\partial x^i}\right|_p .$$

It follows that

$$\varphi_*\left(\sum_{i=1}^n a^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_{j=1}^k b^j \left.\frac{\partial}{\partial y^j}\right|_{\varphi(p)},$$

as required. ∎

Let $\varphi: M \to N$ be a smooth map between smooth manifolds $M$ and $N$, and let $p$ be a point of $M$. Lemma 2.12 shows that the derivative $\varphi_*: T_pM \to T_{\varphi(p)}N$ of the map $\varphi$ at the point $p$ is represented, with respect to the bases of the tangent spaces $T_pM$ and $T_{\varphi(p)}N$ determined by smooth coordinate systems around $p$ and $\varphi(p)$, by the Jacobian matrix of the smooth map between open sets in Euclidean spaces that represents the map $\varphi$ around $p$ with respect to these coordinate systems defined around $p$ and $\varphi(p)$.

# 3 Submanifolds of Smooth Manifolds

## 3.1 Submanifolds

**Definition** Let $M$ be a smooth manifold of dimension $n$, let $S$ be a subset of $M$, and let $r$ be an integer satisfying $0 \leq r \leq n$. The subset $S$ of $M$ is said to be a *smooth submanifold* of $M$ of dimension $r$ if, given any point $p$ of $S$, there exists a smooth chart $(U, \varphi)$ for $M$, defined over some open set $U$ in $M$, such that $p \in U$ and

$$U \cap S = U \cap \varphi^{-1}(P_r),$$

where

$$P_r = \{(s^1, s^2, \ldots, s^n) \in \mathbb{R}^n : s^j = 0 \text{ when } r < j \leq n\}.$$

**Lemma 3.1** *Let $M$ be a smooth manifold of dimension $n$, let $S$ be a subset of $M$, and let $r$ be an integer satisfying $0 \leq r \leq n$. The subset $S$ is a smooth $r$-dimensional submanifold of $M$ if and only if, given any point $p$ of $S$, there exists an open set $U$ and a smooth coordinate system $y^1, y^2, \ldots, y^n$ for $M$ defined over $U$ such that*

$$U \cap S = \{u \in U : y^j(u) = 0 \text{ when } r < j \leq n\}.$$

**Proof** Let $p$ be a point of $M$. Let $(U, \varphi)$ be a smooth chart for $M$ and let

$$\varphi(u) = (y^1(u), y^2(u), \ldots, y^n(u))$$

for all $u \in U$. Then $U \cap S = U \cap \varphi^{-1}(P_r)$, where

$$P_r = \{(s^1, s^2, \ldots, s^n) \in \mathbb{R}^n : s^j = 0 \text{ when } r < j \leq n\},$$

if and only if $S$ is the subset of $U$ which is the set of common zeros of the coordinate functions $y^{r+1}, \ldots, y^n$ on $U$. The result follows. ∎

**Proposition 3.2** *Let $M$ be a smooth manifold of dimension $n$ and let $S$ be a smooth submanifold of $M$ of dimension $r$. Then $S$ is itself a smooth manifold of dimension $r$. The topology on this smooth manifold $S$ is the subspace topology. A smooth coordinate system for $S$ around a point $p$ of $S$ may be obtained by taking a coordinate system $y^1, y^2, \ldots, y^n$ for $M$ defined over some open subset $U$ of $M$, where $p \in U$ and*

$$U \cap S = \{u \in U : y^j(u) = 0 \text{ when } r < j \leq n\},$$

*and then restricting the coordinate functions $y^1, y^2, \ldots, y^r$ to $U \cap S$ so as to obtain a coordinate system for $S$ defined around the point $p$.*

**Proof** A subset of a Hausdorff space is itself a Hausdorff space (with the subspace topology). Also there is some countable collection of open sets in $M$ that is a basis for the topology on $M$. (Every open set in $M$ is then a countable union of open sets belonging to this countable basis.) The intersection of the open sets in this countable basis with the submanifold $S$ constitute a countable basis for the subspace topology on $S$. Now any point $p$ of $S$, there exists a smooth chart $(U, \varphi)$ for $M$ such that $U \cap S = U \cap \varphi^{-1}(P_r)$. Then the restriction $\varphi_S$ of $\varphi: U \to \mathbb{R}^n$ maps $U \cap S$ homeomorphically onto an open subset of $P_r$. We conclude that $S$ is a topological manifold of dimension $r$.

Moreover there is a smooth atlas for $M$ consisting of those continuous charts that are the restrictions $(U \cap S, \varphi_S)$ to $S$ of smooth charts $(U, \varphi)$ for $M$ that satisfy $U \cap S = U \cap \varphi^{-1}(P_r)$. Indeed if $(U, \varphi)$ and $(V, \psi)$ are smooth charts for $M$, and if $U \cap S = U \cap \varphi^{-1}(P_r)$ and $V \cap S = V \cap \psi^{-1}(P_r)$, then the charts $(U, \varphi)$ and $(V, \psi)$ are smoothly compatible, and therefore their restrictions to $S$ are also smoothly compatible. Thus $S$, with the subspace topology, and with the smooth structure determined by the smooth atlas just described, as a smooth manifold of dimension $r$. ∎

## 3.2 The Inverse Function Theorem

**Theorem 3.3** Inverse Function Theorem. *Let $\varphi: U \to \mathbb{R}^n$ be a smooth function defined over some open set $U$ of $\mathbb{R}^n$, and let $\mathbf{p} \in U$. Suppose that the Jacobian matrix representing the derivative of $\varphi$ at the point $\mathbf{p}$ is non-singular. Then there exists an open set $W$ in $\mathbb{R}^n$, where $\mathbf{p} \in W$ and $W \subset U$, which is mapped diffeomorphically by $\varphi$ onto an open set $\varphi(W)$ in $\mathbb{R}^n$.*

We do not present a proof of this theorem here. The following two results (Theorem 3.4 and Corollary 3.5) are results concerning smooth manifolds and smooth maps that are equivalent to the Intermediate Value Theorem.

**Theorem 3.4** *Let $M$ and $N$ be smooth manifolds of the same dimension, let $\varphi: M \to N$ be a smooth map, and let $p \in M$. Suppose that the derivative $\varphi_*: T_pM \to T_{\varphi(p)}N$ of $\varphi$ at the point $p$ is an isomorphism. Then there exists an open set $W$ in $M$, where $p \in W$, which is mapped diffeomorphically by $\varphi$ onto an open set $\varphi(W)$ in $N$.*

**Proof** Let $n$ be the common dimension of $M$ and $N$, let $x^1, x^2, \ldots, x^n$ be a smooth local coordinate system for the smooth manifold $M$, defined throughout some open neighbourhood of the point $p$, and let $y^1, y^2, \ldots, y^n$ be a smooth local coordinate system for the smooth manifold $N$ defined around the point $\varphi(p)$. Then there exist smooth functions $F^1, F^2, \ldots, F^n$, defined around the point $(x^1(p), x^2(p), \ldots, x^n(p)$, such that

$$y^j(\varphi(u)) = F^j(x^1(u), x^2(u), \ldots, x^n(u))$$

for all points $u$ of some sufficiently small open neighbourhood of $U$ in $p$. But then

$$\varphi_* \left( \left. \frac{\partial}{\partial x^k} \right|_p \right) = \sum_{j=1}^n \left. \frac{\partial F^j}{\partial x^k} \right|_p \left. \frac{\partial}{\partial y^j} \right|_{\varphi(p)}$$

(see Lemma 2.12). Thus if $\varphi_*: T_pM \to T_{\varphi(p)}N$ is an isomorphism then the Jacobian matrix

$$\left. \frac{\partial F^j}{\partial x^k} \right|_p$$

is non-singular. The result then follows on applying the Inverse Function Theorem (Theorem 3.3).

**Corollary 3.5** *Let $M$ be a smooth manifold of dimension $n$, let $U$ be an open set in $M$, and let $x^1, x^2, \ldots, x^n$ be smooth real-valued functions defined over $U$. Suppose that the differentials*

$$dx^1_p,\ dx^2_p, \ldots,\ dx^n_p$$

*of these functions at some point* **p** *of $M$ are linearly independent and thus constitute a basis of the cotangent space $T^*_pM$ to $M$ at the point $p$. Then there exists an open set $W$ in $M$, where $p \in W$, such that the restrictions of the smooth functions $x^1, x^2, \ldots, x^n$ to $W$ constitute a smooth coordinate system defined over $M$.*

**Proof** Let $\varphi: U \to \mathbb{R}^n$ be defined such that

$$\varphi(p) = (x^1(p), x^2(p), \ldots, x^n(p))$$

for all $p \in U$, and let $y^1, y^2, \ldots, y^n$ be a smooth coordinate system for $M$ defined around $p$. Then

$$dx^j_p = \sum_{k=1}^n \left. \frac{\partial x^j}{\partial y^k} \right|_p dy^k_p.$$

(see Lemma 2.9). Now the differentials

$$dy^1_p,\ \ dy^2_p, \ldots,\ \ dy^n_p$$

constitute a basis of the cotangent space $T_p^*M$ to $M$ at the point $p$. It follows that if the differentials

$$dx_p^1,\ dx_p^2, \ldots,\ dx_p^n$$

also constitute a basis of this vector space then the Jacobian matrix with entries $\left.\dfrac{\partial x^j}{\partial y^k}\right|_p$ is a non-singular matrix. It then follows from the Inverse Function Theorem (Theorem 3.3) that the map expressing $x^1, x^2, \ldots, x^n$ in terms of $y^1, y^2, \ldots, y^n$ maps some open neighbourhood of $(y^1(p), y^2(p), \ldots, y^m(p))$ diffeomorphically onto some open neighbourhood of $(x^1(p), x^2(p), \ldots, x^m(p))$ in $\mathbb{R}^n$. It follows from this that the restrictions of the smooth real-valued functions $x^1, x^2, \ldots, x^n$ to some sufficiently small open neighbourhood $W$ of $p$ are the components of some smooth chart for $M$ with domain $W$, as required. ∎

## 3.3 Zero Sets of Smooth Functions

**Proposition 3.6** *Let $M$ be a smooth manifold of dimension $n$, and let*

$$S = \{p \in M : f^j(p) = 0 \text{ for } j = 1, 2, \ldots, k\},$$

*where $f^1, f^2, \ldots, f^k$ are smooth real-valued functions on $M$. Suppose that, given any point $p$ of $S$, the differentials*

$$df_p^1,\ df_p^2, \ldots,\ df_p^k$$

*of $f_1, f_2, \ldots, f_k$ at the point $p$ are linearly independent elements of the cotangent space $T_p^*M$. Then $S$ is a smooth submanifold of $M$ of dimension $n - k$.*

**Proof** Let $p$ be a point of $S$, and let $x^1, x^2, \ldots, x^n$ be smooth local coordinates defined around the point $p$. Then the differentials

$$dx_p^1,\ dx_p^2, \ldots,\ dx_p^n$$

constitute a basis of the cotangent space. Now the differentials

$$df_p^1,\ df_p^2, \ldots,\ df_p^k$$

are linearly independent. It follows from basic linear algebra that there exist distinct indices $i_1, i_2, \ldots, i_{n_k}$ such that the differentials

$$dx_p^{i_1},\ ,dx_p^{i_2}, \ldots,\ dx_p^{i_{n-k}},\ df_p^1,\ df_p^2, \ldots,\ df_p^k$$

constitute a basis of the cotangent space $T_p^*M$. We may relabel the smooth coordinate functions $x^1, x^2, \ldots, x^n$ so that $i_j = j$ for $j = 1, 2, \ldots, n - k$.

It then follows from Corollary 3.5 that there exists an open set $W$, where $p \in W$, such that the restrictions of the smooth functions

$$x^1, x^2, \ldots, x^{n-k}, f^1, f^2, \ldots, f^k$$

This proves that $S$ is a smooth submanifold of $M$, of dimension $n - k$ (see Lemma 3.1). ∎

**Proposition 3.7** *Let $M$ and $N$ be smooth manifolds of dimensions $m$ and $n$ respectively, let $\varphi\colon M \to N$ be a smooth map, let $Q$ be a smooth submanifold of $N$ of dimension $k$, and let $P = \varphi^{-1}(Q)$. Suppose that*

$$T_{\varphi(p)}N = \varphi_*(T_pM) + T_{\varphi(p)}Q$$

*for all $p \in P$. Then $P$ is a smooth submanifold of $M$ of dimension $m+k-n$.*

**Proof** Let $p_0$ be a point of $P$. It follows from the definition of submanifolds that there exists an open set $U$ in $N$, where $\varphi(p_0) \in U$, and smooth functions $g^1, g^2, \ldots, g^{n-k}$ on $U$ such that

$$dg^1, dg^2, \ldots, dg^{n-k}$$

are linearly independent at each point of $U$ and

$$U \cap Q = \{u \in U : g^i(u) = 0 \text{ for } i = 1, 2, \ldots, n-k\}.$$

Let $f^i$ be the smooth real-valued function on $\varphi^{-1}(U)$ defined such that $f^i = g^i \circ \varphi$. Then

$$P \cap \varphi^{-1}(U) = \{v \in \varphi^{-1}(U) : f^i(v) = 0 \text{ for } i = 1, 2, \ldots, n-k\}.$$

We now show that the differentials

$$df^1, df^2, \ldots, df^{n-k}$$

are linearly independent at each point $p$ of $P \cap \varphi^{-1}(U)$. Now the differentials $dg^1, dg^2, \ldots, dg^{n-k}$ are linearly independent at $\varphi(p)$ and therefore there exist tangent vectors

$$(Z_1)_{\varphi(p)}, (Z_1)_{\varphi(p)}, \ldots (Z_{n-k})_{\varphi(p)}$$

to $N$ at $\varphi(p)$ which satisfy

$$\langle dg^i_{\varphi(p)}, (Z_j)_{\varphi(p)} \rangle = \delta^i_j,$$

where $\delta^i_j$ is the Kronecker delta, equal to 1 when $i = j$, and equal to zero otherwise. Now $T_{\varphi(p)}N = \varphi_*(T_pM) + T_{\varphi(p)}Q$. It follows that there exist tangent vectors

$$(X_1)_p, (X_1)_p, \ldots (X_{n-k})_p$$

to $M$ at $p$ and tangent vectors

$$(Y_1)_{\varphi(p)}, (Y_1)_{\varphi(p)}, \ldots (Y_{n-k})_{\varphi(p)}$$

to $N$ at $\varphi(p)$ such that

$$(Z_i)_{\varphi(p)} = \varphi_*(X_i)_p + (Y_i)_{\varphi(p)} \quad \text{for } i = 1, 2, \ldots, n-k.$$

Then

$$\delta^i_j = \langle dg^i_{\varphi(p)}, (Z_j)_{\varphi(p)}\rangle = \langle dg^i_{\varphi(p)}, \varphi_*(X_j)_p\rangle + \langle dg^i_{\varphi(p)}, (Y_j)_{\varphi(p)}\rangle = \langle df^i_p, (X_j)_p\rangle$$

for $i, j = 1, 2, \ldots, n-k$, because

$$\langle dg^i_{\varphi(p)}, \varphi_*(X_j)_p\rangle = \langle d(g \circ \varphi)^i_p, (X_j)_p\rangle = \langle df^i_p, (X_j)_p\rangle$$

and

$$\langle dg^i_{\varphi(p)}, \varphi_* Y_{\varphi(p)}\rangle = 0 \text{ for all } Y_{\varphi(p)} \in T_{\varphi(p)}N.$$

Thus if $c_1, c_2, \ldots, c_{n-k}$ are real numbers which satisfy $\sum_{i=1}^{n-k} c_i \, df^i_p = 0$ then

$$0 = \sum_{i=1}^{n-k} c_i \, \langle df^i_p, (X_j)_p = \sum_{i=1}^{n-k} c_i \delta^i_j = c_j$$

for $j = 1, 2, \ldots, n-k$. Thus the differentials $df^1, df^2, \ldots, df^{n-k}$ are linearly independent at the point $p$ of $P \cap \varphi^{-1}(U)$. It then follows from Proposition 3.6 that $P \cap \varphi^{-1}(U)$ is a submanifold of $M$ of dimension $m - n + k$. We have thus shown that each point $p_0$ of $P$ has an open neighbourhood $V$ in $M$ such that $P \cap V$ is a submanifold of $M$ of dimension $m - n + k$. It then follows from the definition of submanifolds that $P$ is itself a submanifold of $M$ of dimension $m - n + k$, as required. ∎

**Corollary 3.8** *Let $M$ and $N$ be smooth manifolds of dimensions $m$ and $n$ respectively, let $\varphi: M \to N$ be a smooth map, let $q$ be a point of $N$, and let $P = \varphi^{-1}(\{q\})$. Suppose that the derivative $\varphi_*: T_pM \to T_qN$ of $\varphi$ is surjective at each point $p$ of $P$. Then $P$ is a smooth submanifold of $M$ of dimension $m - n$.*

# 4 Fibre Bundles and Vector Bundles

## 4.1 Introduction to Tangent Bundles

Let $M$ be a smooth manifold of dimension $n$. Given a point $p$ of $M$, the tangent space $T_pM$ to $M$ at $p$ is defined to be the real vector space of dimension $n$ whose elements are operators $X_p$ that associate a real number $X_p[f]$ to each smooth real-valued function $f$ defined around $p$, and that satisfy the following three conditions:—

(i) $X_p[\alpha f+\beta g] = \alpha X_p[f]+\beta X_p[g]$ for all real numbers $\alpha$ and $\beta$ and smooth functions $f$ and $g$ defined around the point $p$;

(ii) $X_p[f \cdot g] = X_p[f]\, g(p) + f(p)\, X_p[g]$ for all smooth functions $f$ and $g$ defined around the point $p$;

(iii) if $f$ and $g$ are smooth real-valued functions defined around $p$, and if $f|V = g|V$ for some open set $V$ that contains the point $p$, then $X_p[f] = X_p[g]$.

(Here $f \cdot g$ denotes the product of the functions $f$ and $g$, defined such that $(f \cdot g)(m) = f(m)g(m)$ for all $m \in M$, and $f|V$ and $g|V$ denote the restrictions of the functions $f$ and $g$ to the open set $V$.)

If $(x^1, x^2, \ldots, x^n)$ is a smooth local coordinate system defined around the point $p$, and if $X_p$ is a tangent vector at the point $p$, then there exist real numbers $v^1, v^2, \ldots, v^n$ such that

$$X_p = v^1 \left.\frac{\partial}{\partial x^1}\right|_p + v^2 \left.\frac{\partial}{\partial x^2}\right|_p + \cdots + v^n \left.\frac{\partial}{\partial x^n}\right|_p .$$

Then

$$X_p[f] = v^1 \left.\frac{\partial f}{\partial x^1}\right|_p + v^2 \left.\frac{\partial f}{\partial x^2}\right|_p + \cdots + v^n \left.\frac{\partial f}{\partial x^n}\right|_p$$

for all smooth functions $f$ defined around the point $p$. Moreover the function from $\mathbb{R}^n$ to $T_pM$ that sends each element $(v^1, v^2, \ldots, v^n)$ of $\mathbb{R}^n$ to the tangent vector $\displaystyle\sum_{j=1}^n v^j \left.\frac{\partial}{\partial x^j}\right|_p$ is an isomorphism of real vector spaces.

Let $p$ and $q$ be points of $M$ where $p \neq q$. Then the tangent spaces $T_pM$ and $T_qM$ are disjoint, since the elements of $T_pM$ and $T_qM$ are operators acting on smooth real-valued functions defined around the points $p$ and $q$ respectively. Let $TM$ be the union $\bigcup_{p\in M} T_pM$ of all the tangent spaces of the smooth manifold $M$. Then there is a surjective function $\pi_{TM}: TM \to M$, where $\pi_{TM}(X_p) = p$ for all $p \in M$ and $X_p \in T_pM$. Note that $\pi_{TM}^{-1}(\{p\}) = T_pM$ for all $p \in M$.

Now $M$ is a smooth manifold, and therefore there exists a smooth atlas for $M$ which we can represent as a collection $((U_\alpha, \varphi_\alpha) : \alpha \in A)$ of smooth charts for $M$, indexed by some set $A$. The domains of the charts in this smooth atlas cover the manifold $M$ and thus $M = \bigcup_{\alpha \in A} U_\alpha$. Moreover, for each $\alpha \in A$, the function $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ maps the open set $U_\alpha$ diffeomorphically onto an open set in $\mathbb{R}^n$, and therefore there are smooth real-valued functions $x^1_{[\alpha]}, x^2_{[\alpha]}, \ldots, x^n_{[\alpha]}$ defined over $U_\alpha$ such that

$$\varphi_\alpha(p) = (x^1_{[\alpha]}(p), x^2_{[\alpha]}(p), \ldots, x^n_{[\alpha]}(p))$$

for all $p \in U_\alpha$. The partial derivative operators determined by this smooth coordinate system then determine a basis

$$\left.\frac{\partial}{\partial x^1_{[\alpha]}}\right|_p, \quad \left.\frac{\partial}{\partial x^2_{[\alpha]}}\right|_p, \quad \ldots, \quad \left.\frac{\partial}{\partial x^n_{[\alpha]}}\right|_p$$

for the tangent space $T_pM$ at each point $p$ of $U_\alpha$. It follows that the smooth chart $(U_\alpha, \varphi_\alpha)$ determines a function $\psi_\alpha \colon U_\alpha \times \mathbb{R}^n \to TM$ from $U_\alpha \times \mathbb{R}^n$ to $TM$, where

$$\psi_\alpha(p, (v^1, v^2, \ldots, v^n)) = v^1 \left.\frac{\partial}{\partial x^1_{[\alpha]}}\right|_p + v^2 \left.\frac{\partial}{\partial x^2_{[\alpha]}}\right|_p + \cdots + v^n \left.\frac{\partial}{\partial x^n_{[\alpha]}}\right|_p$$

for all $p \in U_\alpha$ and $(v^1, v^2, \ldots, v^n) \in \mathbb{R}^n$. Then $\pi_{TM}(\psi_\alpha(p, \mathbf{v})) = p$ for all $p \in U_\alpha$ and $\mathbf{v} \in \mathbb{R}^n$. Moreover the function $\psi_\alpha$ determines a one-to-one correspondence between elements of the set $U_\alpha \times \mathbb{R}^n$ and elements of the set $\pi_{TM}^{-1}(U_\alpha)$ of tangent vectors to $M$ at points of $U_\alpha$.

Let $p$ be a point of $U_\alpha \cap U_\beta$ for some $\alpha, \beta \in A$, and let $X_p$ be a tangent vector to $M$ at $p$. Then there exist real numbers $v^1, v^2, \ldots, v^n$ and $w^1, w^2, \ldots, w^n$ such that

$$X_p = \psi_\alpha(p, (w^1, w^2, \ldots, w^n)) = \sum_{i=1}^n w^i \left.\frac{\partial}{\partial x^i_{[\alpha]}}\right|_p$$

and

$$X_p = \psi_\beta(p, (v^1, v^2, \ldots, v^n)) = \sum_{j=1}^n v^j \left.\frac{\partial}{\partial x^j_{[\beta]}}\right|_p.$$

Now

$$\frac{\partial}{\partial x^j_{[\beta]}} = \sum_{k=1}^n \frac{\partial x^k_{[\alpha]}}{\partial x^j_{[\beta]}} \frac{\partial}{\partial x^k_{[\alpha]}},$$

and therefore

$$\sum_{k=1}^{n} w^i \left.\frac{\partial}{\partial x^i_{[\alpha]}}\right|_p = X_p = \sum_{i=1}^{n}\sum_{j=1}^{n} v^j \left.\frac{\partial x^i_{[\alpha]}}{\partial x^j_{[\beta]}}\right|_p \left.\frac{\partial}{\partial x^i_{[\alpha]}}\right|_p .$$

It follows that

$$w^i = \sum_{j=1}^{n} (J_{\alpha\beta}(p))^i_j v^j,$$

and thus $\mathbf{w} = J_{\alpha\beta}(p)(\mathbf{v})$, where

$$\mathbf{v} = (v^1, v^2, \ldots, v^n) \quad \text{and} \quad \mathbf{w} = (w^1, w^2, \ldots, w^n)$$

and where $J_{\alpha\beta}(p)\colon \mathbb{R}^n \to \mathbb{R}^n$ is the invertible linear operator on $\mathbb{R}^n$ represented by the non-singular $n \times n$ matrix whose entry $(J_{\alpha\beta}(p))^i_j$ in the $i$th row and $j$th column is the value of the partial derivative $\dfrac{\partial x^i_{[\alpha]}}{\partial x^j_{[\beta]}}$ at the point $p$. Thus

$$\psi_\beta(p, \mathbf{v}) = X_p = \psi_\alpha(p, \mathbf{w}) = \psi_\alpha(p, J_{\alpha\beta}(p)(\mathbf{v})).$$

We deduce from this that

$$\psi_\beta(p, \mathbf{v}) = \psi_\alpha(p, \tau_{\alpha\beta}(p, \mathbf{v}))$$

for all $p \in U_\alpha \cap U_\beta$ and $\mathbf{v} \in \mathbb{R}^n$, where $\tau_{\alpha\beta}\colon (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \mathbb{R}^n$ is the smooth map defined such that

$$\tau_{\alpha\beta}(p, \mathbf{v}) = J_{\alpha\beta}(p)(\mathbf{v})$$

for all $p \in U_\alpha \cap U_\beta$ and $\mathbf{v} \in \mathbb{R}^n$.

Let us now summarize the main features of the configuration that we have been investigating. Given a smooth manifold $M$, we have constructed a set $TM$ and a surjective function $\pi_{TM}\colon TM \to M$. The elements of the set $TM$ are the tangent vectors to the smooth manifold $M$, and the function $\pi_{TM}$ is defined such that $\pi_{TM}(X_p) = p$ for all $p \in M$ and for all $X_p \in T_pM$. Also the subset $\pi_{TM}^{-1}(\{p\})$ of $\{p\}$ is a real vector space for all $p \in M$.

Given a smooth atlas for $M$, represented by an indexed family $((U_\alpha, \varphi_\alpha) : \alpha \in A)$ of smooth charts for $M$, we have constructed functions $\psi_\alpha\colon U_\alpha \times \mathbb{R}^n \to TM$ that satisfy $\pi_{TM}(\psi_\alpha(p, \mathbf{v})) = p$ for all $p \in U_\alpha$ and $\mathbf{v} \in \mathbb{R}^n$. Moreover, given any point $p$ of $U_\alpha$, the function from $\mathbb{R}^n$ to $T_pM$ that sends $\mathbf{v} \in \mathbb{R}^n$ to $\psi_\alpha(p, \mathbf{v})$ is an isomorphism of real vector spaces. We have also shown that, given $\alpha, \beta \in A$ for which $U_\alpha \cap U_\beta$ is non-empty, there exists a smooth map $\tau_{\alpha\beta}: (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\psi_\beta(p, \mathbf{v}) = \psi_\alpha(p, \tau_{\alpha\beta}(p, \mathbf{v}))$$

for all $p \in U_\alpha \cap U_\beta$ and $\mathbf{v} \in \mathbb{R}^n$. Moreover the function sending $\mathbf{v}$ to $\tau(p, \mathbf{v})$ is an invertible linear operator on $\mathbb{R}^n$ for all for all $p \in U_\alpha \cap U_\beta$.

We have not so far introduced any topology or smooth structure on the set $TM$ of tangent vectors. However one can define a natural topology on the set $TM$ which gives this set the structure of a topological manifold of dimension $2n$, where $n$ is the dimension of the smooth manifold $M$. Moreover this manifold admits a natural smooth atlas which gives it the structure of a smooth manifold. Let $TM$ therefore be regarded as a smooth manifold, with this natural topology and smooth structure. Then the surjective function $\pi_{TM}: TM \to M$ is a smooth function, and, for each $\alpha \in A$, the function $\psi_\alpha: U_\alpha \times \mathbb{R}^n \to TM$ is smooth, and moreover it maps its domain $U_\alpha \times \mathbb{R}^n$ diffeomorphically onto the open subset $\pi_{TM}^{-1}(U_\alpha)$ of $TM$. The smooth manifold $TM$ and the smooth map $\pi_{TM}: TM \to M$ constitute the *tangent bundle* of the smooth manifold $M$.

It is not difficult to describe a collection of charts on $TM$ that constitutes a smooth atlas defining the smooth structure on $TM$. Indeed let $\varphi: U \to \mathbb{R}^n$ be a smooth chart, mapping some subset $U$ of $M$ diffeomorphically onto an open set in $\mathbb{R}^n$, and let $x^1, x^2, \ldots, x^n$ be the corresponding coordinate functions on $U$, where

$$\varphi_\alpha(p) = (x^1_{[\alpha]}(p), x^2_{[\alpha]}(p), \ldots, x^n_{[\alpha]}(p))$$

for all $p \in U$. Then there is a function $\tilde{\varphi}: \pi_{TM}^{-1}(U) \to \mathbb{R}^{2n}$ defined such that

$$\tilde{\varphi}\left( \sum_{i=1}^n v^i \left. \frac{\partial}{\partial x^i} \right|_p \right) = (x^1(p), x^2(p), \ldots, x^n(p), v^1, v^2, \ldots, v^n)$$

for all $p \in U$ and $(v^1, v^2, \ldots, v^n) \in \mathbb{R}^n$. This function $\tilde{\varphi}$ is then a smooth chart for the smooth manifold $TM$ defined over $\pi_{TM}^{-1}(U)$, where $TM$ is provided with natural topology and smooth structure referred to above.

Now we could continue this discussion of the tangent bundle of a smooth manifold by introducing directly the definition of the natural topology and smooth structure on the set $TM$ and proving that $TM$, with this topology and smooth structure, is indeed a smooth manifold of dimension $2m$. However the tangent bundle of a smooth manifold is just one example of a smooth *vector bundle* over that manifold. It makes sense therefore to establish the existence and properties of the topology and smooth structure on $TM$ as a special case of a more general result concerning vector bundles. Moreover a *vector bundle* is a special type of *fibre bundle*. We therefore develop the theory of smooth fibre bundles, which can then be applied to vector bundles in general and, in particular, to the tangent bundle of a smooth manifold.

## 4.2 Products of Smooth Manifolds

Let $M$ and $N$ be topological manifolds of dimensions $m$ and $n$ respectively. The Cartesian product $M \times N$ of $M$ and $N$ carries a topology, known as the *product topology*, which is determined by the topologies on $M$ and $N$. A subset of $M \times N$ is open with respect to the product topology if and only if it is a union of subsets of $M \times N$ that are of the form $U \times V$, where $U$ is open in $M$ and $V$ is open in $N$. Thus if $U$ and $V$ are open sets in $M$ and $N$ respectively, then $U \times V$ is an open set in $M \times N$. An open set in $M \times N$ need not itself be of the form $U \times V$, where $U$ and $V$ are open in $M$ and $N$ respectively, but it will always be a union of sets of this form.

**Proposition 4.1** *Let $M$ and $N$ be topological manifolds of dimensions $m$ and $n$ respectively. Then the Cartesian product $M \times N$ of $M$ and $N$, with the product topology, is a topological manifold of dimension $m + n$.*

**Proof** First we verify that $M \times N$ is a Hausdorff space. Now $M$ and $N$ are both Hausdorff spaces. Let $(p_1, q_1)$ and $(p_2, q_2)$ be points of $M \times N$. Suppose that $(p_1, q_1) \neq (p_2, q_2)$. Then either $p_1 \neq p_2$ or else $q_1 \neq q_2$. If $p_1 \neq p_2$, then there exist open sets $U_1$ and $U_2$ in $M$ such that $p_1 \in U_1$, $p_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$, because $M$ is a Hausdorff space. But then $(p_1, q_1) \in U_1 \times N$, $(p_2, q_2) \in U_2 \times N$, the sets $U_1 \times N$ and $U_2 \times N$ are open in $M \times N$, and $(U_1 \times N) \cap (U_2 \times N) = \emptyset$. Similarly if $q_1 \neq q_2$ then there exist open sets $V_1$ and $V_2$ in $N$ such that $q_1 \in V_1$, $q_2 \in V_2$ and $V_1 \cap V_2 = \emptyset$, because $N$ is a Hausdorff space. But then $(p_1, q_1) \in M \times V_1$, $(p_2, q_2) \in M \times V_2$, the sets $M \times V_1$ and $M \times V_2$ are open in $M \times N$, and $(M \times V_1) \cap (M \times V_2) = \emptyset$. We conclude that the product space $M \times N$ is indeed a Hausdorff space.

The definition of topological manifolds also ensures that there are countable collections $\mathcal{C}$ and $\mathcal{D}$ of open sets in $M$ and $N$ respectively which cover these manifolds, where each of the open sets in the collection $\mathcal{C}$ is homeomorphic to an open set in $\mathcal{R}^m$, where $m = \dim M$, and where each of the open sets in the collection $\mathcal{D}$ is homeomorphic to an open set in $\mathcal{R}^n$, where $n = \dim N$. The collection of subsets of $M \times N$ that are of the form $U \times V$, where $U \in \mathcal{C}$ and $V \in \mathcal{D}$, is then a countable collection of open sets which covers the product space $M \times N$, and moreover $U \times V$ is homeomorphic to an open set in $\mathbb{R}^{m+n}$ for all $U \in \mathcal{C}$ and $V \in \mathcal{D}$. Thus the product $M \times N$ of the topological manifolds $M$ and $N$ it itself a topological manifold of dimension $m+n$, as required. ∎

**Lemma 4.2** *Let $M$ and $N$ be topological manifolds of dimensions $m$ and $n$ respectively, let $\mathcal{A}$ be a smooth atlas on $M$, let $\mathcal{B}$ a smooth atlas on $N$, and let $\mathcal{C}$ be the collection of charts on the product manifold $M \times N$ of the form $(U \times V, \varphi \times \psi)$, where $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$, and where $\varphi \times \psi: U \times V \to \mathbb{R}^m \times \mathbb{R}^n$ is defined such that $(\varphi \times \psi)(u, v) = (\varphi(u), \psi(v))$ for all $u \in U$ and $v \in V$. Then $\mathcal{C}$ is a smooth atlas on $M \times N$.*

**Proof** The product space $\mathbb{R}^m \times \mathbb{R}^n$ is isomorphic, as a real vector space, to $\mathbb{R}^{m+n}$, and moreover the natural isomorphism between these real vector spaces of dimension $m+n$ is also a homeomorphism. Each map $(U \times V, \varphi \times \psi)$ belonging to the collection $\mathcal{C}$ is a continuous chart mapping the open set $U \times V$ in $M \times V$ homeomorphically onto an open set in $\mathbb{R}^m \times \mathbb{R}^n$. The domains of these charts cover the product space $M \times N$. It only remains to verify that any two continuous charts in the atlas $\mathcal{C}$ are smoothly compatible.

Let $(U_1 \times V_1, \varphi_1 \times \psi_1)$ and $(U_2 \times V_2, \varphi_2 \times \psi_2)$ be charts belonging to the atlas $\mathcal{C}$ on $M \times N$, where $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ are smooth charts on $M$ belonging to the atlas $\mathcal{A}$, and where $(V_1, \psi_1)$ and $(V_2, \psi_2)$ are smooth charts on $N$ belonging to the atlas $\mathcal{B}$. The charts $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ are also smoothly compatible, and therefore there exists a diffeomorphism $\sigma: \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ such that $\sigma(\varphi_1(u)) = \varphi_2(u)$ for all $u \in U_1 \cap U_2$. Similarly there exists a diffeomorphism $\tau: \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2)$ such that $\tau(\psi_1(v)) = \psi_2(v)$ for all $v \in V_1 \cap V_2$. Let $(\sigma \times \tau)(\mathbf{x}, \mathbf{y}) = (\sigma(\mathbf{x}), \tau(\mathbf{y}))$ for all $\mathbf{x} \in \varphi_1(U_1 \cap U_2)$ and for all $\mathbf{y} \in \psi_1(V_1 \cap V_2)$. Then

$$\sigma \times \tau: \varphi_1(U_1 \cap U_2) \times \psi_1(V_1 \cap V_2) \to \varphi_2(U_1 \cap U_2) \times \psi_2(V_1 \cap V_2)$$

is a diffeomorphism. This diffeomorphism is the transition function between the continuous charts $(U_1 \times V_1, \varphi_1 \times \psi_1)$ and $(U_2 \times V_2, \varphi_2 \times \psi_2)$ on $M \times N$. Therefore these continuous charts belonging to $\mathcal{C}$ are smoothly compatible. It follows that $\mathcal{C}$ is a smooth atlas on $M \times N$. ∎

Let $M$ and $N$ be smooth manifolds of dimension $m$ and $n$ respectively. It follows from Proposition 4.1 and Lemma 4.2 that the Cartesian product $M \times N$ may be regarded as a smooth manifold of dimension $m + n$. The topology on this smooth manifold is the product topology. Let $U$ be an open set in $M$ which is the domain of a smooth chart represented by smooth local coordinate functions $x^1, x^2, \ldots, x^m$. Also let $V$ be an open set in $N$ which is the domain of a smooth chart represented by smooth local coordinate functions $y^1, y^2, \ldots, y^n$. Let $z^1, z^2, \ldots, z^{m+n}$ be the real-valued functions on $U \times V$ defined such that

$$z^j(u, v) = \begin{cases} x^j(u) & \text{if } 1 \leq j \leq m; \\ y^{j-m}(v) & \text{if } m+1 \leq j \leq m+n. \end{cases}$$

Then $U \times V$ is the domain of a smooth chart on $M \times N$ represented by the smooth local coordinate functions $z^1, z^2, \ldots, z^{m+n}$.

## 4.3 Topological Fibre Bundles

**Definition** Let $F$, $E$ and $B$ be a topological spaces, and let $\pi_E: E \to B$ be a continuous surjective map. The topological space $E$ and the continuous map $\pi_E: E \to B$ constitute a *topological fibre bundle* over $B$ with *total space* $E$, *base space* $B$, *fibre* $F$ and *projection map* $\pi_E: E \to B$ provided that, given any point $p$ of $B$, there exists an open set $U$ containing $p$ and a continuous map $\psi: U \times F \to E$ which satisfies the following conditions:

(i) the function $\psi$ maps $U \times F$ homeomorphically onto $\pi_E^{-1}(U)$;

(ii) $\pi_E(\psi(u, f)) = u$ for all $u \in U$ and $f \in F$;

**Proposition 4.3** *Let $B$ and $F$ be topological spaces, let $E$ be a set, let $\pi_E: E \to B$ be a surjective function, let $(U_\alpha : \alpha \in A)$ be collection of open sets in $B$ indexed by a set $A$, and, for all $\alpha, \beta \in A$, let $\psi_\alpha: U_\alpha \times F \to E$ and $\tau_{\alpha\beta}: (U_\alpha \cap U_\beta) \times F \to F$ be functions that satisfy the following conditions:—*

(i) $\bigcup_{\alpha \in A} U_\alpha = B$,

(ii) $\pi_E(\psi_\alpha(u, f)) = u$ *for all* $\alpha \in A$, $u \in U_\alpha$ *and* $f \in F$;

(iii) *the function* $\psi_\alpha: U_\alpha \times F \to E$ *maps* $U \times F$ *bijectively onto* $\pi_E^{-1}(U_\alpha)$ *for all* $\alpha \in A$;

(iv) $\psi_\beta(u, f) = \psi_\alpha(u, \tau_{\alpha\beta}(u, f))$ *for all* $\alpha, \beta \in A$, $u \in U_\alpha \cap U_\beta$ *and* $f \in F$;

(v) *the function* $\tau_{\alpha\beta}: (U_\alpha \cap U_\beta) \times F \to F$ *is continuous for all* $\alpha, \beta \in A$.

*Then there exists a topology on the set $E$ characterized by the property that, for each $\alpha \in A$, the function $\psi_\alpha: U_\alpha \times F \to E$ maps $U_\alpha \times F$ homeomorphically onto $\pi_E^{-1}(U_\alpha)$. The surjective function $\pi_E: E \to B$ is continuous with respect to this topology, and the topological space $E$ and the continuous map $\pi_E: E \to B$ together constitute a topological fibre bundle over the topological space $B$.*

**Proof** We first define a topology on $E$. The open sets in $E$ in this topology are those subsets $W$ of $E$ with the property that $\psi_\alpha^{-1}(W)$ is open in $U_\alpha \times F$ for all $\alpha \in A$. Note that the empty set and the whole set are open in $E$. For each $\alpha \in A$ the preimage under $\psi_\alpha$ of any union of open sets in $E$ is the union of those preimages and is thus a union of open sets in $U_\alpha \times F$. But any union of open sets in a topological space is an open set. It follows that the preimage of any union of open sets in $E$ under each function $\psi_\alpha: U_\alpha \times F \to E$ is an open set in $U_\alpha \times F$. Therefore any union of open sets in $E$ is itself an open set. Also, for each $\alpha \in A$, the preimage under $\psi_\alpha$ of any finite intersection of open sets in $E$ is the intersection of those preimages and is thus an open set in $U_\alpha \times F$, and therefore any finite intersection of open sets in $E$ is itself an open set. We have thus verified that there is a well-defined topology on $E$, where a subset $W$ of $E$ is open in $E$ if and only if $\psi_\alpha^{-1}(W)$ is open in $U_\alpha \times F$ for all $\alpha \in A$.

The composition function $\pi_E \circ \psi_\alpha: U_\alpha \times F \to U$ is continuous, since

$$\pi_E(\psi_\alpha(u, f)) = u$$

for all $u \in U_\alpha$. It follows that $\psi_\alpha^{-1}(\pi_E^{-1}(V))$ is open in $U_\alpha \times F$ for every open set $V$ in $B$. The definition of the topology on $E$ then ensures that $\pi_E^{-1}(U)$ is open in the topological space $E$ for every open set $V$ in $B$. We conclude from this that the function $\pi_E: E \to B$ is continuous.

Next we show that, for each $\alpha \in A$, the function $\psi_\alpha: U_\alpha \times F \to E$ maps its domain $U_\alpha \times F$ homeomorphically onto $\pi_E^{-1}(U_\alpha)$. Now the definition of the topology on $E$ ensures that the function $\psi_\alpha$ is continuous. This function also maps $U_\alpha \times F$ bijectively onto $\pi_E^{-1}(U_\alpha)$. A continuous bijection is a homeomorphism if and only if it maps open sets to open sets. Thus, in order to prove that $\psi_\alpha$ maps $U_\alpha \times F$ homeomorphically onto its image in $E$, it only remains to show that $\psi_\alpha(V)$ is open in $E$ for every open set $V$ in $U_\alpha \times F$.

Now, given $\alpha, \beta \in A$, there is a continuous map $\tau_{\alpha\beta}: (U_\alpha \cap U_\beta) \times F \to F$ such that $\psi_\beta(u, f) = \psi_\alpha(u, \tau_{\alpha\beta}(u, f))$ for all $u \in U_\alpha \cap U_\beta$ and $f \in F$. The map $\tau_{\alpha\beta}$ then determines a continuous map $\chi_{\alpha\beta}: (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ from $(U_\alpha \cap U_\beta) \times F$ to itself, where $\chi(u, f) = (u, \tau_{\alpha\beta}(u, f))$ for all $u \in U_\alpha \cap U_\beta$ and $f \in F$. Now if $(u, f) \in U_\beta \times F$, $(u', f') \in U_\alpha \times F$ and if $\psi_\beta(u, f) = \psi_\alpha(u', f')$ then

$$u = \pi_E(\psi_\beta(u, f)) = \pi_E(\psi_\alpha(u', f')) = u'$$

and

$$\psi_\alpha(u, f') = \psi_\beta(u, f) = \psi_\alpha(u, \tau_{\alpha\beta}(u, f)),$$

and therefore $u \in U_\alpha \cap U_\beta$ and $f' = \tau_{\alpha\beta}(u, f)$. Thus if $V$ is an open set in $U_\alpha \times F$ then

$$\psi_\beta^{-1}(\psi_\alpha(V)) = \{(u, f) \in (U_\alpha \cap U_\beta) \times F : (u, \tau_{\alpha\beta}(u, f)) \in V\} = \chi_{\alpha\beta}^{-1}(V)$$

for all $\beta \in A$, and therefore $\psi_\beta^{-1}(\psi_\alpha(V))$ is an open set in $U_\beta \times F$ for all $\beta \in A$. It follows from the definition of the topology on $E$ that $\psi_\alpha(V)$ is an open set in $E$ for all $\alpha \in A$ and for all open sets $V$ in $U_\alpha \times F$. Thus the continuous injective function $\psi_\alpha: U_\alpha \times F \to E$ maps open subsets of $U_\alpha \times F$ to open subsets of $E$. This function therefore maps $U_\alpha \times F \to E$ homeomorphically onto its range $\pi_E^{-1}(U_\alpha)$. ∎

**Lemma 4.4** *Let $B$ and $F$ be Hausdorff spaces, let $E$ be a topological space, and let $\pi_E: E \to B$ be a topological fibre bundle over $B$ with fibre $F$ and total space $E$. Then the total space $E$ of this fibre bundle is a Hausdorff space.*

**Proof** Let $e_1$ and $e_2$ be points of $E$, where $e_1 \neq e_2$. Suppose that $\pi_E(e_1) \neq \pi_E(e_2)$. The topological space $B$ is a Hausdorff space. Therefore there exist open sets $V_1$ and $V_2$ in $B$ such that $\pi_E(e_1) \in V_1$, $\pi_E(e_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Now we have already shown that the function $\pi_E: E \to B$ is continuous. It follows that $\pi_E^{-1}(V_1)$ and $\pi_E^{-1}(V_2)$ are open sets in $E$. Moreover $e_1 \in \pi_E^{-1}(V_1)$, $e_2 \in \pi_E^{-1}(V_2)$ and $\pi_E^{-1}(V_1) \cap \pi_E^{-1}(V_2) = \emptyset$. Next suppose that $e_1 \neq e_2$ but $\pi_E(e_1) = \pi_E(e_2)$. Then there exists some open set $U$ in $B$ containing the point $p$, where $p = \pi_E(e_1) = \pi_E(e_2)$, and a continuous map $\psi: U \times F \to E$ from $U \times F$ to $E$ that maps $U \times F$ homeomorphically onto $\pi_E^{-1}(U)$ and satisfies $\pi_E(\psi(u, f)) = u$ for all $u \in U$ and $f \in F$. Then there exist $f_1, f_2 \in F$ such that $e_1 = \psi(p, f_1)$ and $e_2 = \psi(p, f_2)$. Moreover $f_1 \neq f_2$. Also the fibre $F$ of the bundle is a Hausdorff space. It follows that there exist open sets $W_1$ and $W_2$ in $F$ such that $f_1 \in W_1$, $f_2 \in W_2$ and $W_1 \cap W_2 = \emptyset$. Then $e_1 \in \psi(U \times W_1)$, $e_2 \in \psi(U \times W_2)$ and $\psi(U \times W_1) \cap \psi(U \times W_2) = \emptyset$. Moreover $\psi(U \times W_1)$ and $\psi(U \times W_2)$ are open sets in $E$. We have thus shown that, given $e_1, e_2 \in E$, where $e_1 \neq e_2$, there exist two disjoint open sets in $E$, where one of these open sets contains the point $e_1$ and the other contains $e_2$. It follows that $E$ is a Hausdorff space, as required. ∎

## 4.4 Smooth Fibre Bundles

**Definition** Let $F$, $E$ and $M$ be a smooth manifolds, and let $\pi_E: E \to M$ be a smooth surjective map. The smooth manifold $E$ and the smooth map $\pi_E: E \to M$ constitute a *smooth fibre bundle* over $M$ with *total space* $E$, *base space* $M$, *fibre* $F$ and *projection map* $\pi_E: E \to M$ provided that, given any point $p$ of $M$, there exists an open set $U$ containing $p$ and a smooth map $\psi: U \times F \to E$ which satisfies the following conditions:

(i) the function $\psi$ maps $U \times F$ diffeomorphically onto $\pi_E^{-1}(U)$;

(ii) $\pi_E(\psi(u, f)) = u$ for all $u \in U$ and $f \in F$;

**Proposition 4.5** *Let $M$ and $F$ be topological manifolds, let $E$ be a topological space, and let $\pi_E: E \to M$ be a topological fibre bundle over $M$ with fibre $F$ and total space $E$. Then the total space $E$ of this fibre bundle is a topological manifold of dimension $n + k$, where $n = \dim M$ and $k = \dim F$.*

**Proof** The topological manifolds $M$ and $F$ are Hausdorff spaces. It follows from Lemma 4.4 that the total space $E$ of the fibre bundle is a Hausdorff space. In order to show that $E$ is a topological manifold of dimension $n+k$, where $n = \dim M$ and $k = \dim F$, we must show that $E$ can be covered by a countable collection of open sets, where each of these open sets is homeomorphic to an open set in a Euclidean space of dimension $n + k$, where $n = \dim M$ and $k = \dim F$.

Now there exists a countable basis that generates the topology of the topological manifold $M$. This countable basis is by definition a countable collection of open sets in $M$, and any open set in $M$ is a union of subsets that belong to the countable basis. Also there exists a collection $(U_\alpha : \alpha \in A)$ of open sets in $M$ that covers $M$, where $\pi_E^{-1}(U_\alpha)$ is homeomorphic to $U_\alpha \times F$ for all $\alpha \in A$. Now each open set $U_\alpha$ in the indexed collection $(U_\alpha : \alpha \in A)$ is a union of open sets that belong to the countable basis. It follows that there exists an infinite sequence of open sets $B_1, B_2, B_3, \ldots$ in $M$ that belong to the countable basis, and a corresponding infinite sequence $\alpha(1), \alpha(2), \alpha(3), \ldots$ of elements of the indexing set $A$ such that $M = \bigcup_{i=1}^{+\infty} B_i$ and $B_i \subset U_{\alpha(i)}$ for all positive integer $i$. Then $M = \bigcup_{i=1}^{+\infty} U_{\alpha(i)}$. Now, for each positive integer $i$, the preimage $\pi_E^{-1}(U_{\alpha(i)})$ of $U_{\alpha(i)}$ in $E$ is an open subset of $E$ that is homeomorphic to the product manifold $U_\alpha \times F$. This product manifold has dimension $n+k$. It follows that, for each positive integer $i$, the open subset $\pi_E^{-1}(U_{\alpha(i)})$ of $E$ is a union of a countable collection of open sets in $E$, where each of open sets in the collection is homeomorphic to an open set in $\mathbb{R}^{n+k}$. Therefore $E$ can itself be covered by a countable collection of open sets, where each of the open sets in the collection is homeomorphic to an open set in $\mathbb{R}^{n+k}$. This completes the proof that $E$ is a topological manifold of dimension $n+k$. ∎

**Proposition 4.6** *Let $M$ and $F$ be smooth manifolds, let $E$ be a set, let $\pi_E: E \to M$ be a surjective function, let $(U_\alpha : \alpha \in A)$ be collection of open sets in $M$ indexed by a set $A$, and, for all $\alpha, \beta \in A$, let $\psi_\alpha: U_\alpha \times F \to E$ and $\tau_{\alpha\beta}: (U_\alpha \cap U_\beta) \times F \to F$ be functions that satisfy the following conditions:—*

(i) $\bigcup_{\alpha \in A} U_\alpha = M$,

(ii) $\pi_E(\psi_\alpha(u, f)) = u$ *for all* $\alpha \in A$, $u \in U_\alpha$ *and* $f \in F$;

(iii) *the function* $\psi_\alpha: U_\alpha \times F \to E$ *maps* $U \times F$ *bijectively onto* $\pi_E^{-1}(U_\alpha)$ *for all* $\alpha \in A$*;*

(iv) $\psi_\beta(u, f) = \psi_\alpha(u, \tau_{\alpha\beta}(u, f))$ *for all* $\alpha, \beta \in A$*,* $u \in U_\alpha \cap U_\beta$ *and* $f \in F$*;*

(v) *the function* $\tau_{\alpha\beta}: (U_\alpha \cap U_\beta) \times F \to F$ *is smooth for all* $\alpha, \beta \in A$*.*

*Then there exists a topology and smooth structure on the set* $E$ *with respect to which* $E$ *is a smooth manifold,* $\pi_E: E \to M$ *is a smooth map and the function* $\psi_\alpha: U_\alpha \times F \to E$ *maps* $U_\alpha \times F$ *diffeomorphically onto* $\pi_E^{-1}(U_\alpha)$ *for all* $\alpha \in A$*. The smooth manifold* $E$ *and the smooth map* $\pi_E: E \to M$ *then constitute a smooth fibre bundle over the smooth manifold* $M$*. Moreover* $\dim E = n + k$*, where* $n = \dim M$ *and* $k = \dim F$*.*

**Proof** It follows from Proposition 4.3 and Proposition 4.5 that there exists a topology on the set $E$ with respect to which $E$ is a topological manifold, $\pi_E: E \to M$ is a continuous map and the function $\psi_\alpha: U_\alpha \times F \to E$ maps $U_\alpha \times F$ homeomorphically onto $\pi_E^{-1}(U_\alpha)$ for all $\alpha \in A$. The topological manifold $E$ and the continuous map $\pi_E: E \to M$ then constitute a topological fibre bundle over the topological manifold $M$. Moreover $\dim E = n+k$, where $n = \dim M$ and $k = \dim F$.

Let $\mathcal{A}$ be the collection consisting of all continuous charts $\varphi: V \to \mathbb{R}^{n+k}$, where $V$ is some open set in $E$, and where, for each $\alpha \in A$, the composition function that sends $(u, f)$ to $\varphi(\psi_\alpha(u, f))$ for all $(u, f) \in \psi_\alpha^{-1}(V)$ is a smooth chart for $U_\alpha \times F$ defined over the open set $\psi_\alpha^{-1}(V)$. We show that $\mathcal{A}$ is a smooth atlas on $E$.

Let $\alpha \in A$, and let $(W, \xi)$ be a smooth chart for the smooth manifold $U_\alpha \times F$. The domain $W$ of this chart is then an open set in $U_\alpha \times F$. Moreover the function $\psi_\alpha: U_\alpha \times F \to E$ maps this open set homeomorphically onto an open set $V$ in $E$, where $W = \psi_\alpha(U_\alpha \times F)$ in $E$. It follows that there is a continuous chart $(V, \varphi)$ for $E$, defined over the open set $V$, which is characterized by the property that $\varphi(\psi_\alpha(w)) = \xi(w)$ for all $(u, f) \in W$. Let $\beta \in A$. Then

$$\varphi(\psi_\beta(u, f)) = \varphi(\psi_\alpha(u, \tau_{\alpha\beta}(u, f))) = \xi(u, \tau_{\alpha\beta}(u, f))$$

for all $(u, f) \in \psi_\beta^{-1}(V)$. The smoothness of the maps $\tau_{\alpha\beta}: (U_\alpha \cap U_\beta) \times F \to F$ and $\xi: W \to \mathbb{R}^{n+k}$ then ensures that $\varphi(\psi_\beta(u, f)$ is a smooth function of $(u, f)$ on $\psi_\beta^{-1}(V)$. We conclude that the continuous chart $(V, \varphi)$ belongs to the collection $\mathcal{A}$. It follows from this that the domains of the continuous charts belonging to the collection $\mathcal{A}$ cover the topological manifold $E$. Moreover it follows easily from the definition of $\mathcal{A}$ that any two continuous charts for the topological manifold $E$ that belong to the atlas $\mathcal{A}$ are smoothly compatible. It follows that The atlas $\mathcal{A}$ is a smooth atlas. It is in fact a maximal smooth atlas, and thus gives the total space $E$ of the topological vector bundle $\pi_E: E \to M$ the structure of a smooth manifold. Moreover, for each $\alpha \in A$, the map $\psi_\alpha: U_\alpha \times F \to E$ is smooth and maps its domain $U_\alpha \times F$ diffeomorphically onto an open set in $E$. Moreover the projection map $\pi_E: E \to M$ is smooth, since its composition with $\psi_\alpha$ is the smooth map sending $(u, f)$ to $u$ for all $u \in U_\alpha$ and $f \in F$. We have therefore shown that $\pi_E: E \to M$ is a smooth fibre bundle, as required. ∎

**Remark** Note that conditions (i)–(iv) are identical in the statements of Propositions 4.3 and 4.6. With regard to condition (v) of those propositions, we note that continuity of the functions $\tau_{\alpha\beta}$ is the basic requirement in order to obtain a topological fibre bundle, whereas smoothness of those functions is the basic requirement in order to obtain a smooth vector bundle.