Modules MA3427 and MA3428: Annual Examination
Course outline and worked solutions

David R. Wilkins

February 14, 2013

Course Website

The module websites, with online lecture notes, problem sets. etc. are located at

```
http://www.maths.tcd.ie/~dwilkins/Courses/MA3427/
http://www.maths.tcd.ie/~dwilkins/Courses/MA3428/
```

Course Outline: MA3427

Course Outline: MA3428

1. (a) [Definition.] Let X and $\tilde{X}$ be topological spaces and let $p: \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p: \tilde{X} \to X$ is said to be a *covering map* if $p: \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) [Bookwork.] The map $p: \tilde{X} \to X$ is a covering map; therefore there exists an open cover $\mathcal{U}$ of X such that each open set U belonging to X is evenly covered by the map p. Now the collection consisting of the preimages $\gamma^{-1}(U)$ of the open sets U belonging to $\mathcal{U}$ is an open cover of the interval $[0,1]$. But $[0,1]$ is compact, by the Heine-Borel Theorem. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that every subinterval of length less than δ is mapped by γ into one of the open sets belonging to $\mathcal{U}$. Partition the interval $[0,1]$ into subintervals $[t_{i-1}, t_i]$, where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, and where the length of each subinterval is less than δ. Then each subinterval $[t_{i-1}, t_i]$ is mapped by γ into some open set in X that is evenly covered by the map p. It follows that once $\tilde{\gamma}(t_{i-1})$ has been determined, we can extend $\tilde{\gamma}$ continuously over the ith subinterval $[t_{i-1}, t_i]$. Indeed suppose that $\gamma([t_{i-1}, t_i]) \subset U$ and that U is evenly covered. Then $p^{-1})(U)$ is a disjoint union of open sets in $\tilde{X}$, where each of these open sets is mapped by p homeomorphically onto U. One of these sets contains the point $\gamma(t_{i-1})$: let that open set be $\tilde{U}$. Then there exists a continuous map $s: U \to \tilde{U}$ that inverts the restriction of the covering map p to $\tilde{U}$. We can then define $\tilde{\gamma}(t) = s(\gamma(t))$ for all $t \in [t_{i-1}, t_i]$. Thus by extending $\tilde{\gamma}$ successively over $[t_0, t_1]$, $[t_1, t_2]$,..., $[t_{n-1}, t_n]$, we can lift the path $\gamma: [0,1] \to X$ to a path $\tilde{\gamma}: [0,1] \to \tilde{X}$ starting at w.

(c) [Not bookwork, though special cases such as the helicoidal covering of the punctured plane are discussed extensively in the lecture notes.]

Let $t_0 \in \mathbb{R}$, let

$$J_{t_0} = \{t \in \mathbb{R} : |t - t_0| < \frac{1}{2}, \}$$

and let $A_{t_0} = q(J_{t_0})$, where $q: \mathbb{R} \to S^1$ is defined such that $q(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in \mathbb{R}$. Then there exists a unique map $s_{t_0}: A_{t_0} \to J_{t_0}$ such that $t = s_{t_0}(q(t))$ for all $t \in J_{t_0}$. Let U_{t_0} be the subset of X defined such that $U_{t_0} = h^{-1}(A_{t_0})$. Then $U_{t,0}$ is an open set in X, and $p^{-1}(U_{t_0}) = \bigcup_{n \in \mathbb{Z}} \tilde{U}_{t_0,n}$, where

$$\tilde{U}_{t_0,n} = \{(x,t) \in \tilde{X} : x \in U_{t_0} \quad \text{and} \quad t = s_{t_0}(h(x)) + n\}.$$

Each set $\tilde{U}_{t_0}$ is an open set in $\tilde{X}$ that is mapped homeomorphically onto U_{t_0} by the map $p \colon \tilde{X} \to X$. It follows that the open set U_{t_0} in X is evenly covered by the map $p \colon \tilde{X} \to X$. Moreover, given any $x \in X$, there exists $t_0 \in \mathbb{R}$ such that $h(x) = q(t_0)$. Then $x \in U_{t_0}$. Thus the open sets U_{t_0} for $t_0 \in \mathbb{R}$ cover the topological space X. We have thus verified that $p \colon \tilde{X} \to X$ is a covering map.

2. (a) [Bookwork.] Let X and Y be topological spaces, and let A be a subset of X. Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps from X to some topological space Y, where $f|A = g|A$ (i.e., $f(a) = g(a)$ for all $a \in A$). We say that f and g are *homotopic relative to* A (denoted by $f \simeq g$ rel A) if and only if there exists a (continuous) homotopy $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$ and $H(a,t) = f(a) = g(a)$ for all $a \in A$.

(b) [Standard definition, but not stated exactly as below in lecture notes.] Let X be a topological space, let x_0 be some chosen point of X, and let $\pi_1(X, x_0)$ be the set of all based homotopy classes of loops based at the point x_0, where two loops γ_1 and γ_2 are in the same based homotopy class if and only if $\gamma_1 \simeq \gamma_2$ rel $\{0,1\}$. Then $\pi_1(X, x_0)$ is a group, the group multiplication on $\pi_1(X, x_0)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1.\gamma_2]$ for all loops γ_1 and γ_2 based at x_0, where $\gamma_1.\gamma_2$ denotes the concatenation of the loops γ_1 and γ_2. This group is the *fundamental group* of X based at the point x_0. The identity element of the fundamental loop is represented by the constant loop at the basepoint x_0. The inverse of a loop $\gamma\colon [0,1] \to X$ is represented by the loop $\gamma^{-1}\colon [0,1] \to X$, where $\gamma^{-1}(t) = \gamma(1-t)$ for all $t \in [0,1]$.

(c) [Stated without proof in the lecture notes.] Let γ_1 and γ_2 be loops in X that start and end at the basepoint x_0. Suppose that $[\gamma_1] = [\gamma_2]$ in $\pi_1(X, x_0)$. Then $\gamma_1 \simeq \gamma_2$ rel $\{0,1\}$, and thus there exists a homotopy $H\colon [0,1] \times [0,1] \to X$, where $H(t,0) = \gamma_1(t)$, $H(t,1) = \gamma_2(t)$ $H(0,\tau) = x_0$ and $H(1,\tau) = x_0$ for all $t, \tau \in [0,1]$. But then $f \circ H$ is a homotopy between the loops γ_1 and γ_2. Moreover $f(H(0,\tau)) = y_0$ and $f(H(1,\tau)) = y_0$ for all $\tau \in [0,1]$. Thus $f \circ \gamma_1 \simeq f \circ \gamma_2$ rel $\{0,1\}$, and therefore $[f \circ \gamma_1] = [f \circ \gamma_2]$ in $\pi_1(Y, y_0)$. Thus $f\colon X \to Y$ induces a well-defined function from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$. This function is a homomorphism, because

$$f_\#([\gamma_1.\gamma_2]) = [f\circ(\gamma_1.\gamma_2)] = [(f\circ\gamma_1).(f\circ\gamma_2)] = [f\circ\gamma_1][f\circ\gamma_2] = f_\#([\gamma_1])f_\#([\gamma_2]).$$

for all loops γ_1 and γ_2 based at x_0.

(d) [Not bookwork. There are several reasonably obvious approaches to the details.] We choose the basepoint within X to be the point , where $= (1,0,0)$. Let

$$r(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

for all $(x, y, z) \in X$. Then the line segment joining (x, y, z) to $r(x, y, z)$ is contained within the set X for all (x, y, z) in X. Let $H((x, y, z), t) = (1 - t)(x, y, z) + tr(x, y, z)$ for all $(x, y, z) \in X$. Then the map H is a homotopy between the identity map $I_X \colon X \to X$ and the map r. Moreover $I_X \simeq r$ rel $\{\}$, and therefore $\gamma \simeq r \circ \gamma$ rel $\{0, 1\}$ for all loops γ in X based at x_0. It follows that the induced homomorphism $r_\# \colon \pi_1(X,) \to \pi_1(r(X),)$ is the inverse of the homomorphism induced by the inclusion map from $r(X),$ to X. But the topological space $r(X)$ is a circle. It follows that $\pi_1(X,) \cong \pi_1(r(X),) \cong \mathbb{Z}$.

3. (a) [Bookwork.] Let $\gamma\colon [0,1] \to X/G$ be a loop in the orbit space with $\gamma(0) = \gamma(1) = q(x_0)$. It follows from the Path Lifting Theorem for covering maps that there exists a unique path $\tilde\gamma\colon [0,1] \to X$ for which $\tilde\gamma(0) = x_0$ and $q \circ \tilde\gamma = \gamma$. Now $\tilde\gamma(0)$ and $\tilde\gamma(1)$ must belong to the same orbit, since $q(\tilde\gamma(0)) = \gamma(0) = \gamma(1) = q(\tilde\gamma(1))$. Therefore there exists some element g of G such that $\tilde\gamma(1) = \theta_g(x_0)$. This element g is uniquely determined, since the group G acts freely on X. Moreover the value of g is determined by the based homotopy class $[\gamma]$ of γ in $\pi_1(X/G, q(x_0))$. Indeed it follows from a basic result (stated on the examination paper) that if σ is a loop in X/G based at $q(x_0)$, if $\tilde\sigma$ is the lift of σ starting at x_0 (so that $q \circ \tilde\sigma = \sigma$ and $\tilde\sigma(0) = x_0$), and if $[\gamma] = [\sigma]$ in $\pi_1(X/G, q(x_0))$ (so that $\gamma \simeq \sigma$ rel $\{0,1\}$), then $\tilde\gamma(1) = \tilde\sigma(1)$. We conclude therefore that there exists a well-defined function

$$\lambda\colon \pi_1(X/G, q(x_0)) \to G,$$

which is characterized by the property that $\tilde\gamma(1) = \theta_{\lambda([\gamma])}(x_0)$ for any loop γ in X/G based at $q(x_0)$, where $\tilde\gamma$ denotes the unique path in X for which $\tilde\gamma(0) = x_0$ and $q \circ \tilde\gamma = \gamma$.

Now let $\alpha\colon [0,1] \to X/G$ and $\beta\colon [0,1] \to X/G$ be loops in X/G based at x_0, and let $\tilde\alpha\colon [0,1] \to X$ and $\tilde\beta\colon [0,1] \to X$ be the lifts of α and β respectively starting at x_0, so that $q\circ\tilde\alpha = \alpha$, $q\circ\tilde\beta = \beta$ and $\tilde\alpha(0) = \tilde\beta(0) = x_0$. Then $\tilde\alpha(1) = \theta_{\lambda([\alpha])}(x_0)$ and $\tilde\beta(1) = \theta_{\lambda([\beta])}(x_0)$. Then the path $\theta_{\lambda([\alpha])} \circ \tilde\beta$ is also a lift of the loop β, and is the unique lift of β starting at $\tilde\alpha(1)$. Let $\alpha.\beta$ be the concatenation of the loops α and β, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then the unique lift of $\alpha.\beta$ to X starting at x_0 is the path $\sigma\colon [0,1] \to X$, where

$$\sigma(t) = \begin{cases} \tilde\alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \theta_{\lambda([\alpha])}(\tilde\beta(2t-1)) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

It follows that

$$\begin{aligned} \theta_{\lambda([\alpha][\beta])}(x_0) &= \theta_{\lambda([\alpha.\beta])}(x_0) = \sigma(1) = \theta_{\lambda([\alpha])}(\tilde\beta(1)) \\ &= \theta_{\lambda([\alpha])}(\theta_{\lambda([\beta])}(x_0)) = \theta_{\lambda([\alpha])\lambda([\beta])}(x_0) \end{aligned}$$

and therefore $\lambda([\alpha][\beta]) = \lambda([\alpha])\lambda([\beta])$. Therefore the function

$$\lambda\colon \pi_1(X/G, q(x_0)) \to G$$

is a homomorphism.

Let $g \in G$. Then there exists a path α in X from x_0 to $\theta_g(x_0)$, since the space X is path-connected. Then $q \circ \alpha$ is a loop in X/G based at $q(x_0)$, and $g = \lambda([q \circ \alpha])$. This shows that the homomorphism λ is surjective.

(b) [Bookwork.] Let $\gamma\colon [0, 1] \to X/G$ be a loop in X/G based at $q(x_0)$. Suppose that $[\gamma] \in \ker \lambda$. Then $\tilde{\gamma}(1) = \theta_e(x_0) = x_0$, and therefore $\tilde{\gamma}$ is a loop in X based at x_0. Moreover $[\gamma] = q_\#[\tilde{\gamma}]$, and therefore $[\gamma] \in q_\#(\pi_1(X, x_0))$. On the other hand, if $[\gamma] \in q_\#(\pi_1(X, x_0))$ then $\gamma = q \circ \tilde{\gamma}$ for some loop $\tilde{\gamma}$ in X based at x_0. But then $x_0 = \tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(x_0)$, and therefore $\lambda([\gamma]) = e$, where e is the identity element of G. Thus $\ker \lambda = q_\#(\pi_1(X, x_0))$, as required.

4. (a) [Not Bookwork.]

$$\begin{aligned}
&(ui + vj)(xi + yj + zk)(ui + vj) \\
&\quad = (uxi^2 + uyij + uzik + vxji + vyj^2 + vzjk)(ui + vj) \\
&\quad = (-ux - vy + vzi - uzj + (uy - vx)k)(ui - vj) \\
&\quad = -u^2xi - uvyi + uvzi^2 - u^2zji + (u^2y + uvx)ki \\
&\qquad\quad - uvxj - v^2yj + v^2zij - uvzj^2 + (uvy - v^2x)kj \\
&\quad = -u^2xi - uvyi - uvz + u^2zk + (u^2y - uvx)j \\
&\qquad\quad - uvxj - v^2yj + v^2zk + uvz - (uvy - v^2x)i \\
&\quad = -(u^2 - v^2)xi - 2uvyi - 2uvxj + (u^2 - v^2)yj + (u^2 + v^2)zk
\end{aligned}$$

(b) [Not Bookwork.]

$$\begin{aligned}
R(t)(x, y, z) &= 2(x \cos \pi t + y \sin \pi t)(\cos \pi t, \sin \pi t, 0) - (x, y, z) \\
&= (\hat{x}(t), \hat{y}(t), \hat{z}(t))
\end{aligned}$$

where

$$\begin{aligned}
\hat{x}(t) &= x(2 \cos^2 \pi t - 1) + 2y \sin \pi t \cos \pi t, \\
\hat{y}(t) &= 2x \sin \pi t \cos \pi t + y(2 \sin^2 \pi t - 1), \\
\hat{z}(t) &= -z
\end{aligned}$$

But

$$\begin{aligned}
\hat{x}(t) &= R_{11}(t)x + R_{12}(t)y + R_{13}(t)z \\
\hat{y}(t) &= R_{21}(t)x + R_{22}(t)y + R_{23}(t)z \\
\hat{z}(t) &= R_{31}(t)x + R_{32}(t)y + R_{33}(t)z.
\end{aligned}$$

It follows that

$$\begin{aligned}
R(t) &= \begin{pmatrix} 2\cos^2 \pi t - 1 & 2 \sin \pi t \cos \pi t & 0 \\ 2 \sin \pi t \cos \pi t & 2 \sin^2 \pi t - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \cos 2\pi t & \sin 2\pi t & 0 \\ \sin 2\pi t & -\cos 2\pi t & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{aligned}$$

We now find $q(t)$. Suppose that we can express $q(t)$ in the form $q(t) = u(t)i + v(t)j$. Then the functions $u(t)$ and $v(t)$ must satisfy

$u(t)^2 + v(t)^2 = 1$. Also $\overline{q(t)} = -q(t)$. It then follows from (a) that $u(t)$ and $v(t)$ must satisfy

$$R_{11}(t) = -R_{22}(t) = u(t)^2 - v(t)^2, \quad R_{12}(t) = R_{21}(t) = 2u(t)v(t),$$

$$R_{33}(t) = -(u(t)^2 + v(t)^2) = -1,$$

$$R_{13}(t) = R_{23}(t) = R_{31}(t) = R_{32}(t) = 0.$$

These requirements are satisfied on taking $u(t) = \cos \pi t$ and $v(t) = \sin \pi t$. Thus the required continuous map sends $t \in [0,1]$ to $q(t)$, where

$$q(t) = (\cos \pi t)i + (\sin \pi t)j.$$

(c) [Not Bookwork.] There does not exist such a continuous map from D to SO(3). Were such a map to exist, it would follow that the loop $\gamma\colon [0,1] \to SO(3)$ defined such that $\gamma(t) = F(\cos 2\pi t, \sin 2\pi t)$ for all $t \in [0,1]$ would represent the identity element of the fundamental group $\pi_1(\mathrm{SO}(3), \gamma(0))$. But $\gamma(t) = R(t)$ for all $t \in [0,1]$. Let Sp(1) denote the group consisting of all quaternions q satisfying $q\overline{q} = 1$, with the operation of quaternion multiplication. The group homomorphism that sends $q \in \mathrm{Sp}(1)$ to the rotation sending $r \in \mathrm{V}(\mathbb{H})$ to qrq^{-1} is a covering map from Sp(1) to SO(3). Let $\tilde{\gamma}(t) = q(t)$ for all $t \in [0,1]$, where $q(t) = (\cos \pi t)i + (\sin \pi t)j$. Then the path $\tilde{\gamma}$ is a lift of the loop γ with respect to the covering map. Were it the case that $\gamma \simeq \varepsilon_{R(0)}$ rel $\{0,1\}$ then the Monodromy Theorem would ensure that $\tilde{\gamma}\colon [0,1] \to \mathrm{Sp}(1)$ was a loop in Sp(1). However $\tilde{\gamma}(0) = i$ and $\tilde{\gamma}(1) = -i$. Therefore it cannot be the case that $\gamma \simeq \varepsilon_{R(0)}$ rel $\{0,1\}$.

5. (a) [Standard Definitions.]

(i) Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *geometrically independent* if the only solution of the linear system

$$\left\{ \begin{array}{rcl} \sum\limits_{j=0}^{q} t_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum\limits_{j=0}^{q} t_j & = & 0 \end{array} \right.$$

is the trivial solution $t_0 = t_1 = \cdots = t_q = 0$.

(ii) A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent points of $\mathbb{R}^k$.

(iii) The *barycentre* of a q-simplex σ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. The is defined to be the point $\hat{\sigma}$, where

$$\hat{\sigma} = \frac{1}{q+1}(\mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_q).$$

(iv) The *barycentric coordinates* of a point $\mathbf{x}$ of a simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are the unique real numbers $t_0, t_1, \ldots, t_q$ for which

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x} \quad \text{and} \quad \sum_{j=0}^{q} t_j = 1.$$

(v) Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a q-simplex σ in some Euclidean space $\mathbb{R}^k$. The *interior* of the simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ is the set of all points that simplex that are of the form $\sum\limits_{j=0}^{q} t_j \mathbf{v}_j$, where $t_j > 0$ for $j = 0, 1, \ldots, q$ and $\sum\limits_{j=0}^{q} t_j = 1$.

(vi) A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

(vii) The *polyhedron* $|K|$ of a simplical complex K is the union of all the simplices of the complex.

(b) [Not bookwook. Basic principles explained in lecture notes.] The vertices of τ may be determined by ordering the vertices of σ to ensure that the sequence of barycentric coordinates is non-increasing. It follows that $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_0$ is an ordering consistent with this requirement, and that the vertices $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ of τ are the barycentres

$$\mathbf{w}_0 = \frac{1}{2}(\mathbf{v}_1+\mathbf{v}_3), \quad \mathbf{w}_1 = \frac{1}{3}(\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3), \quad \mathbf{w}_2 = \frac{1}{4}(\mathbf{v}_0+\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3).$$

Then

$$\mathbf{x} = \frac{1}{6}\mathbf{w}_0 + \frac{1}{2}\mathbf{w}_1 + \frac{1}{3}\mathbf{w}_2.$$

The coefficients in this formula are the barycentric coordinates of $\mathbf{x}$ with respect to the vertices of τ, and moreover the formula confirms that τ is the simplex of K'_σ that contains the point $\mathbf{x}$ in its interior (since any point of the polyhedron of a simplicial complex belongs to the interior of a unique simplex of that complex).

(c) [Bookwork.] The *star* $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ in K is the union of the interiors of all simplices of K that contain the point $\mathbf{x}$.

Every point of $|K|$ belongs to the interior of a unique simplex of K. It follows that the complement $|K| \setminus \mathrm{st}_K(\mathbf{x})$ of $\mathrm{st}_K(\mathbf{x})$ in $|K|$ is the union of the interiors of those simplices of K that do not contain the point $\mathbf{x}$. But if a simplex of K does not contain the point $\mathbf{x}$, then the same is true of its faces. Moreover the union of the interiors of all the faces of some simplex is the simplex itself. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is the union of all simplices of K that do not contain the point $\mathbf{x}$. But each simplex of K is closed in $|K|$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is a finite union of closed sets, and is thus itself closed in $|K|$. We deduce that $\mathrm{st}_K(\mathbf{x})$ is open in $|K|$. Also $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$, since $\mathbf{x}$ belongs to the interior of at least one simplex of K.

6. (a)

$$\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle.$$

$$\begin{aligned}
\partial_{q-1}\partial_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle\right) &= \sum_{j=0}^{q} (-1)^j \partial_{q-1}\left(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle\right) \\
&= \sum_{j=1}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle \\
&\quad + \sum_{j=0}^{q-1} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q\rangle \\
&= 0
\end{aligned}$$

(b) Now $\partial_1(D_0(\langle \mathbf{v}\rangle)) = \langle \mathbf{v}\rangle - \langle \mathbf{w}\rangle$ for all vertices $\mathbf{v}$ of K. It follows that

$$\sum_{k=1}^{s} r_k \langle \mathbf{v}_k\rangle - \left(\sum_{k=1}^{s} r_k\right) \langle \mathbf{w}\rangle = \sum_{k=1}^{s} r_k(\langle \mathbf{v}_k\rangle - \langle \mathbf{w}\rangle) \in B_0(K; R)$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of K. It follows that

$$z - \varepsilon(z)\langle w\rangle \in B_0(K; R)$$

for all $z \in C_0(K; R)$, where $\varepsilon\colon C_0(K; R) \to R$ is the R-module homomorphism from $C_0(K; R)$ to R defined such that

$$\varepsilon\left(\sum_{k=1}^{s} r_k \langle \mathbf{v}_k\rangle\right) = \sum_{k=1}^{s} r_k$$

for all $r_1, r_2, \ldots, r_s \in R$ and for all vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ of K. It follows that $\ker \varepsilon \subset B_0(K; R)$. But

$$\varepsilon(\partial_1(\langle \mathbf{u}, \mathbf{v}\rangle)) = \varepsilon(\langle \mathbf{v}\rangle - \langle \mathbf{u}\rangle) = 0$$

for all edges $\mathbf{u}\,\mathbf{v}$ of K, and therefore $B_0(K; R) \subset \ker \varepsilon$. We conclude therefore that $B_0(K; R) = \ker \varepsilon$. Now $H_0(K; R) \cong C_0(K; R)/B_0(K; R)$. It follows that the homomorphism ε induces an isomorphism from $H_0(K; R)$ to the ring R of coefficients.

Also

$$
\begin{aligned}
\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle)) \\
&= \partial_{q+1}(\langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle) \\
&= \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle \\
&\qquad + \sum_{j=0}^{q} (-1)^{j+1} \langle \mathbf{w}, \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle \\
&= \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle - D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle))
\end{aligned}
$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Thus

$$\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$$

for all $c \in C_q(K; R)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K; R)$, and hence $Z_q(K; R) = B_q(K; R)$. It follows that $H_q(K; R)$ is the zero group for all $q > 0$.

7. (a) [Not bookwork.]

$$\begin{aligned}
\partial_2\sigma_1 &= \langle \mathbf{v}_2\,\mathbf{v}_3\rangle - \langle \mathbf{v}_1\,\mathbf{v}_3\rangle + \langle \mathbf{v}_1\,\mathbf{v}_2\rangle = \rho_5 - \rho_2 + \rho_1\\
\partial_2\sigma_2 &= \langle \mathbf{v}_5\,\mathbf{v}_6\rangle - \langle \mathbf{v}_1\,\mathbf{v}_6\rangle + \langle \mathbf{v}_1\,\mathbf{v}_5\rangle = \rho_{12} - \rho_4 + \rho_3\\
\partial_2\sigma_3 &= \langle \mathbf{v}_2\,\mathbf{v}_6\rangle - \langle \mathbf{v}_4\,\mathbf{v}_6\rangle + \langle \mathbf{v}_4\,\mathbf{v}_2\rangle = \rho_7 - \rho_{11} - \rho_6\\
\partial_2\sigma_4 &= \langle \mathbf{v}_5\,\mathbf{v}_3\rangle - \langle \mathbf{v}_4\,\mathbf{v}_3\rangle + \langle \mathbf{v}_4\,\mathbf{v}_5\rangle = -\rho_9 + \rho_8 + \rho_{10}
\end{aligned}$$

Therefore

$$\begin{aligned}
\partial_2\left(\sum_{i=0}^{4} m_i\sigma_i\right) &= m_1\rho_1 - m_1\rho_2 + m_2\rho_3 - m_2\rho_4\\
&\qquad + m_1\rho_5 - m_3\rho_6 + m_3\rho_7 + m_4\rho_8\\
&\qquad - m_4\rho_9 + m_4\rho_{10} - m_3\rho_{11} + m_2\rho_{12}.
\end{aligned}$$

(b) [Not bookwork.] Let $z = \sum_{i=1}^{12} n_i\rho_i$. Then

$$\begin{aligned}
0 &= \partial_1 z = (-n_1 - n_2 - n_3 - n_4)\langle \mathbf{v}_1\rangle + (n_1 - n_5 - n_6 - n_7)\langle \mathbf{v}_2\rangle\\
&\quad + (n_2 + n_5 - n_8 - n_9)\langle \mathbf{v}_3\rangle + (n_6 + n_8 - n_{10} - n_{11})\langle \mathbf{v}_4\rangle\\
&\quad + (n_3 + n_9 + n_{10} - n_{12})\langle \mathbf{v}_5\rangle + (n_4 + n_7 + n_{11} + n_{12})\langle \mathbf{v}_6\rangle
\end{aligned}$$

The coefficients of the vertices above must therefore all be zero. Let $z_i = \partial_2\sigma_i$ for $i = 1, 2, 3, 4$. Then

$$\begin{aligned}
\sum_{i=1}^{7} m_i z_i &= m_1\rho_1 + (m_5 + m_7 - m_1)\rho_2 + m_2\rho_3\\
&\quad - (m_2 + m_5 + m_7)\rho_4 + (m_1 + m_6 - m_7)\rho_5 - m_3\rho_6\\
&\quad + (m_3 - m_6 + m_7)\rho_7 + (m_4 + m_5)\rho_8 + (m_6 - m_4)\rho_9\\
&\quad + m_4\rho_{10} + (m_5 - m_3)\rho_{11} + (m_2 + m_6)\rho_{12}.
\end{aligned}$$

Suppose that $n_1, n_2, \ldots, n_{12}$ are chosen so that the coefficients of ρ_1, ρ_2, ρ_3, ρ_6, ρ_8, ρ_9 and ρ_{10} in $\sum_{i=1}^{7} m_i z_i$ are equal to n_1, n_2, n_3, n_6, n_8, n_9 and n_{10} respectively. Then $m_1 = n_1$, $m_2 = n_3$, $m_3 = -n_6$, $m_4 = n_{10}$, $m_5 = n_8 - n_{10}$, $m_6 = n_9 + n_{10}$ and $m_7 = n_1 + n_2 - n_8 + n_{10}$, and

$$\begin{aligned}
\sum_{i=1}^{7} m_i z_i &= n_1\rho_1 + n_2\rho_2 + n_3\rho_3 - (n_1 + n_2 + n_3)\rho_4\\
&\quad + (-n_2 + n_8 + n_9)\rho_5 + n_6\rho_6\\
&\quad + (n_1 + n_2 - n_6 - n_8 - n_9)\rho_7 + n_8\rho_8\\
&\quad + n_9\rho_9 + n_{10}\rho_{10} + (n_6 + n_8 - n_{10})\rho_{11}\\
&\quad + (n_3 + n_9 + n_{10})n_{12}
\end{aligned}$$

The coefficients $m_1, m_2, \ldots, m_7$ are uniquely determined by the above conditions and equations. But the requirement that $\partial_1 z = 0$ ensures that $-(n_1 + n_2 + n_3) = n_4$, $-n_2 + n_8 + n_9 = n_5$, $n_6 + n_8 - n_{10} = n_{11}$, $n_3 + n_9 + n_{10} = n_{12}$, and

$$n_1 + n_2 - n_6 - n_8 - n_9 = n_1 - n_5 - n_6 = n_7.$$

Thus $\sum_{i=1}^{7} m_i z_i = \sum_{i=1}^{12} n_i \rho_i$, as required.

(c) [Not bookwork, but standard technique.] It follows from (b) that there is a well-defined homomorphism $\varphi\colon Z_1(K;\mathbb{Z}) \to \mathbb{Z}^3$, where

$$\varphi\left(\sum_{i=1}^{7} m_i z_i\right) = (z_5, z_6, z_7).$$

This homomorphism is surjective, and its kernel is $B_1(K;\mathbb{Z})$. It follows that

$$\begin{aligned} H_1(K;\mathbb{Z}) &= Z_1(K;\mathbb{Z})/B_1(K;\mathbb{Z}) = Z_1(K;\mathbb{Z})/\ker\varphi \\ &\cong \varphi(Z_1(K;\mathbb{Z})) = \mathbb{Z}^3, \end{aligned}$$

as required.

8. (a) [Definitions. From printed lecture notes.] A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of R-modules, together with homomorphisms $\partial_i: C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient module $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i: C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1}: C_{i+1} \to C_i$.

Let C_* and D_* be chain complexes. A *chain map* $f: C_* \to D_*$ is a sequence $f_i: C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_*: A_* \to B_*$ and $q_*: B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

(b) [From printed lecture notes.] Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i: B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2}: A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then $q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_i$, by exactness. But then

$$\begin{aligned} p_{i-1}(w + \partial_i(a)) &= p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) \\ &= \partial_i(b') = p_{i-1}(w'), \end{aligned}$$

and $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i \colon Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function $\tilde{\alpha}_i$ is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1} \colon B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i \colon Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism

$$\alpha_i \colon H_i(C_*) \to H_{i-1}(A_*),$$

as required.