Modules MA3427 and MA3428: Annual Examination
Course outline and worked solutions

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Course Website

The module websites, with online lecture notes, problem sets. etc. are located at

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http://www.maths.tcd.ie/~dwilkins/Courses/MA3427/
http://www.maths.tcd.ie/~dwilkins/Courses/MA3428/
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Course Outline: MA3427

Course Outline: MA3428

Provisional—module under development, teaching not yet completed.

1. (a) [Definition.] Let X and $\tilde{X}$ be topological spaces and let $p: \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p: \tilde{X} \to X$ is said to be a *covering map* if $p: \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) [Bookwork.] Let $Z_0 = \{z \in Z : g(z) = h(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of Z. There exists an open set U in X containing the point $p(g(z))$ which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(z)$; let this open set be denoted by $\tilde{V}$. Let $N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. Then $g(z) = h(z)$, and therefore $\tilde{V} = \tilde{U}$. It follows from this that both g and h map the open set N_z into $\tilde{U}$. But $p \circ g = p \circ h$, and $p|\tilde{U}: \tilde{U} \to U$ is a homeomorphism. Therefore $g|N_z = h|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$, since $g(z) \neq h(z)$. But $g(N_z) \subset \tilde{U}$ and $h(N_z) \subset \tilde{V}$. Therefore $g(z') \neq h(z')$ for all $z' \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of Z is therefore both open and closed. Also Z_0 is non-empty by hypothesis. We deduce that $Z_0 = Z$, since Z is connected. Thus $g = h$, as required.

(d) [Not bookwork.] The number of such paths is m. The paths are $\alpha_k: [0,1] \to \mathbb{C}$ for $k = 0, 1, \ldots, m-1$, where $\alpha_k(t) = \sqrt[n]{t}e^{2\pi ik/m}$ for all $t \in [0,1]$. (Here $i = \sqrt{-1}$, and $\sqrt[n]{t}$ denotes the non-negative real number that is the non-negative nth root of the non-negative real number t.)

(e) [Not bookwork.] The map f is not a covering map. If $p: \tilde{X} \to X$ is a covering map then, given any path $\gamma: [0,1] \to X$, and given any

$w \in \tilde{X}$ for which $p(w) = \gamma(0)$, there exists a uniquely-determined path $\tilde{\gamma}\colon [0,1] \to \tilde{X}$ satisfying $\tilde{\gamma}(0) = w$ and $p \circ \tilde{\gamma} = \gamma$. This property is not possessed by the map f. Indeed when $\gamma\colon [0,1] \to \mathbb{C}$ is defined such that $\gamma(t) = t$ for all $t \in [0,1]$, and if $w = 0$ then there are m distinct paths $\tilde{\gamma}$ satisfying $f \circ \tilde{\gamma} = \gamma$.

2. [Based on lecture notes.] Let X be a topological space, and let x_0 and x_1 be points of X. A *path* in X from x_0 to x_1 is defined to be a continuous map $\gamma\colon [0,1] \to X$ for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$. A *loop* in X based at x_0 is defined to be a continuous map $\gamma\colon [0,1] \to X$ for which $\gamma(0) = \gamma(1) = x_0$.

   We can concatenate paths. Let $\gamma_1\colon [0,1] \to X$ and $\gamma_2\colon [0,1] \to X$ be paths in some topological space X. Suppose that $\gamma_1(1) = \gamma_2(0)$. We define the *product path* $\gamma_1.\gamma_2\colon [0,1] \to X$ by

$$(\gamma_1.\gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

   If $\gamma\colon [0,1] \to X$ is a path in X then we define the *inverse path* $\gamma^{-1}\colon [0,1] \to X$ by $\gamma^{-1}(t) = \gamma(1-t)$.

   Let X be a topological space, and let $x_0 \in X$ be some chosen point of X. We define an equivalence relation on the set of all (continuous) loops based at the basepoint x_0 of X, where two such loops γ_0 and γ_1 are equivalent if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0,1\}$. We denote the equivalence class of a loop $\gamma\colon [0,1] \to X$ based at x_0 by $[\gamma]$. This equivalence class is referred to as the *based homotopy class* of the loop γ. The set of equivalence classes of loops based at x_0 is denoted by $\pi_1(X, x_0)$.

   Let X be a topological space, let x_0 be some chosen point of X, and let $\pi_1(X, x_0)$ be the set of all based homotopy classes of loops based at the point x_0. We show $\pi_1(X, x_0)$ is a group, the group multiplication on $\pi_1(X, x_0)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1.\gamma_2]$ for all loops γ_1 and γ_2 based at x_0. This group is the *fundamental group* of the topological space X based at x_0.

   First we show that the group operation on $\pi_1(X, x_0)$ is well-defined. Let γ_1, γ_1', γ_2 and γ_2' be loops in X based at the point x_0. Suppose that $[\gamma_1] = [\gamma_1']$ and $[\gamma_2] = [\gamma_2']$. Let the map $F\colon [0,1] \times [0,1] \to X$ be defined by

$$F(t,\tau) = \begin{cases} F_1(2t, \tau) & \text{if } 0 \le t \le \frac{1}{2}, \\ F_2(2t-1, \tau) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

   where $F_1\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_1 and γ_1', $F_2\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_2 and γ_2', and where the homotopies F_1 and F_2 map $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Then F is itself a homotopy from $\gamma_1.\gamma_2$ to $\gamma_1'.\gamma_2'$, and maps $(0,\tau)$ and $(1,\tau)$ to x_0 for all

$\tau \in [0,1]$. Thus $[\gamma_1.\gamma_2] = [\gamma_1'.\gamma_2']$, showing that the group operation on $\pi_1(X, x_0)$ is well-defined.

Next we show that the group operation on $\pi_1(X, x_0)$ is associative. Let γ_1, γ_2 and γ_3 be loops based at x_0, and let $\alpha = (\gamma_1.\gamma_2).\gamma_3$. Then $\gamma_1.(\gamma_2.\gamma_3) = \alpha \circ \theta$, where

$$\theta(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t \leq \frac{1}{2}; \\ t - \frac{1}{4} & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}; \\ 2t - 1 & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$

Thus the map $G\colon [0,1] \times [0,1] \to X$ defined by $G(t,\tau) = \alpha((1-\tau)t + \tau\theta(t))$ is a homotopy between $(\gamma_1.\gamma_2).\gamma_3$ and $\gamma_1.(\gamma_2.\gamma_3)$, and moreover this homotopy maps $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. It follows that $(\gamma_1.\gamma_2).\gamma_3 \simeq \gamma_1.(\gamma_2.\gamma_3)$ rel $\{0,1\}$ and hence $([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]([\gamma_2][\gamma_3])$. This shows that the group operation on $\pi_1(X, x_0)$ is associative.

Let $\varepsilon\colon [0,1] \to X$ denote the constant loop at x_0, defined by $\varepsilon(t) = x_0$ for all $t \in [0,1]$. Then $\varepsilon.\gamma = \gamma \circ \theta_0$ and $\gamma.\varepsilon = \gamma \circ \theta_1$ for any loop γ based at x_0, where

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2t - 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \qquad \theta_1(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

for all $t \in [0,1]$. But the continuous map $(t,\tau) \mapsto \gamma((1-\tau)t + \tau\theta_j(t))$ is a homotopy between γ and $\gamma \circ \theta_j$ for $j = 0, 1$ which sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\varepsilon.\gamma \simeq \gamma \simeq \gamma.\varepsilon$ rel $\{0,1\}$, and hence $[\varepsilon][\gamma] = [\gamma] = [\gamma][\varepsilon]$. We conclude that $[\varepsilon]$ represents the identity element of $\pi_1(X, x_0)$.

It only remains to verify the existence of inverses. Now the map $K\colon [0,1] \times [0,1] \to X$ defined by

$$K(t,\tau) = \begin{cases} \gamma(2\tau t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma(2\tau(1-t)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

is a homotopy between the loops $\gamma.\gamma^{-1}$ and ε, and moreover this homotopy sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\gamma.\gamma^{-1} \simeq \varepsilon \operatorname{rel}\{0,1\}$, and thus $[\gamma][\gamma^{-1}] = [\gamma.\gamma^{-1}] = [\varepsilon]$. On replacing γ by γ^{-1}, we see also that $[\gamma^{-1}][\gamma] = [\varepsilon]$, and thus $[\gamma^{-1}] = [\gamma]^{-1}$, as required.

3. (a) [Bookwork.] Let x_0 and x_1 be the points of X given by

$$x_0 = \alpha(0) = \beta(0), \qquad x_1 = \alpha(1) = \beta(1).$$

Now $\alpha \simeq \beta$ rel $\{0, 1\}$, and therefore there exists a homotopy $F\colon [0, 1] \times [0, 1] \to X$ such that

$$F(t, 0) = \alpha(t) \quad \text{and} \quad F(t, 1) = \beta(t) \quad \text{for all } t \in [0, 1],$$

$$F(0, \tau) = x_0 \quad \text{and} \quad F(1, \tau) = x_1 \quad \text{for all } \tau \in [0, 1].$$

It then follows from the Monodromy Theorem that there exists a continuous map $G\colon [0, 1] \times [0, 1] \to \tilde{X}$ such that $p \circ G = F$ and $G(0, 0) = \tilde{\alpha}(0)$. Then $p(G(0, \tau)) = x_0$ and $p(G(1, \tau)) = x_1$ for all $\tau \in [0, 1]$. A basic result concerning uniqueness of lifts of continuous paths ensures that any continuous lift of a constant path must itself be a constant path. Therefore $G(0, \tau) = \tilde{x}_0$ and $G(1, \tau) = \tilde{x}_1$ for all $\tau \in [0, 1]$, where

$$\tilde{x}_0 = G(0, 0) = \tilde{\alpha}(0), \qquad \tilde{x}_1 = G(1, 0).$$

However

$$G(0, 0) = G(0, 1) = \tilde{x}_0 = \tilde{\alpha}(0) = \tilde{\beta}(0),$$

$$p(G(t, 0)) = F(t, 0) = \alpha(t) = p(\tilde{\alpha}(t))$$

and

$$p(G(t, 1)) = F(t, 1) = \beta(t) = p(\tilde{\beta}(t))$$

for all $t \in [0, 1]$. It follows that the map that sends $t \in [0, 1]$ to $G(t, 0)$ is a lift of the path α that starts at $\tilde{x}_0$, and the map that sends $t \in [0, 1]$ to $G(t, 1)$ is a lift of the path β that also starts at $\tilde{x}_0$. However the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β are uniquely determined by their starting points. It follows that $G(t, 0) = \tilde{\alpha}(t)$ and $G(t, 1) = \tilde{\beta}(t)$ for all $t \in [0, 1]$. In particular,

$$\tilde{\alpha}(1) = G(1, 0) = \tilde{x}_1 = G(1, 1) = \tilde{\beta}(1).$$

Moreover the map $G\colon [0, 1] \times [0, 1] \to \tilde{X}$ is a homotopy between the paths $\tilde{\alpha}$ and $\tilde{\beta}$ which satisfies $G(0, \tau) = \tilde{x}_0$ and $G(1, \tau) = \tilde{x}_1$ for all $\tau \in [0, 1]$. It follows that $\tilde{\alpha} \simeq \tilde{\beta}$ rel $\{0, 1\}$, as required.

(b) [Bookwork: part of larger proof concerning the fundamental group of the circle.]

Let $F\colon [0,1]\times[0,1] \to S^1$ be the homotopy between α and β defined by

$$F(t,\tau) = p\left((1-\tau)\tilde{\alpha}(t) + \tau\tilde{\beta}(t)\right),$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the lifts of α and β respectively satisfying $\tilde{\alpha}(0) = \tilde{\beta}(0) = 0$. Then

$$F(0,\tau) = p\left((1-\tau)\tilde{\alpha}(0) + \tau\tilde{\beta}(0)\right) = p(\tilde{\alpha}(0)) = \alpha(0) = \mathbf{b}$$

for all $\tau \in [0,1]$, because $\tilde{\beta}(0) = \tilde{\alpha}(0)$. Similarly

$$F(1,\tau) = p\left((1-\tau)\tilde{\alpha}(1) + \tau\tilde{\beta}(1)\right) = p(\tilde{\alpha}(1)) = \alpha(1) = \mathbf{b}$$

for all $\tau \in [0,1]$. Thus $\alpha \simeq \beta$ rel $\{0,1\}$, and thus the loops α and β represent the same element of the fundamental group $\pi_1(S^1, \mathbf{b})$.

4. (a) [Bookwork.] A two-dimensional simplicial complex is a finite collection of triangles, edges and vertices in some ambient Euclidean space. Each of those triangles, edges and vertices is a closed subset of the ambient Euclidean space, and therefore the union of any finite collection of such triangles, edges and vertices is a closed subset of the ambient Euclidean space.

   Now, given any point $\mathbf{p}$ of $|K|$, the complement $|K| \setminus \mathrm{st}_K(\mathbf{p})$ of the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ of $\mathbf{p}$ in $|K|$ is by definition the union of all triangles, edges and vertices belonging to K that do not contain the point $\mathbf{p}$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{p})$ is closed in $|K|$, and $\mathbf{p} \notin |K| \setminus \mathrm{st}_K(\mathbf{p})$. Therefore $\mathrm{st}_K(\mathbf{p})$ is open in $|K|$, and $\mathbf{p} \in \mathrm{st}_K(\mathbf{p})$, as required.

   (b) [Bookwork.] Let σ_0 be a triangle in K, and let F be the subset of the polyhedron $|K|$ of K which is the union of all triangles that can be joined to σ_0 by a finite sequence of triangles belonging to K, where successive triangles in this sequence intersect along a common edge. Then F is a finite union of triangles, and those trianges are closed subsets of $|K|$, and therefore F is itself a closed subset of $|K|$.

   Let $\mathbf{p}$ be a point of F. If $\mathbf{p}$ does not lie on any edge belonging to K then the star neighbourhood $\mathrm{st}_K(\mathbf{p})$ belongs to just one triangle belonging to K, and moreover this triangle must then be a subset of F (or else the point $\mathbf{p}$ would not belong to F). Thus if $\mathbf{p} \in F$ does not like on any edge belonging to K then $\mathrm{st}_K(\mathbf{p}) \subset F$.

   Next suppose that the point $\mathbf{p}$ of F lies on some edge belonging to K but is not an endpoint of that edge. Then the point $\mathbf{p}$ belongs to exactly two triangles of K that intersect along a common edge (because the two-dimensional simplicial complex represents a closed surface). At least one of these triangles must be contained in the set F (since $\mathbf{p} \in F$) and therefore both triangles are contained in F. But the star neighbourhood of the point $\mathbf{p}$ is contained in the union of those two triangles. Therefore $\mathrm{st}_K(\mathbf{p}) \subset F$ in this case also.

   Finally suppose that the point $\mathbf{p}$ is a vertex of K. Then the requirement that the two-dimensional simplicial complex K represent a triangulated closed surface ensures that if at least one of the triangles belonging to K with a vertex at $\mathbf{p}$ is contained in F then every triangle belonging to K with a vertex at $\mathbf{v}$ must be contained in F. It follows that $\mathrm{st}_K(\mathbf{p}) \subset F$.

   We have now shown that, given any point $\mathbf{p}$ of F, the star neighbourhood $\text{st}_K(\mathbf{p})$ of $\mathbf{p}$ in $|K|$ is a subset of F. But this star neighbourhood is an open subset of $|K|$ (by the result of (a)). Therefore the subset F of $|K|$ is both open and closed in $|K|$. Thus if the topological space $|K|$ is connected then $F = |K|$.

Every point of a topological space belongs to unique connected component which is the union of all connected subsets of the topological space that contain the given point. It follows that every triangle belonging to K is contained in a some connected component of $|K|$, and if two triangles belonging to K intersect along a common edge, or at a common vertex, then both belong to the same connected component of $|K|$. It follows that the set F is contained in some connected component of $|K|$. Thus if the topological space $|K|$ is not connected then F is a proper subset of $|K|$. We deduce that $F = |K|$ if and only if $|K|$ is a connected topological space. The result follows.

Note: *the above proof is given in the distributed lecture notes. There are other ways to arrive at the result which may depend, to a greater or lesser extent, on the course material.*

5. (a) [Definition.] Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *geometrically independent* if the only solution of the linear system

$$\left\{ \begin{array}{rcl} \sum_{j=0}^{q} \lambda_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum_{j=0}^{q} \lambda_j & = & 0 \end{array} \right.$$

is the trivial solution $\lambda_0 = \lambda_1 = \cdots = \lambda_q = 0$.

(b) [Bookwork.] Suppose that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent. Let $\lambda_1, \lambda_2, \ldots, \lambda_q$ be real numbers which satisfy the equation

$$\sum_{j=1}^{q} \lambda_j (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

Then $\sum_{j=0}^{q} \lambda_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} \lambda_j = 0$, where $\lambda_0 = -\sum_{j=1}^{q} \lambda_j$, and therefore

$$\lambda_0 = \lambda_1 = \cdots = \lambda_q = 0.$$

It follows that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Conversely, suppose that these displacement vectors are linearly independent. Let $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_q$ be real numbers which satisfy the equations $\sum_{j=0}^{q} \lambda_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} \lambda_j = 0$. Then $\lambda_0 = -\sum_{j=1}^{q} \lambda_j$, and therefore

$$\mathbf{0} = \sum_{j=0}^{q} \lambda_j \mathbf{v}_j = \lambda_0 \mathbf{v}_0 + \sum_{j=1}^{q} \lambda_j \mathbf{v}_j = \sum_{j=1}^{q} \lambda_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows from the linear independence of the displacement vectors $\mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$ that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_q = 0.$$

But then $\lambda_0 = 0$ also, because $\lambda_0 = -\sum_{j=1}^{q} \lambda_j$. It follows that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent, as required.

(c) [Definition.] A *simplex* in $\mathbb{R}^k$ of dimension q with vertices

$$\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$$

is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \le t_j \le 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent points of $\mathbb{R}^k$.

(d) [Definitions.] A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The *dimension* of a simplicial complex K is the maximum of the dimensions of the simplices of K.

A subset L of a simplicial complex K is said to be a *subcomplex* of K if every face of every simplex of L belongs to L.

(e) [Essentially bookwork: principle employed in proofs, though not isolated as a result in its own right.] Let ρ be a simplex of L and let τ be a proper face of ρ. Then $\dim \tau < \dim \rho \le \dim K = \dim \sigma$, and therefore $\tau \neq \sigma$, and thus $\tau \in L$. It follows that if $\rho \in L$ then $\tau \in L$ for all proper faces τ of ρ. Thus L is a subcomplex of K.

6. [Entire question is a problem. Not bookwork.]

(a) By inspection the boundary of the 2-chain is given by

$$\begin{aligned}(a+d+g)\langle \mathbf{v}_1\mathbf{v}_2\rangle + (b+e+g)\langle \mathbf{v}_2\mathbf{v}_3\rangle + (c+f+g)\langle \mathbf{v}_3\mathbf{v}_1\rangle \\ + (c-a)\langle \mathbf{v}_1\mathbf{v}_4\rangle + (a-b)\langle \mathbf{v}_2\mathbf{v}_4\rangle + (b-c)\langle \mathbf{v}_3\mathbf{v}_4\rangle \\ + (f-d)\langle \mathbf{v}_1\mathbf{v}_5\rangle + (d-e)\langle \mathbf{v}_2\mathbf{v}_5\rangle + (e-f)\langle \mathbf{v}_3\mathbf{v}_5\rangle\end{aligned}$$

Thus the boundary of the 2-chain is zero if and only if $a = b = c$, $d = e = f$ and $a + d + g = 0$. It follows that the 2-chain is a 2-cycle if and only if it is of the form $mz_1 + nz_2$ for some integers m and n. (Indeed z_1 and z_2 are 2-cycles, and if the 2-chain of *(a)* is a 2-cycle then it is of the form $mz_1 + nz_2$ with $a = b = c = m$, $d = e = f = n$ and $g = -m - n$.)

Now $H_2(K) = Z_2(K)$ since $B_2(K) = 0$. The function sending $mz_1 + nz_2$ to (m, n) is an isomorphism from $Z_2(K)$ to $\mathbb{Z} \oplus \mathbb{Z}$. Thus $H_2(K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(b)

(i) In order that the 1-chain

$$\langle \mathbf{v}_1\mathbf{v}_2\rangle + \langle \mathbf{v}_2\mathbf{v}_4\rangle + \langle \mathbf{v}_4\mathbf{v}_3\rangle + \langle \mathbf{v}_3\mathbf{v}_5\rangle + \langle \mathbf{v}_5\mathbf{v}_1\rangle$$

be the boundary of the 2-chain specified in (a), there must exist integers a, b, c, d, e and f such that

$$\begin{aligned}&\langle \mathbf{v}_1\mathbf{v}_2\rangle + \langle \mathbf{v}_2\mathbf{v}_4\rangle + \langle \mathbf{v}_4\mathbf{v}_3\rangle + \langle \mathbf{v}_3\mathbf{v}_5\rangle + \langle \mathbf{v}_5\mathbf{v}_1\rangle \\ &\quad = (a+d+g)\langle \mathbf{v}_1\mathbf{v}_2\rangle + (b+e+g)\langle \mathbf{v}_2\mathbf{v}_3\rangle \\ &\qquad + (c+f+g)\langle \mathbf{v}_2\mathbf{v}_3\rangle \\ &\qquad + (c-a)\langle \mathbf{v}_1\mathbf{v}_4\rangle + (a-b)\langle \mathbf{v}_2\mathbf{v}_4\rangle + (b-c)\langle \mathbf{v}_3\mathbf{v}_4\rangle \\ &\qquad + (f-d)\langle \mathbf{v}_1\mathbf{v}_5\rangle + (d-e)\langle \mathbf{v}_2\mathbf{v}_5\rangle + (e-f)\langle \mathbf{v}_3\mathbf{v}_5\rangle.\end{aligned}$$

We thus require that

$$a+d+g = 1, \quad b+e+g = 0, \quad c+f+g = 0,$$

$$c-a = 0, \quad a-b = 1, \quad b-c = -1,$$

$$f-d = -1, \quad d-e = 0, \quad e-f = 1.$$

Then $b = a-1$, $c = a$, $e = d$, $f = d-1$ and $g = 1-a-d$. Each pair of integers a, d determines a solution to these equations.

In particular if $a = d = 0$ then $c = e = 0$, $b = -1$, $f = -1$ and $g = 1$. It follows that the 1-chain

$$\langle \mathbf{v}_1\mathbf{v}_2 \rangle + \langle \mathbf{v}_2\mathbf{v}_4 \rangle + \langle \mathbf{v}_4\mathbf{v}_3 \rangle + \langle \mathbf{v}_3\mathbf{v}_5 \rangle + \langle \mathbf{v}_5\mathbf{v}_1 \rangle$$

is the boundary of

$$\langle \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 \rangle - \langle \mathbf{v}_2\mathbf{v}_3\mathbf{v}_4 \rangle - \langle \mathbf{v}_3\mathbf{v}_1\mathbf{v}_5 \rangle.$$

(ii) The 1-chain $3\langle \mathbf{v}_1\mathbf{v}_2 \rangle + 4\langle \mathbf{v}_2\mathbf{v}_3 \rangle - \langle \mathbf{v}_3\mathbf{v}_1 \rangle$ is not a 1-boundary since it is not a 1-cycle:

$$\partial_1(3\langle \mathbf{v}_1\mathbf{v}_2 \rangle + 4\langle \mathbf{v}_2\mathbf{v}_3 \rangle - \langle \mathbf{v}_3\mathbf{v}_1 \rangle) = -4\langle \mathbf{v}_1 \rangle - \langle \mathbf{v}_2 \rangle + 5\langle \mathbf{v}_3 \rangle.$$

7. (a) [From printed lecture notes.] A sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of R-modules and R-module homomorphisms is said to be *exact* at G if and only if $\text{image}(p\colon F \to G) = \ker(q\colon G \to H)$. A sequence of R-modules and R-module homomorphism is said to be *exact* if it is exact at each R-module occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

(b) $\phi \circ \psi_1 = \psi_2 \circ \theta$

(c) [Based on printed lecture notes.] First we prove that if ψ_2 and ψ_4 are monomorphisms and if ψ_1 is a epimorphism then ψ_3 is an monomorphism, Suppose that ψ_2 and ψ_4 are monomorphisms and that ψ_1 is an epimorphism. We wish to show that ψ_3 is a monomorphism. Let $x \in G_3$ be such that $\psi_3(x) = 0$. Then $\psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0$, and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover

$$\phi_2(\psi_2(y)) = \psi_3(\theta_2(y)) = \psi_3(x) = 0,$$

hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since ψ_1 is an epimorphism. Then

$$\psi_2(\theta_1(w)) = \phi_1(\psi_1(w)) = \psi_2(y),$$

and hence $\theta_1(w) = y$, since ψ_2 is a monomorphism. But then

$$x = \theta_2(y) = \theta_2(\theta_1(w)) = 0$$

by exactness. Thus ψ_3 is a monomorphism.

Next we prove that if ψ_2 and ψ_4 are epimorphisms and if ψ_5 is a monomorphism then ψ_3 is an epimorphism. Thus suppose that ψ_2 and ψ_4 are epimorphisms and that ψ_5 is a monomorphism. We wish to show that ψ_3 is an epimorphism. Let a be an element of H_3. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since ψ_4 is an epimorphism. Now

$$\psi_5(\theta_4(b)) = \phi_4(\psi_4(b)) = \phi_4(\phi_3(a)) = 0,$$

hence $\theta_4(b) = 0$, since ψ_5 is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then

$$\phi_3(\psi_3(c)) = \psi_4(\theta_3(c)) = \psi_4(b),$$

hence $\phi_3\left(a-\psi_3(c)\right)=0$, and thus $a-\psi_3(c)=\phi_2(d)$ for some $d\in H_2$, by exactness. But ψ_2 is an epimorphism, hence there exists $e\in G_2$ such that $\psi_2(e)=d$. But then

$$\psi_3\left(\theta_2(e)\right)=\phi_2\left(\psi_2(e)\right)=a-\psi_3(c).$$

Hence $a=\psi_3\left(c+\theta_2(e)\right)$, and thus a is in the image of ψ_3. This shows that ψ_3 is an epimorphism.

It follows that if ψ_1, ψ_2, ψ_4 and ψ_5 are isomorphisms, then so is ψ_3.

8. (a) [Based on lecture notes.] The qth chain group $C_q(K, L; R)$ of the simplicial pair is defined to be the quotient group

$$C_q(K; R)/C_q(L; R),$$

where $C_q(K; R)$ and $C_q(L; R)$ denote the groups of q-chains of K and L respectively.

The boundary homomorphism $\partial_q: C_q(K; R) \to C_{q-1}(L; R)$ maps the subgroup $C_q(L; R)$ into $C_{q-1}(L; R)$, and therefore induces a homomorphism $\partial_q: C_q(K, L; R) \to C_{q-1}(K, L; R)$. We define

$$H_q(K, L; R) = Z_q(K, L; R)/B_q(K, L; R),$$

where

$$\begin{aligned} Z_q(K, L; R) &= \ker(\partial_q: C_q(K, L; R) \to C_{q-1}(K, L; R)) \\ &= \{c + C_q(L; R) : c \in C_q(K; R) \\ &\qquad \text{and } \partial_q c \in C_{q-1}(L; R)\}, \\ B_q(K, L; R) &= \text{image}(\partial_{q+1}: C_{q+1}(K, L; R) \to C_q(K, L; R)) \\ &= \{\partial_{q+1}(e) + C_q(L; R) : e \in C_{q+1}(K; R)\}. \end{aligned}$$

(b) [Based on lecture notes.] The homology exact sequence of the simplicial pair (K, L) is the exact sequence

$$\cdots \xrightarrow{\partial_*} H_q(L; R) \xrightarrow{i_*} H_q(K; R) \xrightarrow{u_*} H_q(K, L; R) \xrightarrow{\partial_*} H_{q-1}(L; R)$$
$$\xrightarrow{i_*} H_{q-1}(K; R) \xrightarrow{u_*} \cdots$$

of homology groups is exact, where the homomorphisms i_*, u_* and ∂_* are induced by the inclusion homomorphisms i_q, quotient homomorphisms u_q and boundary homomorphisms ∂_q respectively between the relevant chain groups.

(c) [Problem. Not bookwork.] Let K be the kernel of the homomorphism $\partial_*: H_3(K, L, \mathbb{Z}) \to H_2(L; \mathbb{Z})$. Then the sequence

$$H_3(L; \mathbb{Z}) \xrightarrow{i_*} H_3(K; \mathbb{Z}) \xrightarrow{u_*} K \to 0$$

is exact, as this is a portion of the homology exact sequence of the simplicial pair (K, L). But $H_3(L; \mathbb{Z}) = 0$. It follows from exactness that $H_3(K; \mathbb{Z}) \cong K$. Now $\partial_*(m\alpha + n\beta) = 15m + 21n\gamma$ for all $m, n \in \mathbb{Z}$. It follows that

$$\begin{aligned} K &= \{m\alpha + n\beta : m, n \in \mathbb{Z} : 15m + 21n = 0\} \\ &= \{j(7\alpha - 5\beta) : j \in \mathbb{Z}\}. \end{aligned}$$

Thus $H_3(K, \mathbb{Z}) \cong K \cong \mathbb{Z}$.

Also the sequence

$$H_3(K, L; \mathbb{Z}) \xrightarrow{\partial_*} H_2(L; \mathbb{Z}) \xrightarrow{i_*} H_2(K; \mathbb{Z}) \xrightarrow{u_*} H_2(K, L; \mathbb{Z})$$

is exact. But $H_2(K, L; \mathbb{Z}) = 0$. It follows that the sequence

$$0 \to \partial_*(H_3(K, L; \mathbb{Z})) \to H_2(L; \mathbb{Z}) \xrightarrow{i_*} H_2(K; \mathbb{Z}) \to 0$$

is exact. But

$$H_2(L; \mathbb{Z}) = \{k\gamma : k \in \mathbb{Z}\}$$

and

$$\partial_*(H_3(K, L; \mathbb{Z})) = \{(15m + 27n)\gamma : m, n \in \mathbb{Z}\}.$$

It follows that $H_2(K; Z) \cong \mathbb{Z}/I$, where

$$I = \{15m + 27n : m, n \in \mathbb{Z}\}.$$

Now the highest common fact of 15 and 27 is 3. It follows that there exist integers m_1 and n_1 such that $3 = 15m_1 + 27n_1$, and therefore I is the subgroup $3\mathbb{Z}$ of $\mathbb{Z}$ generated by the positive integer 3. Thus

$$H_2(K; \mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3,$$

as required.